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Various standing waves in a Timoshenko beam

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Abstract

Vibration modes of a finite-length Timoshenko beam are studied as standing waves using the wave-train closure principle, in order to obtain a complete picture of various vibration modes in a beam and to understand the mechanism of their formulations. In particular, the existence of degenerate modes in a beam is investigated. Firstly, it is shown that the two degenerate flexural waves accommodated by an infinite Timoshenko beam are derived from the in-phase and the out-of-phase relations between transverse vibrations due to bending and shear deformations, respectively. A wave representation of beam vibration is thus developed. Secondly, wave reflection behavior at an elastically supported boundary is analyzed. It is shown that while these two waves are degenerate in an infinite beam, they have to be superposed at the boundary in general, but remain degenerate for certain special boundary conditions. Based on these results, the expression of wave-train closure principle for a finite-length Timoshenko beam is derived, and used to study different standing waves in a beam. It is shown that three types of standing waves (vibration modes) exist in Timoshenko beams, namely, superposed, degenerate, and single. A condition of space synchronization must be satisfied for superposed standing waves. For the other two types of standing waves, this condition is satisfied naturally. While the superposed standing wave is the most general form of vibration mode, vibration modes of elastically supported beams at specific frequencies or beams with sliding and/or simply supported boundary conditions are single standing waves. When additional conditions are satisfied, two single standing waves could exist at the same natural frequency to formulate degenerate standing waves.

1. Introduction

Free vibration is usually treated by the mode approach as an eigenvalue problem; however, more physical understanding could be obtained by using the wave propagation approach. In the

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wave propagation approach, the wave-train closure principle [1] gives the condition for a propagating wave to formulate a standing wave. The principle states that, in order to formulate a standing wave (vibration mode), a propagating wave must return to its starting point after completing one complete circuit of the system with the same amplitude and phase, i.e., it closes on itself.

This principle was also termed the phase closure principle and has been applied to free vibration analysis of Euler–Bernoulli beams [2]. According to Euler–Bernoulli theory, there are one propagating wave and one evanescent wave in an infinite beam with the same wave number at a given frequency. Mead [2] showed that each of these two waves could be used in the wave-train closure principle, leading to the same frequency equation.

Two waves sharing the same wave number is the result of ignoring the effect of shear deformation in the Euler–Bernoulli beam model. This approximation leads to a physically incorrect dispersion behavior [3]. The effects of rotary inertia and shear deformation were considered by Timoshenko [4,5]. According to the Timoshenko theory, beam vibration is described by a vector $\mathbf{s}(z,t) = \{ {w(z,t) \atop \phi(z,t)} \}$, where $w(z,t)$ is transverse displacement and $\phi(z,t)$ is the bending rotation, which is related through the shear deformation, $\gamma(z,t)$, as $\partial w/\partial z = \phi + \gamma$ [6].

Wave propagation in an infinite Timoshenko beam has been analyzed by Mead [7] in an attempt to understand energy transmission associated with propagating waves and the behavior of wave reflection and transmission at a constraint in an infinite beam. Chan et al. [8] showed that Timoshenko beam vibration is a phenomenon of superposed standing waves; the two waves predicted by the Euler–Bernoulli beam theory are actually two types of wave with different wave number at a given frequency, corresponding to the first two flexural waves predicted by the exact theory of elasticity. These two types of wave were termed the k_a- and the k_b-waves in Ref. [8], respectively; the former is translation-dominated with anomalous dispersion behavior, and the latter is rotation-dominated with normal dispersion behavior.

From the point of view of wave mechanics, these two waves in an infinite Timoshenko beam are said to be degenerate, that is, they exist at the same frequency but do not interact with one another in free space. Interactions can only occur at boundaries or discontinuities. For a finite beam with general elastic supports at the ends, in order to satisfy the boundary conditions, these two waves have to be superposed to formulate a vibration mode. For the simply supported beam, however, these two waves are found to be separated and each formulates a vibration mode, resulting in the so-called ‘second spectrum’. This suggests that there may exist different modes in a finite-length Timoshenko beam, and their formulation is related to the effect of boundaries on the degeneracy of the two waves.

In this paper, the expression of wave-train closure principle for a Timoshenko beam is examined in an attempt to understand the mechanism for the formulation of superposed and separated vibration modes, thus obtain a complete picture of various standing waves in a beam. The emphasis is on whether and under what conditions degenerate modes exist in a Timoshenko beam.

