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Robust stabilization to non-linear delayed systems via delayed state feedback: the averaging method

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Abstract

The problem of robust stabilization to dynamical systems via delayed feedback is important in applications. Due to the fact that the characteristic quasi-polynomial of a delayed system is difficult to analyze when uncertain parameters are involved, this problem has been most frequently solved on the basis of the method of Lyapunov functional, by solving Riccati equations, or by solving linear matrix inequalities. By applying the averaging method that reduces the delay differential equation of infinite dimensional to an ordinary differential equation, this paper presents a simple method to stabilize the trivial solution or periodic solutions of a type of non-linear delayed vibration systems via delayed state feedback. In particular, this method is applied to the robust stabilization of those systems when the system parameters are uncertain, but fall into given intervals, respectively. In addition, an extension is made to this problem for a general class of delayed systems that result from a small perturbation of a linear delay system with characteristic roots of non-positive real parts only. This can serve as a straightforward application to the Hopf bifurcation control of delayed systems with weak non-linearity.

1. Introduction

The active control technique has drawn much attention over the past decades. One of important control objectives is to stabilize engineering systems if their equilibrium positions or steady-state motions of the control plants are not asymptotically stable. To reach good performance of stabilization, two factors should be taken into consideration in the design phase of a control. One comes from the inevitable time delays in the feedback path or deliberately introduced time delays

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in control, and the other from the uncertainty of both time delay and system parameters due to measurement, modelling errors, etc. The feedback gains, thus, should be chosen so as to render the controlled system asymptotically stable for all parameter combinations in a given admissible set. This problem falls into the category of robust stabilization and has been intensively studied over the past decades, see for example Refs. [1–3] and references therein.

When a delayed feedback control is performed, the controlled system is described by a delay differential equation (DDE), a special class of functional differential equations (FDE) and has been intensively studied, see for examples, Refs. [2,4–6]. Generally speaking, the stability analysis or the stabilization problem of a DDE can be solved by investigating the root location of the characteristic function or by using the method of Lyapunov functional. It is usually very difficult to achieve a simple relationship for the stability/stabilization condition in terms of the system parameters by using the method of characteristic roots, so in the literature, the problem of robust stabilization to time delayed systems, or the stabilization problem via delayed feedback control, is most frequently solved on the basis of the method of Lyapunov functional, by solving Riccati equations [7,8], or by solving linear matrix inequalities [2,9,10]. Such techniques depend heavily on the estimation technique and thus usually give conservative results.

In many applications, the system can be considered as a small perturbation of a simple system, so it is possible to apply singular perturbation methods to solve this stabilization problem. Among the singular perturbation methods for DDEs, the method of multiple scales is frequently used to determine the periodic solution coming either from Hopf bifurcation or from a periodic excitation, see for example, Refs. [5,11,12]. Another widely used method is the averaging method. For example, it has been successfully implemented to study the complex dynamics of two coupled van der Pol oscillators with delay [13]. In addition, the averaging technique has also been applied in Ref. [14] to stabilize planar biological systems via delayed feedback of displacement, and it has been proved that a linear delayed feedback can always stabilize the trivial solution. In the literature regarding the averaging method, however, the uncontrolled systems are usually ordinary differential equations (ODE) that result from a small perturbation of an undamped vibration system such as in Refs. [13,14]. This limits its applications in practice.

The objective of this study is to develop a simple method, on the basis of the averaging method, for the robust stabilization to the unstable equilibrium of a class of delayed vibration systems via delayed feedback control. The system equation under study is described by a much broader class of DDEs than that discussed in the previous literature. The averaging method is actually not only applicable to the robust stabilization problem of the trivial solution, but works also for the problem of Hopf bifurcation control of delayed systems with weak non-linearity, a problem to annihilate the periodic solution or to obtain an asymptotically stable periodic solution with desired properties such as oscillation at a given amplitude. The problem is formulated in Section 2. In order to derive a simple stability condition for the trivial solution so that the problem of robust stabilization can be solved easily, the averaged equation is first derived in Section 3. In Section 4, the problem of robust stabilization to the vibration system is discussed in detail. A generalization of the results is made in Section 5, which can also be applied to study the problem of Hopf bifurcation control. Finally in Section 6, several concluding remarks are drawn from the discussion.

2. The problem formulation

In this paper, we study the problem of robust stabilization to a delayed s.d.o.f. vibration system via delayed feedback in general form

$$\ddot{x}(t) + \omega^2 x(t) + \varepsilon g(x(t), \dot{x}(t), x(t-\tau_1), \dot{x}(t-\tau_1)) = \varepsilon f(x(t), \dot{x}(t), x(t-\tau_2), \dot{x}(t-\tau_2)), \tag{1}$$

where $0 < \varepsilon \ll 1$ is the small parameter, $f(0,0,0,0) = g(0,0,0,0) = 0$, $x \in R$, εf is the control force, $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ are the delays in the control plant and the controller, respectively. The trivial equilibrium of the uncontrolled system (namely $f = 0$) is assumed to be unstable, this is the case when the control plant undergoes Hopf bifurcation at $\varepsilon = 0$. To stabilize the equilibrium or to annihilate the periodic solutions means to perform a delayed feedback control so that the trivial equilibrium of the controlled system is asymptotically stable. In some applications, it is required to maintain a periodic solution with given amplitude.

Eq. (1) is a small perturbation of an undamped s.d.o.f. vibration system. In practice, it is also required to study the robust stabilization problem to the following equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau_1) + \varepsilon\mathbf{g}(\mathbf{x}(t), \mathbf{x}(t-\tau_1)) + \varepsilon\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau_2)), \tag{2}$$

where $\mathbf{g}(\mathbf{0},\mathbf{0}) = \mathbf{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$, $\mathbf{x} \in R^n$ and $0 < \varepsilon \ll 1$. Eq. (2) is a small perturbation of a linear delayed system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau_1). \tag{3}$$

We assume that the linear delayed Eq. (3) has exactly one pair of conjugate eigenvalues: $\lambda = \pm i\omega$, and all the other eigenvalues stay in the open left-half complex plane. This case is encountered when a delayed system with weak non-linearity undergoes Hopf bifurcation at $\varepsilon = 0$. For example, it is easy to show that when $T = 1.2264$, the quasi-polynomial $\lambda^2 - 0.5\lambda e^{-\lambda T} + 1$ has exactly one pair of roots $\lambda = \pm 1.2088i$ and all the other roots stay in the left-half complex plane. Consequently the vibration system

$$\ddot{x}(t) - 0.5(1.2264+\varepsilon)\dot{x}(t-1) + (1.2264+\varepsilon)^2 x(t) + \varepsilon\mu(1.2264+\varepsilon)^2 x^3(t) = 0 \tag{4}$$

undergoes Hopf bifurcation at $\varepsilon = 0$, and for small $0 < \varepsilon \ll 1$, the vibration frequency of the bifurcating periodic solution is $\omega = 1.2088 + O(\varepsilon^3)$, see for example Ref. [5]. An equation of the form Eq. (2) is then obtained after proper transformation if a (weak) linear delayed feedback control $\varepsilon[ux(t-\tau) + v\dot{x}(t-\tau)]$ is performed on Eq. (4). The control objective is to annihilate the periodic solution, or to stabilize the periodic solutions via delayed feedback.

Usually, uncertainty in time delay and system parameters exists due to measurement, modelling errors, etc. So the asymptotical stability should be robust with respect to uncertain delays, or to the uncertain system parameters, or to both of them. In solving the robust stabilization problem, the introduction of the small parameter ε is important and useful so that we can use the singular perturbation methods to examine the dynamics of the corresponding DDE and to derive simple stability conditions such that the robust stabilization problem can be achieved easily.

3. The averaged equation

For small $0<\varepsilon\ll 1$, as in the case of ODE, it is beneficial to introduce the following transformation:

$$x(t) = r(t)\cos(\omega t + \theta(t)),$$
$$\dot{x}(t) = -\omega r(t)\sin(\omega t + \theta(t)), \tag{5}$$

where both $r(t)$ and $\theta(t)$ are to be determined. Substituting Eq. (5) into Eq. (1) yields

$$\omega\dot{r}(t) = \varepsilon\sin(\omega t + \theta(t))(g - f),$$
$$\omega r(t)\dot{\theta}(t) = \varepsilon\cos(\omega t + \theta(t))(g - f), \tag{6}$$

where

$$f := f(r(t)\cos(\omega t + \theta(t)), -r(t)\omega\sin(\omega t + \theta(t)), r(t-\tau_2)\cos(\omega t - \omega\tau_2 + \theta(t-\tau_2)),$$
$$-\ r(t-\tau_2)\omega\sin(\omega t - \omega\tau_2 + \theta(t-\tau_2))),$$

$$g := g(r(t)\cos(\omega t + \theta(t)), -r(t)\omega\sin(\omega t + \theta(t)), r(t-\tau_1)\cos(\omega t - \omega\tau_1 + \theta(t-\tau_1)),$$
$$-\ r(t-\tau_1)\omega\sin(\omega t - \omega\tau_1 + \theta(t-\tau_1))).$$

Let $\Omega := C([-\tau, 0], R^2), (\tau = \max\{\tau_1, \tau_2\})$, be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to R^2, then Eq. (6) can be viewed as a FDE

$$\dot{\mathbf{y}}(t) = \varepsilon\mathbf{Y}(t, \mathbf{y}_t), \quad \mathbf{y}(t) := [r(t)\ \theta(t)]^{\mathrm{T}} \in R^2, \tag{7}$$

where $\mathbf{y}_t(\eta) := \mathbf{y}(t+\eta)$, $(\forall\eta\in[-\tau, 0])$, and $\mathbf{Y}(t,\cdot)$ is periodic with period $T = 2\pi/\omega$. For small ε, $\mathbf{y}(t)$ varies slowly, and $\mathbf{y}_t(\eta) = \mathbf{y}(t) + O(\varepsilon)$ holds for $\forall\eta\in[-\tau, 0]$ with no restriction on the delay interval [15]. This means that in applying the averaging method, $r(t-\tau_1)$ and $\theta(t-\tau_1)$, $r(t-\tau_2)$ and $\theta(t-\tau_2)$, can be replaced by $r(t)$ and $\theta(t)$ over one period, respectively. In this way, the averaging technique reduces the DDE to an ODE.

In fact, we define

$$F(r) := \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} f\,(r\cos(\omega t + \theta), -r\omega\sin(\omega t + \theta), r\cos(\omega t - \omega\tau_2 + \theta),$$
$$-\ r\omega\sin(\omega t - \omega\tau_2 + \theta))\sin(\omega t + \theta)\,\mathrm{d}t,$$

$$G(r) := \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} g\,(r\cos(\omega t + \theta), -r\omega\sin(\omega t + \theta), r\cos(\omega t - \omega\tau_1 + \theta),$$
$$-\ r\omega\sin(\omega t - \omega\tau_1 + \theta))\sin(\omega t + \theta)\,\mathrm{d}t, \tag{8}$$

which satisfies $F(0) = G(0) = 0$, and they can be simplified as to

$$F(r) := \frac{1}{2\pi}\int_\theta^{2\pi+\theta} f(r\cos\phi, -r\omega\sin\phi, r\cos(\phi - \omega\tau_2), -r\omega\sin(\phi - \omega\tau_2))\sin\phi\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\phi, -r\omega\sin\phi, r\cos(\phi - \omega\tau_2), -r\omega\sin(\phi - \omega\tau_2))\sin\phi\,\mathrm{d}t, \tag{9}$$

$$G(r) := \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\phi, -r\omega\sin\phi, r\cos(\phi-\omega\tau_1), -r\omega\sin(\phi-\omega\tau_1))\sin\phi\,\mathrm{d}t. \tag{10}$$

Then the averaged equation of Eq. (6) that governs the variation of $r(t)$ is as follows:

$$\omega\dot{r} = \varepsilon h(r), \quad h(r) := G(r) - F(r). \tag{11}$$

And an equation that describes the variation of $\theta(t)$ can be obtained as well. As is well known, an equilibrium $r = r_0$ of Eq. (11) is asymptotically stable if $h'(r_0) < 0$, and unstable if $h'(r_0) > 0$.

4. Robust stabilization

In this section, we first derive a simple stability condition for the controlled system (1) with fixed parameters, then study the problem of robust stabilization when the system has uncertain parameters. Two simple demonstrative examples will be presented.

4.1. Stability of the averaged equation

One can readily prove that the above claim about the stability of averaged equation is also true for the stability of $x = 0$ of the original Eq. (1).

In fact, denote by D_1, D_2, D_3 and D_4 the partial derivative operators with respect to from the first to the fourth variables, respectively, in f, g, then the characteristic quasi-polynomial $p(\lambda, \varepsilon)$ of the linearized system of Eq. (1) reads

$$p(\lambda,\varepsilon) = \lambda^2 + \omega^2 + \varepsilon[b_1 + b_2\lambda + \mathrm{e}^{-\lambda\tau_1}(b_3 + b_4\lambda)] - \varepsilon[a_1 + a_2\lambda + \mathrm{e}^{-\lambda\tau_2}(a_3 + a_4\lambda)], \tag{12}$$

where

$$a_1 = D_1f(0,0,0,0), \quad a_2 = D_2f(0,0,0,0), \quad a_3 = D_3f(0,0,0,0), \quad a_4 = D_4f(0,0,0,0),$$

$$b_1 = D_1g(0,0,0,0), \quad b_2 = D_2g(0,0,0,0), \quad b_3 = D_3g(0,0,0,0), \quad b_4 = D_4g(0,0,0,0).$$

The equilibrium $x = 0$ of Eq. (1) is asymptotically stable if all the roots of $p(\lambda, \varepsilon)$ stay in the open left-half complex plane. When $\varepsilon = 0$, Eq. (12) has a pair of roots $\pm\mathrm{i}\omega$ on the imaginary axis, and the equilibrium $x = 0$ of Eq. (1) is critically stable. An application of the implicit function theorem shows that the roots $\lambda(\varepsilon)$ of Eq. (12) depend smoothly on ε in a neighborhood of $\varepsilon = 0$. The movement of the above two roots becomes clear if the sign of $S^* = \mathrm{Re}\,\lambda'(0)$ is not zero. For small $\varepsilon > 0$, the equilibrium $x = 0$ of Eq. (1) is asymptotically stable if $S^* < 0$, and unstable if $S^* > 0$.