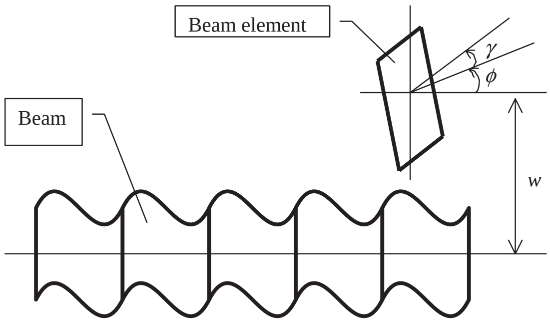
Instead of using the conventional vector $\mathbf{s}(z,t) = \{ {w(z,t) \atop \phi(z,t)} \}$ to describe Timoshenko beam vibration, the vector of bending and shear displacements, $\mathbf{w}(z,t) = \{ {w_\phi(z,t) \atop w_\gamma(z,t)} \}$, where $w_\phi(z,t)$ and $w_\gamma(z,t)$ are transverse displacements due to bending rotation $\phi(z,t)$ and shear angle $\gamma(z,t)$, respectively, is used to represent beam vibration in order to investigate the role of shear deformation in differentiating the k_a- and the k_b-waves. Wave reflection at a general elastically supported boundary is then analyzed to demonstrate the effect of boundary conditions. Classical boundary conditions are also considered as limiting cases of the elastically supported boundary condition. While it has been shown by Mead [7] that an incident k_a- or k_b-wave will in general generate both types of waves upon reflection, the condition at which an incident wave generates a reflected wave of its own type is sought in the present paper. Based on this investigation of wave reflection, the expression of wave-train closure principle for an elastically supported Timoshenko beam, including classical boundary conditions as limiting cases, is derived to study all possible forms of standing waves.

2. Waves in an infinite Timoshenko beam

The wave equation for an infinite Timoshenko beam shown in Fig. 1 can be written as

$$\mathbf{D}\mathbf{w} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{1}$$

where $\mathbf{w}$ is a vector representing Timoshenko beam vibration, and $\mathbf{D}$ is the corresponding operator. As mentioned in the Introduction, instead of the conventional expression $\mathbf{w}(z,t) = \left\{ \begin{smallmatrix} w(z,t) \\ \phi(z,t) \end{smallmatrix} \right\}$, $\mathbf{w}$ is



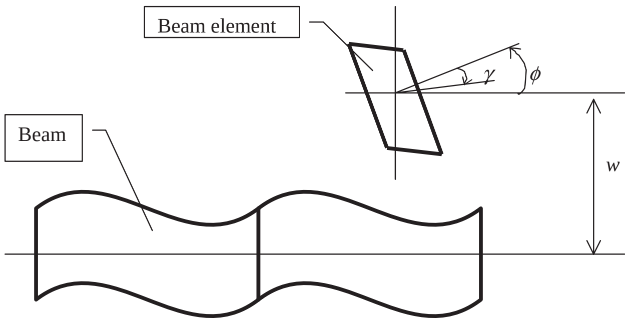


Fig. 1. Illustration of the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves in an infinite beam and the definitions of the quantities describing beam motion in Eq. (1). (a) $\mathbf{w}^{in}$-wave, (b) $\mathbf{w}^{out}$-wave.

expressed as $\mathbf{w}(z,t) = \left\{ {w_\phi(z,t) \atop w_\gamma(z,t)} \right\}$ in the present study, where $w_\phi(z,t)$ and $w_\gamma(z,t)$ are transverse displacements due to bending rotation, $\phi(z,t) = \partial w_\phi(z,t)/\partial z$, and shear angle, $\gamma(z,t) = \partial w_\gamma(z,t)/\partial z$, respectively. Fig. 1 gives an illustration of a beam element showing the definitions of these quantities. These two expressions are related by the following transformation:

$$\begin{Bmatrix} w(z,t) \\ \phi(z,t) \end{Bmatrix} = \begin{Bmatrix} w_\phi(z,t) + w_\gamma(z,t) \\ \partial w_\phi(z,t)/\partial z \end{Bmatrix} = \begin{pmatrix} 1 & 1 \\ \partial/\partial z & 0 \end{pmatrix} \begin{Bmatrix} w_\phi(z,t) \\ w_\gamma(z,t) \end{Bmatrix}.$$

The operator $\mathbf{D}$, accordingly, is given by

$$\mathbf{D} = \begin{bmatrix} m\dfrac{\partial^2}{\partial t^2} & m\dfrac{\partial^2}{\partial t^2} - KGA\dfrac{\partial^2}{\partial z^2} \\ EI\dfrac{\partial^3}{\partial z^3} - J\dfrac{\partial^3}{\partial z \partial t^2} & KGA\dfrac{\partial}{\partial z} \end{bmatrix}, \tag{2}$$

where $m = \rho A$ and $J = \rho I$ are the mass and the mass moment of inertia per unit length of the beam, respectively; ρ is the density, I and A are the second moment of area and the cross-sectional area, respectively, E and G are Young's modulus and shear modulus, respectively, and K is the shear coefficient.

The general solution of the wave equation can be assumed as

$${}_{+}\mathbf{w}(z,t) = \begin{Bmatrix} {}_{+}w_\phi(z,t) \\ {}_{+}w_\gamma(z,t) \end{Bmatrix} = \begin{Bmatrix} {}_{+}\mathrm{W}_{\phi 0} \\ {}_{+}\mathrm{W}_{\gamma 0}\mathrm{e}^{\mathrm{i}\theta} \end{Bmatrix} \mathrm{e}^{\mathrm{i}(\omega t - kz)} \tag{3a}$$

for the forward propagating wave, and

$${}_{-}\mathbf{w}(z,t) = \begin{Bmatrix} {}_{-}w_\phi(z,t) \\ {}_{-}w_\gamma(z,t) \end{Bmatrix} = \begin{Bmatrix} {}_{-}\mathrm{W}_{\phi 0} \\ {}_{-}\mathrm{W}_{\gamma 0}\mathrm{e}^{\mathrm{i}\theta} \end{Bmatrix} \mathrm{e}^{\mathrm{i}(\omega t + kz)} \tag{3b}$$

for the backward propagating wave, where ${}_{\pm}\mathrm{W}_{\phi 0}$ and ${}_{\pm}\mathrm{W}_{\gamma 0}$ are the amplitudes of transverse displacements due to bending rotation and shear deformation, respectively, ω is frequency, k is the wave number, and θ is the phase difference between ${}_{\pm}w_\phi(z,t)$ and ${}_{\pm}w_\gamma(z,t)$. Substituting the assumed solution Equations (3a) and (3b) into Eq. (1) gives

$$\begin{bmatrix} -m\omega^2 & -m\omega^2 + KGAk^2 \\ -\mathrm{i}EIk^3 + \mathrm{i}J\omega^2 k & \mathrm{i}KGA \cdot k \end{bmatrix} \begin{pmatrix} {}_{+}\mathrm{W}_{\phi 0} \\ {}_{+}\mathrm{W}_{\gamma 0}\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4a}$$

$$\begin{bmatrix} -m\omega^2 & -m\omega^2 + KGAk^2 \\ \mathrm{i}EIk^3 - \mathrm{i}J\omega^2 k & -\mathrm{i}KGA \cdot k \end{bmatrix} \begin{pmatrix} {}_{-}\mathrm{W}_{\phi 0} \\ {}_{-}\mathrm{W}_{\gamma 0}\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4b}$$

It is not difficult to deduce that

$$\sin\theta = 0 \quad \text{thus } \theta = 0 \text{ or } \pi. \tag{5}$$

This suggests that ${}_{\pm}w_\gamma(z,t)$ is either in-phase or out-of-phase with respect to ${}_{\pm}w_\phi(z,t)$. For both the in-phase and the out-of-phase cases, however, the same dispersion relations are deduced from Eqs. (4a) and (4b), written as

$$k = 0, \tag{6a}$$

$$k^4 - \left(\frac{J}{EI} + \frac{m}{KGA}\right)\omega^2 k^2 - \left(\frac{\omega^2 m}{EI} - \frac{\omega^4 mJ}{EI \cdot KGA}\right) = 0. \tag{6b}$$

Eq. (6a) represents the rigid-body motion of the beam, thus not considered here. The solutions of Eq. (6b) are written as [8]

$$k_a^2 = \left(\frac{\omega^2}{2c_o^2} + \frac{\omega^2}{2c_2'^2}\right) + \frac{1}{2}\sqrt{\left(\frac{\omega^2}{c_o^2} - \frac{\omega^2}{c_2'^2}\right)^2 + \frac{4\omega^2}{c_o^2 r_g^2}}, \tag{7a}$$

$$k_b^2 = \left(\frac{\omega^2}{2c_o^2} + \frac{\omega^2}{2c_2'^2}\right) - \frac{1}{2}\sqrt{\left(\frac{\omega^2}{c_o^2} - \frac{\omega^2}{c_2'^2}\right)^2 + \frac{4\omega^2}{c_o^2 r_g^2}}, \tag{7b}$$

where $r_g = \sqrt{I/A}$, $c_o = \sqrt{E/\rho}$, $c_2' = \sqrt{KG/\rho}$. The wave solution associated with k_a was termed the k_a-wave, while the wave solution associated with k_b the k_b-wave [8]. From Eqs. (7a) and (7b), the phase speeds of the k_a- and the k_b-waves are obtained as

$$c_{pa}^2 = \frac{2}{(1/c_o^2 + 1/c_2'^2) + \sqrt{(1/c_o^2 - 1/c_2'^2)^2 + 4/(c_o^2 r_g^2 \omega^2)}}, \tag{8a}$$

$$c_{pb}^2 = \frac{2}{(1/c_o^2 + 1/c_2'^2) - \sqrt{(1/c_o^2 - 1/c_2'^2)^2 + 4/(c_o^2 r_g^2 \omega^2)}}. \tag{8b}$$

It can be deduced that $(c_{pa}^2/c_2'^2) < 1$ and $(c_{pb}^2/c_2'^2) > 1$.