Performing implicit differentiation of λ in Eq. (12) with respect to ε at $\varepsilon = 0$ gives

$$S^* = -\frac{b_2}{2} + \frac{b_3\sin(\omega\tau_1)}{2\omega} - \frac{b_4\cos(\omega\tau_1)}{2} + \frac{a_2}{2} - \frac{a_3\sin(\omega\tau_2)}{2\omega} + \frac{a_4\cos(\omega\tau_2)}{2}. \tag{13}$$

On the other hand, straightforward computation tells

$$h'(0) = -\frac{b_2\omega}{2} + \frac{b_3\sin(\omega\tau_1)}{2} - \frac{b_4\omega\cos(\omega\tau_1)}{2} + \frac{a_2\omega}{2} - \frac{a_3\sin(\omega\tau_2)}{2} + \frac{a_4\omega\cos(\omega\tau_2)}{2}. \tag{14}$$

Therefore, one has

$$S^* = \frac{h'(0)}{\omega}. \tag{15}$$

It follows that there is a small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, one has $\mathrm{Re}\,\lambda(\varepsilon) < 0$ if $h'(0) < 0$, or there is at least one pair of characteristic roots with $\mathrm{Re}\,\lambda(\varepsilon) > 0$ if $h'(0) > 0$. That is to say, there is a small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $x = 0$ of Eq. (1) is asymptotically stable if $h'(0) < 0$, and unstable if $h'(0) > 0$. More precisely, we have

Theorem 1. *There is a small* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the equilibrium* $x = 0$ *of Eq.* (1) *is asymptotically stable if*

$$-\omega D_2 g + \sin(\omega\tau_1) D_3 g - \omega\cos(\omega\tau_1) D_4 g + \omega D_2 f - \sin(\omega\tau_2) D_3 f + \omega\cos(\omega\tau_2) D_4 f < 0 \tag{16}$$

and unstable if the revised strict inequality holds, where (*and hereafter*) *all the partial derivatives are evaluated at the origin* (0, 0, 0, 0).

Unlike the current methods that are usually very difficult to use to achieve simple stability conditions for the trivial solution of a delayed system, the averaging method results in a very simple stability condition as shown in Eq. (16). Moreover, it is easy to show that a linear delayed feedback control of the form

$$f(x(t), \dot{x}(t), x(t-\tau_2), \dot{x}(t-\tau_2)) = w\dot{x}(t) + ux(t-\tau_2) + v\dot{x}(t-\tau_2) \tag{17}$$

can always stabilize $x = 0$ of Eq. (1). This claim is also true for the delayed linear feedback of Pyragas type (DLFP), defined as

$$f(x(t), \dot{x}(t), x(t-\tau_2), \dot{x}(t-\tau_2)) = u[x(t) - x(t-\tau_2)] + v[\dot{x}(t) - \dot{x}(t-\tau_2)]. \tag{18}$$

Remark 1. A DLFP control is widely used in stabilizing UPOs (unstable periodic orbits) in nonlinear dynamics, it is also frequently applied to improve the stability of a periodic motion emerged from a periodic excitation. For example, the periodic forced system

$$\ddot{x}(t) - \varepsilon\xi\dot{x}(t) + \alpha^2 x(t) = p_0\cos(\omega t) \tag{19}$$

has an unstable periodic motion

$$\bar{x}(t) = \frac{(\alpha^2 - \omega^2)p_0}{(\varepsilon\xi\omega)^2 + (\alpha^2 - \omega^2)^2}\cos(\omega t) - \frac{\varepsilon\xi\omega p_0}{(\varepsilon\xi\omega)^2 + (\alpha^2 - \omega^2)^2}\sin(\omega t) \tag{20}$$

with period $T = 2\pi/\omega$. Such a mathematical model with negative damping is often used to explain the self-excited vibrations caused by friction [16]. To improve the asymptotic stability of $\bar{x}(t)$, a state feedback control with the same delay T can be introduced to the system

$$\ddot{x}(t) - \varepsilon\xi\dot{x}(t) + \alpha^2 x(t) = p_0\cos(\omega t) + \varepsilon u[x(t) - x(t-T)] + \varepsilon v[\dot{x}(t) - \dot{x}(t-T)]. \tag{21}$$

Let $x(t) = \bar{x}(t) + \delta(t)$, then the disturbance $\delta(t)$ satisfies

$$\ddot{\delta}(t) + \alpha^2\delta(t) - \varepsilon\xi\dot{\delta}(t) = \varepsilon u[\delta(t) - \delta(t-T)] + \varepsilon v[\dot{\delta}(t) - \dot{\delta}(t-T)]. \tag{22}$$

To stabilize the periodic solution $\bar{x}(t)$, it is necessary to stabilize the equilibrium $\delta = 0$ of Eq. (22). For small $\varepsilon > 0$, the stability condition for the equilibrium $\delta = 0$ of Eq. (22) simply reads

$$u \sin(\alpha T) - v\alpha \cos(\alpha T) + \xi\alpha + v\alpha < 0.$$

4.2. Robust stabilization

This subsection deals with the robust stabilization of Eq. (1). We assume that the system has several uncertain parameters

$$\mathbf{q} \in Q := \{(q_1, q_2, \ldots, q_s) \in R^s : \underline{q}_i \leqslant q_i \leqslant \bar{q}_i \ (i = 1, 2, \ldots, s)\}, \quad \underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j \ (j = 1, 2). \tag{23}$$

The objective is to determine the feedback gains such that the trivial equilibrium $x = 0$ of Eq. (1) is asymptotically stable for all parameter combinations.

Though the method of Lyapunov functional can be applied to solve this robust stabilization problem, the averaging method is most preferable. As seen in the above subsection, the averaging technique features extremely simple stability conditions in terms of system parameters, which is very important and advantageous for the stabilization problem of any systems with uncertain parameters. Recall that the stability condition of the equilibrium $x = 0$ of Eq. (1) is $h'(0) < 0$, namely condition (16). This stability condition is very simple and can be easily checked even when uncertain parameters are involved. Now, $h'(0)$ depends on the uncertain parameters $\mathbf{q} \in Q$, and $\underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j$, $(j = 1, 2)$, this leads to

Theorem 2. *There is a small* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the equilibrium* $x = 0$ *of the controlled system Eq.* (1) *is asymptotically stable for all parameters* $\mathbf{q} \in Q$, *and* $\underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j$ $(j = 1, 2)$ *if the feedback gains are chosen such that the maximal value of* $h'(0)$ *over* $\mathbf{q} \in Q$, *and* $\underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j$ $(j = 1, 2)$ *is negative.*

It is worthy of note that the extreme case when $\underline{\tau}_1 = 0$ or $\underline{\tau}_2 = 0$, must be considered separately. In this case, the stability condition consists of the condition of Theorem 2 and the Hurwitz stability conditions that govern the asymptotical stability of the system without delays.

In addition, we can eliminate the delays τ_i from the above inequality to get a stabilization condition that is independent of the delay. In fact, there is a $\phi \in [0, 2\pi)$ such that

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi) \leqslant \sqrt{a^2 + b^2} \tag{24}$$

then there is a small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $x = 0$ of Eq. (1) is asymptotically for all parameters $\mathbf{q} \in Q$, and $\underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j$ $(j = 1, 2)$ if the feedback gains are chosen such that

$$\max_{\mathbf{q} \in Q} \left\{ (D_2 f - D_2 g)\omega + \sqrt{(D_3 f)^2 + (D_4 f)^2 \omega^2} + \sqrt{(D_3 g)^2 + (D_4 g)^2 \omega^2} \right\} < 0. \tag{25}$$

In [17], a different method on the basis of the Hermite–Biehler theorem is given for determining the admissible set of the feedback gains of a PID control that stabilizes a DDE with fixed parameters. It does not work for our problem due to the parametric uncertainty.

The asymptotical stability of a non-trivial equilibrium $r_0 > 0$ of the averaged equation is governed by $h'(r_0) < 0$. The idea used in Ref. [14] can be used to prove that a delayed cubic feedback can always stabilize an unstable periodic solution of Eq. (1) (which corresponds to a $r_0 > 0$ of the averaged system) with any prescribed amplitude. The robustness analysis can be carried out easily as done above for the trivial solution.

4.3. Illustrative examples

To demonstrate the effectiveness of the presented method, two simple examples are given as follows.

Example 1. Robust stabilization of the linear vibration system with negative damping. Let us first consider the following vibration system:

$$\ddot{x}(t) + \omega^2 x(t) - \varepsilon\xi\dot{x}(t) = \varepsilon[ux(t-\tau) + v\dot{x}(t-\tau)], \tag{26}$$

where a negative damping coefficient is considered. The uncontrolled system is obviously unstable since the system has negative damping. In the design phase, the controlled system is usually assumed to be asymptotically stable when the delay disappears. Now, the feedback gains u and v are chosen so as to make the equilibrium of the controlled system asymptotically stable for all $0 < \omega \in [\omega_1, \omega_2]$, $0 < \xi \in [\xi_1, \xi_2]$ and $\tau \in [0, \tau_0]$. Since the condition governing the asymptotical stability of $x = 0$ is

$$v\omega\cos(\omega\tau) - u\sin(\omega\tau) + \xi\omega < 0, \quad \varepsilon u < \omega^2, \tag{27}$$

where the first inequality comes from $h'(0) < 0$, and the second results from the stability condition for the case of $\tau = 0$. So the feedback gains should be chosen such that the above condition holds for all $0 < \omega \in [\omega_1, \omega_2]$, $0 < \xi \in [\xi_1, \xi_2]$ and $\tau \in [0, \tau_0]$.

Let us check the validity of the result in a different routine. Obviously, the characteristic function of the linear system is

$$p(\lambda) = \lambda^2 - \varepsilon\xi\lambda + \omega^2 - \varepsilon(u + v\lambda)\mathrm{e}^{-\lambda\tau}. \tag{28}$$

We choose u, v such that condition (27) holds for all $0 < \omega \in [\omega_1, \omega_2]$, $0 < \xi \in [\xi_1, \xi_2]$ and $\tau \in [0, \tau_0]$, then it is obvious that the system without delay is asymptotically stable for all such parameter combination. Moreover, following the idea used in Ref. [18], one can see that for small ε, the characteristic roots that cross the imaginary axis must be those emerged from $\pm\mathrm{i}\omega$. Let $\lambda = \mathrm{i}(\omega + \varepsilon\sigma)$, then for the problem of robust stabilization it is required that $p(\mathrm{i}(\omega + \varepsilon\sigma)) \neq 0$ for all $0 < \omega \in [\omega_1, \omega_2]$, $0 < \xi \in [\xi_1, \xi_2]$ and $\tau \in [0, \tau_0]$. Straightforward computation shows that

$$\begin{aligned} \operatorname{Re} p(\mathrm{i}(\omega + \varepsilon\sigma)) &= [-2\omega\sigma - v\omega\sin(\omega\tau) - u\cos(\omega\tau)]\varepsilon + O(\varepsilon^2), \\ \operatorname{Im} p(\mathrm{i}(\omega + \varepsilon\sigma)) &= [-\xi\omega - v\omega\cos(\omega\tau) + u\sin(\omega\tau)]\varepsilon + O(\varepsilon^2). \end{aligned} \tag{29}$$

It follows that there is a $\varepsilon_0 > 0$, such that $p(\mathrm{i}(\omega + \varepsilon\sigma)) \neq 0$ holds for all $0 < \omega \in [\omega_1, \omega_2]$, $0 < \xi \in [\xi_1, \xi_2]$ and $\tau \in [0, \tau_0]$, and for all $\varepsilon \in (0, \varepsilon_0)$ if condition (27) holds. Remark 1 tells that the delayed feedback control stabilizes also the periodic solution $\bar{x}(t)$ of Eq. (19) robustly.

In addition, vibration instabilities in machining processes such as chatter in metal cutting and washboarding in wood machining have been shown to involve regenerative cutting force and the regenerative damping force. To improve the stability of the vibration system, the following s.d.o.f. model is frequently used:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -u[x(t) - x(t-T)] - v[\dot{x}(t) - \dot{x}(t-T)], \tag{30}$$

where m, c, k are the mass, damping and stiffness of the system, respectively, T is the tooth passage period, and $-u[x(t) - x(t-T)]$, $-v[\dot{x}(t) - \dot{x}(t-T)]$ are the so-called the regenerative cutting force and the regenerative damping force, respectively [19]. If we assume that the damping and the feedback gains are of order ε, then the robust stability condition for $x = 0$ of Eq. (30) can be obtained easily as done above.

Example 2. Robust stabilization of a non-linear vibration system. Consider the delayed linear feedback control to the delayed van der Pol oscillator

$$\ddot{x}(t) + x(t) - \varepsilon[1 - x^2(t-\tau_1)]\dot{x}(t) = \varepsilon[ux(t-\tau_2) + v\dot{x}(t-\tau_2)]. \tag{31}$$

As is well known, if $0<\varepsilon\ll 1$, then the uncontrolled van del Pol oscillator has an unstable equilibrium $x = 0$ and an asymptotically stable limit cycle with period $1 + o(\varepsilon)$ and with amplitude $2/\sqrt{2 - \cos(2\tau_1)} + O(\varepsilon)$, since $G(r) = [1 - (1 - \cos(2\tau_1)/2)r^2/2]r/2$. For the controlled system, it is easy to show that $F(r) = \frac{1}{2}(v\cos\tau_2 - u\sin\tau_2)r$, so the condition $G'(0) - F'(0)<0$ that governs the asymptotic stability of $x = 0$ of the controlled van del Pol equation is as follows: $1 + v\cos\tau_2 - u\sin\tau_2<0$. To ensure the robustness of the stability in considering $\tau_i\in[\underline{\tau}_i, \bar{\tau}_i]$ $(i = 1, 2)$, the feedback gain (u, v) should be chosen such that

$$1 + v\cos\tau_2 - u\sin\tau_2<0, \quad \forall\tau_i\in[\underline{\tau}_i, \bar{\tau}_i]\,(i = 1, 2). \tag{32}$$

This can be checked easily as above. Let us check the results numerically.