For the in-phase case, the amplitude of the displacement vector $\mathbf{w}$ is given by

$$_\pm\mathbf{W}^{in} = \begin{pmatrix} _\pm \mathrm{W}_{\phi 0}^{in} \\ _\pm \mathrm{W}_{\gamma 0}^{in} \end{pmatrix}, \tag{9}$$

where $_\pm \mathrm{W}_{\gamma 0}^{in} = _\pm \mathrm{W}_{\phi 0}^{in}/[(c_2'/c_p)^2 - 1]$, and the superscript '*in*' represents 'in-phase'. From Eqs. (8a) and (8b), it is not difficult to see that $_\pm \mathrm{W}_{\gamma 0}^{in} = _\pm \mathrm{W}_{\phi 0}^{in}/[(c_2'/c_p)^2 - 1] > 0$ for the k_a-wave while $_\pm \mathrm{W}_{\gamma 0}^{in} = {}_\pm \mathrm{W}_{\phi 0}^{in}/[(c_2'/c_p)^2 - 1] < 0$ for the k_b-wave. Since the amplitudes are positive, it can be concluded that only the k_a-wave is the solution of the in-phase case. Therefore, it would be more appropriate to term the k_a-wave as the $\mathbf{w}^{in}$-wave, written as

$$_\pm\mathbf{w}^{in} = \begin{Bmatrix} r_a \\ 1 \end{Bmatrix} (_\pm \mathrm{W}_{\gamma 0}^{in}) \mathrm{e}^{\mathrm{i}(\omega t \mp k_a z)} = \begin{Bmatrix} r_a \\ 1 \end{Bmatrix} (_\pm w_\gamma^{in}), \tag{10}$$

where $r_a = (c_2'/c_{pa})^2 - 1 > 0$. One can see that the displacements due to bending and shear are not independent but related through the parameter r_a. Consequently, this wave is characterized by $_\pm w_\gamma^{in}(z, t)$.

Analogously, for the out-of-phase case, the amplitude of the displacement vector $\mathbf{w}$ is given by

$$_\pm\mathbf{W}^{out} = \begin{pmatrix} _\pm \mathrm{W}_{\phi 0}^{out} \\ -_\pm \mathrm{W}_{\gamma 0}^{out} \end{pmatrix}, \tag{11}$$

where ${}_{\pm}\mathrm{W}_{\gamma 0}^{out} = {}_{\pm}\mathrm{W}_{\phi 0}^{out}/[1-(c_2'/c_p)^2]$, and the superscript '*out*' represents 'out-of-phase'. From Eqs. (8a) and (8b), it can be deduced that ${}_{\pm}\mathrm{W}_{\gamma 0}^{out} = {}_{\pm}\mathrm{W}_{\phi 0}^{out}/[1-(c_2'/c_p)^2] < 0$ for the k_a-wave, while ${}_{\pm}\mathrm{W}_{\gamma 0}^{out} = {}_{\pm}\mathrm{W}_{\phi 0}^{out}/[1-(c_2'/c_p)^2] > 0$ for the k_b-wave. Therefore, the out-of-phase case leads to the k_b-wave only. The k_b-wave is thus denoted as the $\mathbf{w}^{out}$-wave and expressed as

$$ {}_{\pm}\mathbf{w}^{out} = \begin{Bmatrix} r_b \\ 1 \end{Bmatrix} (-{}_{\pm}\mathrm{W}_{\gamma 0}^{out}) \mathrm{e}^{\mathrm{i}(\omega t \mp k_b z)} = \begin{Bmatrix} r_b \\ 1 \end{Bmatrix} ({}_{\pm}w_{\gamma}^{out}), \tag{12} $$

where $r_b = (c_2'/c_{pb})^2 - 1 < 0$. Similarly, the displacements due to bending and shear are not independent but related through the parameter r_b, and this wave can thus be characterized by ${}_{\pm}\mathrm{w}_{\gamma}^{out}(z,t)$.

The $\mathbf{w}^{in}$-wave propagates in the whole frequency spectrum, but the $\mathbf{w}^{out}$-wave only propagates when a critical value of frequency is exceeded. The critical frequency is determined by the shear rigidity and the radius of gyration, expressed as $\omega_c = c_2'/r_g$. Below this critical frequency, the $\mathbf{w}^{out}$-wave is non-propagating, i.e., evanescent. An evanescent wave can be treated as a special case of a propagating wave. Therefore, at each frequency in the whole spectrum, a $\mathbf{w}^{in}$- and a $\mathbf{w}^{out}$-wave co-exist in an infinite beam. These two waves are said to be degenerate, and it is appropriate to express these degenerate waves as

$$ \begin{aligned} {}_{\pm}\mathbf{w}_d(z,t) &= \left\{ {}_{\pm}w_{\gamma}^{in}(z,t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad {}_{\pm}w_{\gamma}^{out}(z,t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \\ &= \left\{ ({}_{\pm}\mathrm{W}_{\gamma 0}^{in}) \mathrm{e}^{\mp \mathrm{i} k_a z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (-{}_{\pm}\mathrm{W}_{\gamma 0}^{out}) \mathrm{e}^{\mp \mathrm{i} k_b z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{13} $$

where the subscript '*d*' represents 'degenerate'.