Let $\tau_1 = 0.1$, $u = 0$, $\varepsilon = 0.1$, $\tau_2\in[0, 0.2]$, then the admissible set of v is $v< -1/\cos\tau_2\leqslant -1/\cos 0.2 = -1.0203$, due to condition (32). Hence, $x = 0$ is not unstable if $v > -1$. Fig. 1 shows that when $v = -0.9$, $\tau_2 = 0.05$, and $v = -0.9$, $\tau_2 = 0.7$, the zero solution is unstable. Decrease the value of v to -1.4, the trivial equilibrium $x = 0$ is robust asymptotically stable as shown in Fig. 2.

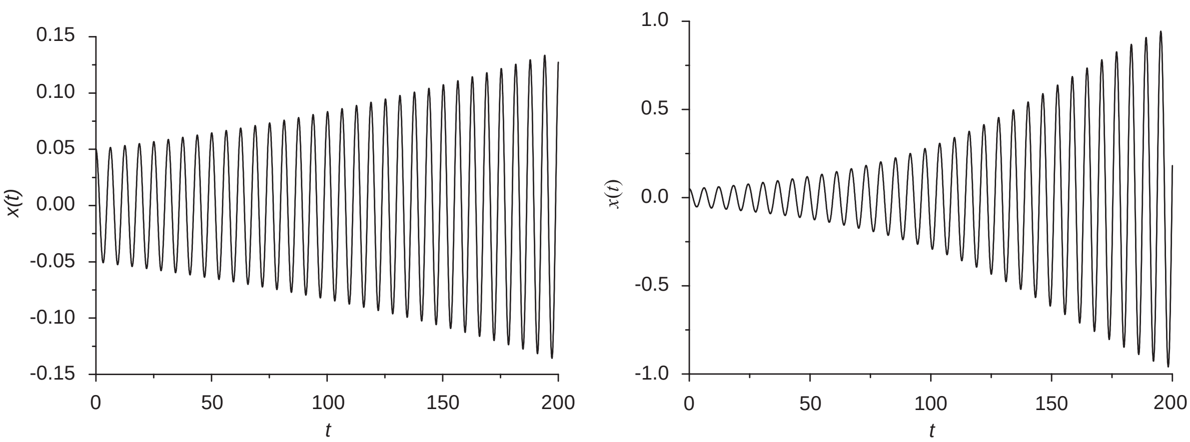

Fig. 1. The instability of the zero solution when $v = -0.9$: (left) $\tau_2 = 0.05$, (right) $\tau_2 = 0.7$.

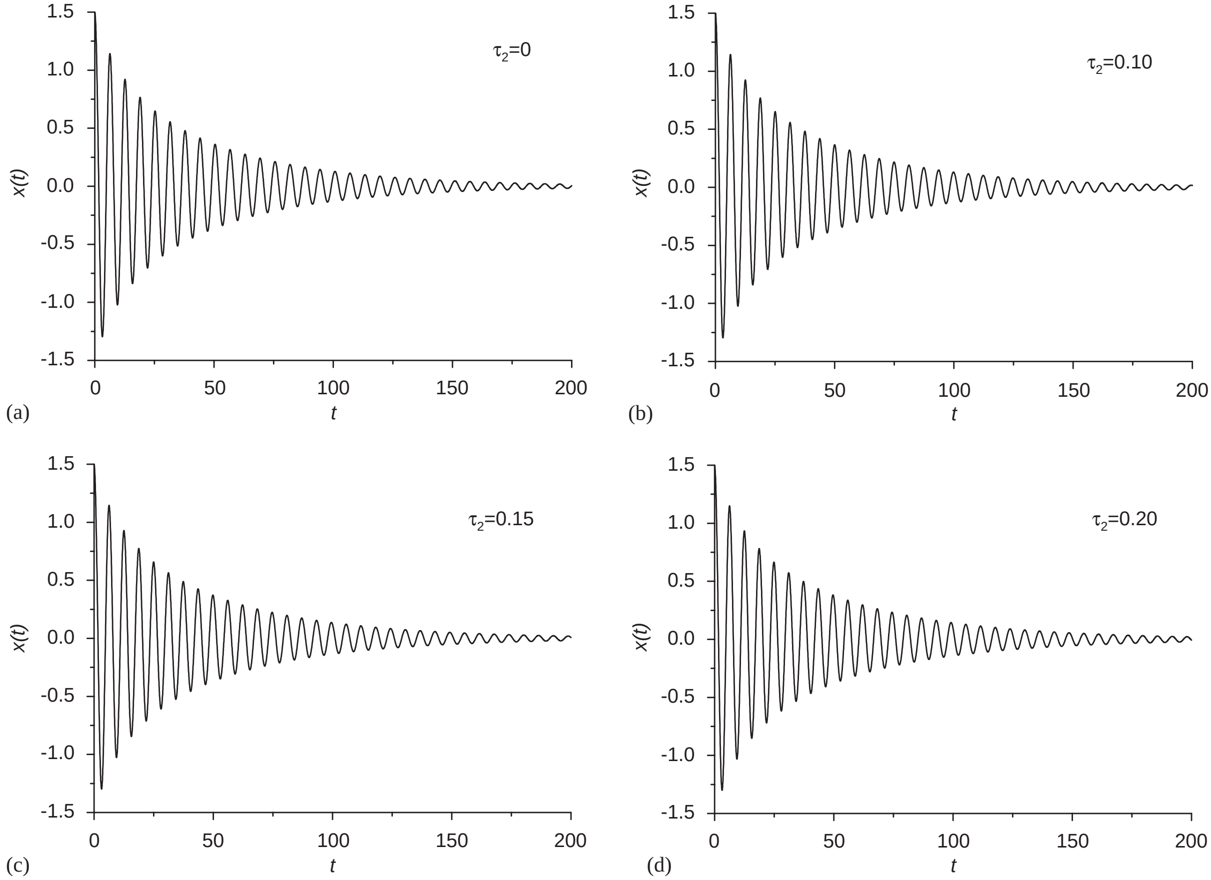

Fig. 2. The robust stability of the zero solution when $v = -1.4$.

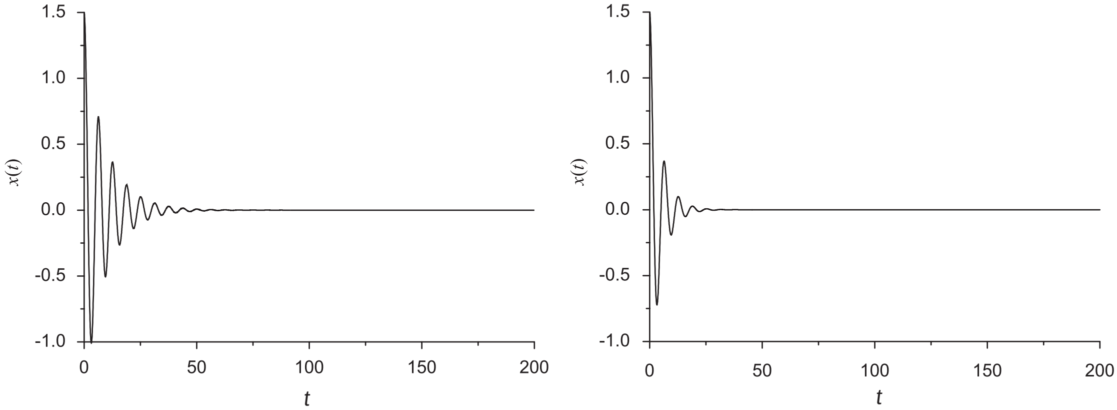

Fig. 3. The asymptotic stability of the zero solution when $\tau_2 = 0.05$: (left) $v = -3$, (right) $v = -5$.