3. Wave reflection at an elastically supported boundary

The application of wave-train closure principle requires the understanding of wave reflection at the boundaries and wave propagation along the beam. In this section, wave reflection at a general elastically supported boundary is considered. The boundary is located at the position $z = 0$ as shown in Fig. 2 and the boundary conditions are expressed as

$$ \left. \begin{pmatrix} K_{z_0} & K_{z_0} + KGA \dfrac{\partial}{\partial z} \\ EI \dfrac{\partial^2}{\partial z^2} - T_{z_0} \dfrac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{w}_{\phi}(z) \\ \mathrm{w}_{\gamma}(z) \end{pmatrix} \right|_{z=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{14} $$

where K_{z_0} and T_{z_0} are translational and rotational spring constants at the boundary. As a first attempt, consider wave reflection at the right-side boundary of the beam as shown in Fig. 2a, where a forward-propagating wave is incident and a backward-propagating wave is reflected.

In the formulation of a standing wave, all the waves involved must have the same frequency. Therefore, only a pair of degenerate $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-waves at the same frequency is considered in the following.

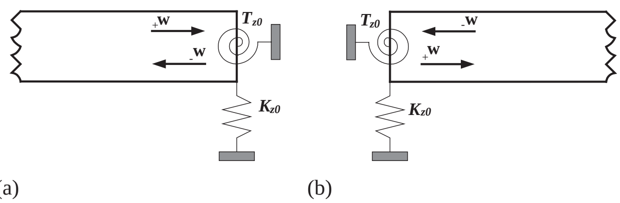


Fig. 2. Wave reflection at a general elastically supported boundary. (a) Right-side boundary, (b) left-side boundary.

3.1. Superposition of $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-waves upon reflection

In a general case of wave reflection, it is known that an incident wave would induce reflected waves of all types. A Timoshenko beam accommodates two types of waves, $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-waves, thus an incident $\mathbf{w}^{in}$- or $\mathbf{w}^{out}$-wave induces reflected waves of both types at a boundary in general, as pointed out by Mead [7]. For an incident $\mathbf{w}^{in}$-wave, the wave reflection can be written as

$$\begin{Bmatrix} {}_+\mathrm{w}_\gamma^{in} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{Bmatrix}. \tag{15}$$

The reflected waves can be expressed as

$$\begin{Bmatrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{Bmatrix} = \begin{Bmatrix} r_{11R}({}_+\mathrm{W}_{\gamma 0}^{in})\mathrm{e}^{\mathrm{i}k_a z} \\ r_{21R}({}_+\mathrm{W}_{\gamma 0}^{in})\mathrm{e}^{\mathrm{i}k_b z} \end{Bmatrix}, \tag{16}$$

where r_{11R}, r_{21R} are wave reflection coefficients. Similarly, for an incident $\mathbf{w}^{out}$-wave, the wave reflection can be written as

$$\begin{Bmatrix} 0 \\ {}_+\mathrm{w}_\gamma^{out} \end{Bmatrix} \Rightarrow \begin{Bmatrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{Bmatrix}. \tag{17}$$

The reflected waves are given by

$$\begin{Bmatrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{Bmatrix} = \begin{Bmatrix} r_{12R}(-{}_+\mathrm{W}_{\gamma 0}^{out})\mathrm{e}^{\mathrm{i}k_a z} \\ r_{22R}(-{}_+\mathrm{W}_{\gamma 0}^{out})\mathrm{e}^{\mathrm{i}k_b z} \end{Bmatrix}, \tag{18}$$

where r_{12R}, r_{22R} are wave reflection coefficients.

Once a standing wave is formulated, both types of wave are present at the boundary. Wave reflection is thus represented by the combination of Eqs. (15) and (17), written here as

$$\begin{Bmatrix} ({}_+\mathrm{w}_\gamma^{in}) \\ ({}_+\mathrm{w}_\gamma^{out}) \end{Bmatrix} \Rightarrow \begin{Bmatrix} ({}_-\mathrm{w}_\gamma^{in}) \\ ({}_-\mathrm{w}_\gamma^{out}) \end{Bmatrix}. \tag{19}$$

The reflected waves are expressed as

$$\begin{Bmatrix} ({}_-\mathrm{w}_\gamma^{in}) \\ ({}_-\mathrm{w}_\gamma^{out}) \end{Bmatrix} = \mathbf{R}_R \begin{Bmatrix} ({}_+\mathrm{W}_{\gamma 0}^{in})\mathrm{e}^{\mathrm{i}k_a z} \\ (-{}_+\mathrm{W}_{\gamma 0}^{out})\mathrm{e}^{\mathrm{i}k_b z} \end{Bmatrix} = \begin{pmatrix} r_{11R} & r_{12R} \\ r_{21R} & r_{22R} \end{pmatrix} \begin{Bmatrix} ({}_+\mathrm{W}_{\gamma 0}^{in})\mathrm{e}^{\mathrm{i}k_a z} \\ (-{}_+\mathrm{W}_{\gamma 0}^{out})\mathrm{e}^{\mathrm{i}k_b z} \end{Bmatrix}. \tag{20}$$