In Fig. 3, the zero solution of the controlled system with $v = -3$, $\tau_2 = 0.05$, and $v = -5$, $\tau_2 = 0.05$ is asymptotically stable, too. If $1 + v\cos\tau_2 - u\sin\tau_2 > 0$, then the trivial equilibrium $x = 0$ of the controlled system is unstable, and the unique non-trivial periodic solution emerges. In fact, the unique non-trivial periodic solution corresponds to the unique non-trivial root of $G(r) - F(r) = 0$, which reads

$$r_0 = 2\sqrt{\frac{1 + v\cos\tau_2 - u\sin\tau_2}{2 - \cos(2\tau_1)}}. \tag{33}$$

Obviously, the amplitude r_0 can be any given positive value by proper chosen (u, v). The unique periodic solution is asymptotically stable since $h'(r_0) = -(1 + v\cos\tau_2 - u\sin\tau_2) < 0$. Considering the uncertainty of the delay and a given admissible interval $\underline{r} \leqslant r_0 \leqslant \bar{r}$ for the amplitude, the problem of robust stabilization to this periodic solution is to choose (u, v) such that

$$1 + v\cos\tau_2 - u\sin\tau_2 > 0, \quad \underline{r} \leqslant 2\sqrt{\frac{1 + v\cos\tau_2 - u\sin\tau_2}{2 - \cos(2\tau_1)}} \leqslant \bar{r}, \quad \forall \tau_i \in [\underline{\tau}_i, \bar{\tau}_i] \ (i = 1, 2). \tag{34}$$

This also can be checked easily as done above.

5. Extension

In the above two sections, as a key step in the problem of robust stabilization of delayed s.d.o.f. vibration system, the polar co-ordinates and the averaging method were combined to derive a simple stability condition of Eq. (1). In a more general case such as Eq. (2), the polar co-ordinates and the averaging method cannot be used directly, and one needs firstly to reduce the infinite-dimensional equation into a set of ODE. To this end, the center manifold reduction for DDE is preferable.

5.1. The reduction procedure

Let $\Omega := C([-\tau, 0], R^n)$ be the Banach space of continuous functions mapping $[-\tau, 0]$ into R^n (the real n-dimensional linear vector space equipped with norm $|\phi|$) with norm $||\phi|| = \max_{-\tau \leqslant \theta \leqslant 0} |\phi(\theta)|$, and $\Omega^* := C([0, \tau], R^{*n})$ the Banach space of continuous functions mapping $[0, \tau]$ into R^{*n} (the linear space of n-dimensional row vectors) with norm $||\psi|| = \max_{0 \leqslant s \leqslant \tau} |\psi(s)|$, where $\tau = \max\{\tau_1, \tau_2\} > 0$. The unperturbed Eq. (3) can be rewritten as

$$\dot{\mathbf{x}}(t) = \int_{-\tau}^{0} \mathrm{d}\boldsymbol{\eta}(\theta)\mathbf{x}(t + \theta), \tag{35}$$

where the function $\boldsymbol{\eta}(\theta)$ of bounded variation is

$$\boldsymbol{\eta}(\theta) = \begin{cases} \mathbf{A}_0, & \theta = 0, \\ \mathbf{A}_1, & \theta = -\tau_1, \\ \mathbf{0}, & \text{otherwise}. \end{cases} \tag{36}$$

Along with Eq. (35), we consider the equation

$$\dot{\mathbf{y}}(t) = -\int_{-\tau}^{0} \mathbf{y}(t-\theta)\,\mathrm{d}\boldsymbol{\eta}(\theta),$$

which is "adjoint" with respect to the bilinear form for all $\phi \in \Omega$ and $\psi \in \Omega^*$

$$(\psi, \phi) := \psi(0)\phi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(\xi-\theta)\,\mathrm{d}\boldsymbol{\eta}(\theta)\phi(\xi)\,\mathrm{d}\xi = \psi(0)\phi(0) - \int_{-\tau}^{0} \psi(\xi+\tau)\mathbf{A}_1\phi(\xi)\,\mathrm{d}\xi. \tag{37}$$

In addition, we define the linear operator $\mathbf{L}: \Omega \to \Omega$ and the "adjoint" linear operator $\mathbf{L}^*: \Omega^* \to \Omega^*$

$$\mathbf{L}(\phi(\theta)) = \begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ \mathbf{A}_0\phi(0) + \mathbf{A}_1\phi(-\tau_1), & \theta = 0, \end{cases} \quad \mathbf{L}^*(\psi(s)) = \begin{cases} -\dfrac{\mathrm{d}\psi}{\mathrm{d}s}, & s \in (0, \tau], \\ \psi(0)\mathbf{A}_0 + \psi(-\tau_1)\mathbf{A}_1, & s = 0, \end{cases} \tag{38}$$

in the sense of $(\mathbf{L}^*\psi, \phi) = (\psi, \mathbf{L}\phi)$ for all $\phi \in \Omega$ and $\psi \in \Omega^*$. The eigenvalues of $\mathbf{L}$ are the same as the characteristic roots of the linear delayed system (35), namely the roots of the characteristic equation $\det[\lambda\mathbf{I}_{n\times n} - \mathbf{A}_0 - \mathbf{A}_1 e^{-\lambda\tau_1}] = 0$.

Now we assume that Eq. (35) has exactly one pair of conjugate characteristic roots, $\lambda = \pm i\omega$ and all the other eigenvalues stay in the open left-half complex plane. In applying the center manifold reduction, it is required to decompose the state space Ω by the two characteristic roots $\lambda = \pm i\omega$. To this end, we define two basis matrices $\boldsymbol{\Phi}(\theta)_{n\times 2}$, $\boldsymbol{\Psi}(s)_{2\times n}$ that satisfy $\mathbf{L}\boldsymbol{\Phi} = \boldsymbol{\Phi}\mathbf{B}$, $\mathbf{L}^*\boldsymbol{\Psi} = \mathbf{B}\boldsymbol{\Psi}$ and $(\boldsymbol{\Psi}, \boldsymbol{\Phi}) = \mathbf{I}_{2\times 2}$, where

$$\mathbf{B} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}. \tag{39}$$

Once the solutions of the eigenvalue problem $\mathbf{L}(\phi(\theta)) = i\omega\phi(\theta)$, $(\phi \in \Omega)$, namely

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = i\omega\phi(\theta), \quad \theta \in [-\tau, 0), \tag{40a}$$

$$\mathbf{A}_0\phi(0) + \mathbf{A}_1\phi(-\tau_1) = i\omega\phi(0) \tag{40b}$$

and that of the "adjoint" eigenvalue problem $\mathbf{L}^*(\psi(s)) = i\omega\psi(s)$, $(\psi \in \Omega^*)$, namely

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = i\omega\psi(s), \quad s \in (0, \tau], \tag{41a}$$

$$\psi(0)\mathbf{A}_0 + \psi(-\tau_1)\mathbf{A}_1 = i\omega\psi(0) \tag{41b}$$

are on hand, $\boldsymbol{\Phi}(\theta)_{n\times 2}$ and $\boldsymbol{\Psi}(s)_{2\times n}$ are found to be

$$\boldsymbol{\Phi}(\theta) = [\operatorname{Im}\phi(\theta)\ \operatorname{Re}\phi(\theta)], \quad \boldsymbol{\Psi}(s) = [\operatorname{Re}\psi^{\mathrm{T}}(s)\ \operatorname{Im}\psi^{\mathrm{T}}(s)]^{\mathrm{T}} \tag{42}$$

then Ω can be decomposed by $\lambda = \pm i\omega$ as $\Omega = P \oplus Q$, where $P = \operatorname{span}\{\operatorname{Im}\phi(\theta), \operatorname{Re}\phi(\theta)\}$.

Now, let $\mathbf{x}(t)$ be the solution of Eq. (2), then we have

$$\mathbf{x}_t = \boldsymbol{\Phi}\mathbf{z} + \mathbf{x}_t^Q, \quad \mathbf{z} = (\boldsymbol{\Psi}, \mathbf{x}_t), \quad \mathbf{x}_t^Q \in Q. \tag{43}$$

From the definition, $\mathbf{z}$ is found to satisfy the following differential equation [15]:

$$\dot{\mathbf{z}} = \mathbf{B}\mathbf{z} + \varepsilon\mathbf{\Psi}(0)(\mathbf{f}+\mathbf{g})(\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q), \tag{44}$$

where $\mathbf{f}((\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)|_{\theta=0}, (\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)|_{\theta=-\tau})$ and $\mathbf{g}((\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)|_{\theta=0}, (\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)|_{\theta=-\tau})$ are simply denoted by $\mathbf{f}(\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)$ and $\mathbf{g}(\mathbf{\Phi}\mathbf{z}+\mathbf{x}_t^Q)$, respectively. And $\mathbf{x}_t^Q$ is governed by a differential equation. In addition, it has been shown that any bounded solution of $\mathbf{x}_t^Q$ must be of such a nature that $\mathbf{x}_t^Q = O(\varepsilon)$ as $\varepsilon \to 0$ [15]. Consequently, if our analysis is based upon an approximation procedure that can be justified by investigating only the terms of order ε, then the basic problem lies in the study of the ODE

$$\dot{\mathbf{z}} = \mathbf{B}\mathbf{z} + \varepsilon\mathbf{\Psi}(0)(\mathbf{f}+\mathbf{g})(\mathbf{\Phi}\mathbf{z}), \quad \mathbf{z} = (\mathbf{\Psi}, \mathbf{x}_t). \tag{45}$$

At this stage, it is convenient to combine the polar co-ordinates and the averaging method to carry out the stability analysis of Eq. (2). In fact, let $\mathbf{z} = [z_1 \ z_2]^{\mathrm{T}}$, then at $\varepsilon = 0$, the above equation has a solution $(z_1, z_2) = (-r\sin(\omega t), r\cos(\omega t))$, hence for small $0 < \varepsilon \ll 1$, the transformation

$$z_1(t) = -r(t)\sin(\omega t + \theta(t)),$$

$$z_2(t) = r(t)\cos(\omega t + \theta(t)), \tag{46}$$

converts Eq. (53) into a set of equations

$$\omega\dot{r}(t) = \varepsilon R(t, r(t), \theta(t)),$$

$$\omega\dot{\theta}(t) = \varepsilon\Theta(t, r(t), \theta(t)), \tag{47}$$

where $R(t, r, \theta)$, $\theta(t, r, \theta)$ are periodic in t with period $T = 2\pi/\omega$, $r(t)$ and $\theta(t)$ can be considered as constants over one period since they vary slowly. The averaged equation reads

$$\omega\dot{r}(t) = \varepsilon\bar{R}(r, \theta) := \frac{\varepsilon}{2\pi/\omega}\int_0^{2\pi/\omega} R(t, r, \theta)\,\mathrm{d}t,$$

$$\omega\dot{\theta}(t) = \varepsilon\bar{\Theta}(r, \theta) := \frac{\varepsilon}{2\pi/\omega}\int_0^{2\pi/\omega} \Theta(t, r, \theta)\,\mathrm{d}t. \tag{48}$$

In particular, we are interested in the case when $h(r) := \omega/(2\pi)\int_0^{2\pi/\omega} R(t, r, \theta)\,\mathrm{d}t$ is independent of θ. For this case, we have the averaged equation

$$\omega\dot{r} = \varepsilon h(r). \tag{49}$$

The main procedures above can be found in Ref. [15], but were written in a form that is very difficult to understand by most engineers. This subsection is to make the theory more understandable and computationally tractable.

It is worth mentioning that the above reduction procedure is valid also for the case when the uncontrolled system has multiple time delays. In addition, Ref. [20] presented alternative ways of averaging which retain the delay term, but the above procedure is enough and also very effective in solving the present problem.

5.2. Eq. (1) revisited

To illustrate the reduction procedure above, let us consider again the reduction for Eq. (1). Solving the eigenvalue problem $\mathbf{L}\phi(\theta) = \mathrm{i}\omega\phi(\theta), (\phi \in \Omega)$, namely

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \mathrm{i}\omega\phi(\theta), \quad \theta \in [-\tau, 0), \tag{50a}$$

$$\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \phi(0) = \mathrm{i}\omega\phi(0), \tag{50b}$$

gives a solution $\phi(\theta) = [1\ \ \mathrm{i}\omega]^{\mathrm{T}} \mathrm{e}^{\mathrm{i}\omega\theta}$, so the basis matrix $\mathbf{\Phi}(\theta)$ can be taken as

$$\mathbf{\Phi}(\theta) = [\mathrm{Im}\,\phi(\theta)\ \ \mathrm{Re}\,\phi(\theta)] = \begin{bmatrix} \sin(\omega\theta) & \cos(\omega\theta) \\ \omega\cos(\omega\theta) & -\omega\sin(\omega\theta) \end{bmatrix}. \tag{51}$$

And similarly, the "adjoint" eigenvalue problem $\mathbf{L}^*\psi(s) = \mathrm{i}\omega\psi(s)$, $(\psi \in \Omega^*)$, namely

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = -\mathrm{i}\omega\psi(s), \quad s \in (0, \tau], \tag{52a}$$

$$\psi(0)\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} = \mathrm{i}\omega\psi(0), \tag{52b}$$

has a solution $\psi(s) = [\mathrm{i}\omega\ \ 1]\mathrm{e}^{-\mathrm{i}\omega s} = [\omega\sin(\omega s)\cos(\omega s)] + \mathrm{i}[\omega\cos(\omega s)\ \ -\sin(\omega s)]$, so the basis matrix $\mathbf{\Psi}(s)$ can be taken as

$$\mathbf{\Psi}(s) = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}^{-1} \begin{bmatrix} \omega\sin(\omega s) & \cos(\omega s) \\ \omega\cos(\omega s) & -\sin(\omega s) \end{bmatrix} = \begin{bmatrix} \sin(\omega s) & \cos(\omega s)/\omega \\ \cos(\omega s) & -\sin(\omega s)/\omega \end{bmatrix}, \tag{53}$$

which satisfies $(\mathbf{\Psi}, \mathbf{\Phi}) = \mathbf{I}_{2\times 2}$. Let

$$\begin{aligned} k(z_1, z_2) :=\, & f(z_2, \omega z_1, -z_1\sin(\omega\tau_2) + z_2\cos(\omega\tau_2), \omega(z_1\cos(\omega\tau_2) + z_2\sin(\omega\tau_2))) \\ & - g(z_2, \omega z_1, -z_1\sin(\omega\tau_1) + z_2\cos(\omega\tau_1), \omega(z_1\cos(\omega\tau_1) + z_2\sin(\omega\tau_1))). \end{aligned} \tag{54}$$