The wave reflection matrix $\mathbf{R}_R$ is determined from the boundary conditions, written as

$$\mathbf{R}_R = \begin{pmatrix} r_{11R} & r_{12R} \\ r_{21R} & r_{22R} \end{pmatrix} = -\{[_-\mathbf{R}]\}^{-1}[_+\mathbf{R}], \tag{21}$$

where

$$[_+\mathbf{R}] = \begin{pmatrix} (r_a+1)K_{z_0} - \mathrm{i}k_a KGA & (r_b+1)K_{z_0} - \mathrm{i}k_b KGA \\ -r_a(k_a^2 EI - \mathrm{i}k_a T_{z_0}) & -r_b(k_b^2 EI - \mathrm{i}k_b T_{z_0}) \end{pmatrix}, \text{ and } [_-\mathbf{R}] = \begin{pmatrix} (r_a+1)K_{z_0} + \mathrm{i}k_a KGA & (r_b+1)K_{z_0} + \mathrm{i}k_b KGA \\ -r_a(k_a^2 EI + \mathrm{i}k_a T_{z_0}) & -r_b(k_b^2 EI + \mathrm{i}k_b T_{z_0}) \end{pmatrix}.$$

This matrix is not diagonal, showing that the two waves interact in the reflection process at the boundary.

3.2. Condition for the waves to remain degenerate upon reflection

Since the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves are degenerate before and after reflection, it would be of interest to investigate whether and under what condition they remain degenerate upon reflection. By "remain degenerate upon reflection", it means that $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-waves reflect at the boundary independently. In other words, the superposition of the incident and the reflected waves $({}_+\mathrm{w}_\gamma^{in} + {}_-\mathrm{w}_\gamma^{in})$ and $({}_+\mathrm{w}_\gamma^{out} + {}_-\mathrm{w}_\gamma^{out})$ satisfy the boundary conditions individually. This yields

$$\begin{pmatrix} K_{z_0} & K_{z_0} - \mathrm{i}k_a KGA \\ -k_a^2 EI + \mathrm{i}k_a T_{z_0} & 0 \end{pmatrix} \begin{pmatrix} r_a \\ 1 \end{pmatrix} {}_+\mathrm{w}_\gamma^{in}(0) + \begin{pmatrix} K_{z_0} & K_{z_0} + \mathrm{i}k_a KGA \\ -k_a^2 EI - \mathrm{i}k_a T_{z_0} & 0 \end{pmatrix} \begin{pmatrix} r_a \\ 1 \end{pmatrix} {}_-\mathrm{w}_\gamma^{in}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22a}$$

and

$$\begin{pmatrix} K_{z_0} & K_{z_0} - \mathrm{i}k_b KGA \\ -k_b^2 EI + \mathrm{i}k_b T_{z_0} & 0 \end{pmatrix} \begin{pmatrix} r_b \\ 1 \end{pmatrix} {}_+\mathrm{w}_\gamma^{out}(0) + \begin{pmatrix} K_{z_0} & K_{z_0} + \mathrm{i}k_b KGA \\ -k_b^2 EI - \mathrm{i}k_b T_{z_0} & 0 \end{pmatrix} \begin{pmatrix} r_b \\ 1 \end{pmatrix} {}_-\mathrm{w}_\gamma^{out}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{22b}$$

From Eqs. (22a) and (22b), the following two equations can be derived:

$$KGA \cdot EI - (r_a + 1)K_{z_0} T_{z_0} = 0, \tag{23a}$$

$$KGA \cdot EI - (r_b + 1)K_{z_0} T_{z_0} = 0 \tag{23b}$$

and they lead to

$$\omega^2 = \frac{K_{z_0} T_{z_0}}{KGA \cdot EI} c_2'^2. \tag{24}$$

Eq. (24) suggests that a $\mathbf{w}^{in}$- and a $\mathbf{w}^{out}$-waves remain degenerate during reflection only at a specific frequency, designated as ω_{spe} in the following. In this case, Eqs. (22a) and (22b) can be combined and written as

$$[_{-}\mathbf{R}]\left\{\begin{array}{c}{}_{-}\mathrm{w}_{\gamma}^{in}\\ {}_{-}\mathrm{w}_{\gamma}^{out}\end{array}\right\}\Bigg|_{z=0}+[_{+}\mathbf{R}]\left\{\begin{array}{c}{}_{+}\mathrm{w}_{\gamma}^{in}\\ {}_{+}\mathrm{w}_{\gamma}^{out}\end{array}\right\}\Bigg|_{z=0}=\left\{\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\right\}, \tag{25}$$

where

$$[_{+}\mathbf{R}]=\begin{pmatrix}\lfloor(r_a+1)K_{z_0}-\mathrm{i}k_aKGA\rfloor & 0\\ 0 & [(r_b+1)K_{z_0}-\mathrm{i}k_bKGA]\end{pmatrix},$$

$$[_{-}\mathbf{R}]=\begin{pmatrix}\lfloor(r_a+1)K_{z_0}+\mathrm{i}k_aKGA\rfloor & 0\\ 0 & [(r_b+1)K_{z_0}+\mathrm{i}k_bKGA]\end{pmatrix}.$$

Eq. (25) can be further simplified to

$$\left\{\begin{array}{c}{}_{-}\mathrm{w}_{\gamma}^{in}\\ {}_{-}\mathrm{w}_{\gamma}^{out}\end{array}\right\}=\mathbf{R}_R\left\{\begin{array}{c}{}_{+}\mathrm{w}_{\gamma}^{in}\\ {}_{+}\mathrm{w}_{\gamma}^{out}\end{array}\right\}, \tag{26}$$

where the wave reflection matrix

$$\mathbf{R}_R=\begin{pmatrix}r_{11R} & 0\\ 0 & r_{22R}\end{pmatrix}=\begin{pmatrix}-\mathrm{e}^{-\mathrm{i}\theta^{in}} & 0\\ 0 & -\mathrm{e}^{-\mathrm{i}\theta^{out}}\end{pmatrix}.$$

In the matrix, r_{11R} and r_{22R} are wave reflection coefficients for the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves, respectively,

$$\theta^{in}=2\,\mathrm{arctg}\left(\frac{c_{pa}}{c_2'^2}\sqrt{\frac{T_{z_0}\cdot KGA}{EI\cdot K_{z_0}}}\right),\qquad \theta^{out}=2\,\mathrm{arctg}\left(\frac{c_{pb}}{c_2'^2}\sqrt{\frac{T_{z_0}\cdot KGA}{EI\cdot K_{z_0}}}\right)$$

and the subscript "R" represents "Right-side boundary". The wave reflection matrix is now diagonal, suggesting that the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves reflect independently.

3.3. Wave reflection matrices for classical boundary conditions

In the above analysis, K_{z_0} and T_{z_0} are assumed to be of finite value. In this section, limiting cases of the general elastic supports, which correspond to classical boundary conditions as listed in Table 1, are discussed.

For the free boundary condition, $K_{z_0}=0$ and $T_{z_0}=0$, while for the clamped boundary condition, $K_{z_0}=\infty$ and $T_{z_0}=\infty$. According to Eq. (24), degenerate waves only occur at 0 and ∞, respectively. Therefore, for a free or a clamped boundary condition, the waves are superposed upon the reflection at all frequencies. The simply supported and the sliding boundary conditions are worthy of further study since $K_{z_0}T_{z_0}$ is indeterminate in these two cases ($K_{z_0}=\infty$ and $T_{z_0}=0$ for the former while $K_{z_0}=0$ and $T_{z_0}=\infty$ for the latter), thus ω_{spe} is indeterminate.

The simply supported boundary conditions are given by

$$\begin{pmatrix}1 & 1\\ EI\dfrac{\partial^2}{\partial z^2} & 0\end{pmatrix}\begin{pmatrix}\mathrm{w}_{\phi}(z)\\ \mathrm{w}_{\gamma}(z)\end{pmatrix}\Bigg|_{z=0}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{27}$$

Table 1
Classical boundary conditions as limiting cases of general elastic support

$K_{z_0} T_{z_0}$	Boundary conditions
0	Free
∞	Clamped
$\infty \cdot 0$ (indeterminate)	Simply supported
$0 \cdot \infty$ (indeterminate)	Sliding

Assuming the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves to remain degenerate upon reflection as given by Eq. (22), superposition of the incident and reflected $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-waves should satisfy the boundary conditions individually. This yields

$$\begin{pmatrix} 1 & 1 \\ -k_a^2 EI & 0 \end{pmatrix} \begin{pmatrix} r_a \\ 1 \end{pmatrix} [{}_+\mathrm{w}_\gamma^{in}(0)] + \begin{pmatrix} 1 & 1 \\ -k_a^2 EI & 0 \end{pmatrix} \begin{pmatrix} r_a \\ 1 \end{pmatrix} [{}_-\mathrm{w}_\gamma^{in}(0)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{28a}$$

$$\begin{pmatrix} 1 & 1 \\ -k_b^2 EI & 0 \end{pmatrix} \begin{pmatrix} r_b \\ 1 \end{pmatrix} [{}_+\mathrm{w}_\gamma^{out}(0)] + \begin{pmatrix} 1 & 1 \\ -k_b^2 EI & 0 \end{pmatrix} \begin{pmatrix} r_b \\ 1 \end{pmatrix} [{}_-\mathrm{w}_\gamma^{out}(0)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{28b}$$