The corresponding simplified Eq. (45) now reads

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 1/\omega \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ k(z_1, z_2) \end{bmatrix}. \tag{55}$$

Hence, for small $0 < \varepsilon \ll 1$, the transformation (46) converts the above equation to the following equations in polar co-ordinates:

$$\begin{aligned} \omega\dot{r}(t) = \varepsilon[& g(r(t)\cos(\omega t + \theta(t)), -r(t)\omega\sin(\omega t + \theta(t)), r(t - \tau_1)\cos(\omega t - \omega\tau_1 + \theta(t - \tau_1)), \\ & - r(t - \tau_1)\omega\sin(\omega t - \omega\tau_1 + \theta(t - \tau_1))) - f(r(t)\cos(\omega t + \theta(t)), -r(t)\omega\sin(\omega t + \theta(t)), \\ & r(t - \tau_2)\cos(\omega t - \omega\tau_2 + \theta(t - \tau_2)), -r(t - \tau_2)\omega\sin(\omega t - \omega\tau_2 + \theta(t - \tau_2)))]\sin(\omega t + \theta(t)), \end{aligned}$$

$$\begin{aligned}\omega r(t)\dot{\theta}(t) = \varepsilon[&g(r(t)\cos(\omega t+\theta(t)), -r(t)\omega\sin(\omega t+\theta(t)), r(t-\tau_1)\cos(\omega t-\omega\tau_1+\theta(t-\tau_1)),\\ &- r(t-\tau_1)\omega\sin(\omega t-\omega\tau_1+\theta(t-\tau_1))) - f(r(t)\cos(\omega t+\theta(t)), -r(t)\omega\sin(\omega t+\theta(t)),\\ &r(t-\tau_2)\cos(\omega t-\omega\tau_2+\theta(t-\tau_2)), -r(t-\tau_2)\omega\sin(\omega t-\omega\tau_2+\theta(t-\tau_2)))]\\ &\times\cos(\omega t+\theta(t)),\end{aligned} \tag{56}$$

which is exactly the same as Eq. (6). As a result, the averaging technique simplifies the above equation to Eq. (11).

5.3. Stabilization

Once the reduced ODE are at hand, the problem of robust stabilization can be solved as in the above two sections. In fact, it has been shown that there is a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $\mathbf{x} = \mathbf{0}$ of Eq. (2) is asymptotically stable if the trivial equilibrium of the averaged equation is asymptotically stable, and is unstable if the trivial equilibrium of the averaged equation is unstable. Note that the feedback gains are linearly appeared in the averaged equation, this observation leads to the following theorem.

Theorem 3. *Assume that Eq. (2) has exactly one pair of conjugate eigenvalues $\lambda = \pm i\omega$ and all the other eigenvalues stay in the open left-half complex plane. Then there is a small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, a delayed state feedback control can always make the equilibrium of Eq. (2) asymptotically stable.*

In fact, given a delayed state feedback control of the form

$$\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau_2)) = \mathbf{K}\mathbf{x}(t-\tau_2), \quad \mathbf{0} \neq \mathbf{K} \in R^{n\times n} \tag{57}$$

we have a non-zero terms $\varepsilon\mathbf{\Psi}(0)\mathbf{f}(\mathbf{\Phi}\mathbf{z})$ in Eq. (45):

$$\mathbf{\Psi}(0)\mathbf{f}(\mathbf{\Phi}\mathbf{z}) = [\mathbf{\Psi}(0)\cdot\mathbf{K}\cdot\mathbf{\Phi}(-\tau_2)]\mathbf{z} \in R^2, \quad \mathbf{\Psi}(0)\cdot\mathbf{K}\cdot\mathbf{\Phi}(-\tau_2) \neq \mathbf{0} \tag{58}$$

since $\mathbf{\Phi}$, $\mathbf{\Psi}$ have two independent column vectors, and two independent row vectors, respectively. Hence, the averaged equation must contain a term cr with the constant c being a linear combination of the feedback gains (the entries of the matrix $\mathbf{K}$). Therefore, the stability condition $h'(0) < 0$ can always be true by proper choice of the feedback gains. As a result, there is a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $\mathbf{x} = \mathbf{0}$ of Eq. (2) is asymptotically stable for all $\mathbf{q} \in Q$ and $\forall\tau_i \in [\underline{\tau}_i, \bar{\tau}_i]$ $(i = 1, 2)$, if $h'(0) < 0$ holds for all $\mathbf{q} \in Q$ and $\forall\tau_i \in [\underline{\tau}_i, \bar{\tau}_i]$ $(i = 1, 2)$.

Remark 2. By following the idea used in Ref. [14], one can also prove that if Eq. (35) has exactly one pair of conjugate eigenvalues $\pm i\omega$ and all the other eigenvalues stay in the open left-half complex plane, then for small $\varepsilon > 0$, a non-linear feedback control

$$\varepsilon\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau_2)) = \varepsilon[\mathbf{K}_1\mathbf{x}(t-\tau_2) + \mathbf{K}_3\mathbf{x}^3(t-\tau_2)], \quad \mathbf{0} \neq \mathbf{K}_3 \in R^{n\times n} \tag{59}$$

can always stabilize an unstable periodic solution of Eq. (2) with any prescribed amplitude. Here $\mathbf{x}^3(t-\tau_2)$ stands for $[x_1^3(t-\tau_2), x_2^3(t-\tau_2), \ldots, x_n^3(t-\tau_3)]^{\mathrm{T}}$.

6. Conclusions

In this paper, the problem of robust stabilization of a type of non-linear delayed systems with uncertain parameters via general delayed feedback control is solved on the basis of the averaging method. The main contributions of this paper are two-fold. Firstly, we have proved that a linear delayed feedback control can always stabilize the trivial equilibrium, as well as the periodic solutions, of a class of delayed systems whose linearized systems have characteristic roots of non-positive real parts only, a class of systems that is much broader than that discussed in the literature. The results generalize some previous results and can also serve as a method for the Hopf bifurcation control of a delayed system with weak non-linearity via delayed feedback control. Secondly, we have applied the averaging technique to study the problem of robust stabilization to general delayed systems that are resulted from small perturbation of linear delayed systems. Comparing with the widely used method of Lyapunov functional, the present method shows more effective and flexible in application. An important feature of the present method is the introduction of a small parameter such that the asymptotic stability of the trivial solution (and the periodic solutions) of the controlled system is the same as that of the equilibriums of the averaged equation obtained by using the averaging method for DDEs. The computation is simple and most importantly, the stability condition resulted from the averaged equation is very simple, for example, the feedback gains are linearly appeared in the stability condition. Thus, the conditions that justify the problem of robust stabilization can be easily verified.

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