Eqs. (28a) and (28b) can be further reduced to

$$\begin{pmatrix} (r_a+1) & (r_a+1) \\ -r_a(k_a^2 EI) & -r_a(k_a^2 EI) \end{pmatrix} \begin{pmatrix} {}_+\mathrm{w}_\gamma^{in}(0) \\ {}_-\mathrm{w}_\gamma^{in}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{29a}$$

$$\begin{pmatrix} (r_b+1) & (r_b+1) \\ -r_b(k_b^2 EI) & -r_b(k_b^2 EI) \end{pmatrix} \begin{pmatrix} {}_+\mathrm{w}_\gamma^{out}(0) \\ {}_-\mathrm{w}_\gamma^{out}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{29b}$$

It can be seen that these two equations hold for all frequencies. Eqs. (29a) and (29b) can then be written as

$$\begin{gathered} [{}_-\mathbf{R}] \left\{ \begin{matrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{matrix} \right\} + [{}_+\mathbf{R}] \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\} \Bigg|_{z=0} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \\ \left\{ \begin{matrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{matrix} \right\} \Bigg|_{z=0} = -[{}_-\mathbf{R}]^{-1} [{}_+\mathbf{R}] \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\} \Bigg|_{z=0}, \end{gathered} \tag{30}$$

where

$$[{}_-\mathbf{R}] = \begin{pmatrix} (r_a+1) & 0 \\ 0 & (r_b+1) \end{pmatrix} \quad \text{and} \quad [{}_+\mathbf{R}] = \begin{pmatrix} (r_a+1) & 0 \\ 0 & (r_b+1) \end{pmatrix}.$$

Eq. (30) is identical to Eq. (26), but the wave reflection matrix is now given by

$$\mathbf{R}_R = -\{[{}_-\mathbf{R}]\}^{-1} [{}_+\mathbf{R}] = -\mathbf{I}. \tag{31}$$

It can be seen that, upon reflection at a simply supported boundary, the phase is shifted for both waves by π with the amplitude unchanged at all frequencies.

The sliding boundary conditions are given by

$$\begin{pmatrix} 0 & KGA\dfrac{\partial}{\partial z} \\ -\dfrac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{w}_\phi(z) \\ \mathrm{w}_\gamma(z) \end{pmatrix} \Bigg|_{z=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{32}$$

Following the same procedure, it can be seen that wave reflection at a sliding boundary is still represented by Eq. (30), but the wave reflection matrix is given by

$$\mathbf{R}_R = -\{[{}_-\mathbf{R}]\}^{-1}[{}_+\mathbf{R}] = \mathbf{I}. \tag{33}$$

This shows that the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves remain degenerate upon the reflection at all frequencies. However, both the phase and the amplitude are unchanged upon reflection at all frequencies.

The reflection of a backward-propagating wave impinging on a left-hand boundary is a mirror image of the forward-propagating wave impinging on a right-hand boundary, so the same conclusions can be drawn. Table 2 summarizes the wave reflection matrices for various boundary conditions when reflection occurs at the left or the right side of the boundary.

4. Wave-train closure principle: the formulation of standing waves

4.1. Expression of the wave-train closure principle

In this section, the wave-train closure principle is used to examine all possible vibration modes of a finite-length Timoshenko beam. The beam is assumed to be of length L with general elastic supports at both ends, as illustrated in Fig. 3.

The phase and amplitude changes of the waves must now be considered as they propagate along the beam, in addition to the changes taking place due to reflection at the boundaries. Without loss of generality, suppose that the wave starts from the location $z = Z$ and propagates in the positive direction. The complex amplitudes are related by

$$\left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=L} = \mathbf{T}_{ZL} \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=Z} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_a(L-Z)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_b(L-Z)} \end{pmatrix} \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=Z} \tag{34a}$$

for the two waves at $z = L$ and Z,

$$\left\{ \begin{matrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=0} = \mathbf{T}_{L0} \left\{ \begin{matrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=L} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_aL} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_bL} \end{pmatrix} \left\{ \begin{matrix} {}_-\mathrm{w}_\gamma^{in} \\ {}_-\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=L} \tag{34b}$$

for the two waves at $z = L$ and 0, and

$$\left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=Z} = \mathbf{T}_{0Z} \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=0} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}k_aZ} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_bZ} \end{pmatrix} \left\{ \begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix} \right\}\Bigg|_{z=0} \tag{34c}$$

for the two waves at $z = 0$ and Z.

Table 2
List of wave reflection matrices for various boundary conditions

	Elastically supported (Superposed at frequencies other than ω_{spe})	Elastically supported (Degenerate at ω_{spe})	Simply supported	Sliding
$[_{+}\mathbf{R}]$	$\begin{pmatrix} [(r_a+1)K_{z_0} - \mathrm{i}k_a KGA] & [(r_b+1)K_{z_0} - \mathrm{i}k_b KGA] \\ -r_a(k_a^2 EI - \mathrm{i}k_a T_{z_0}) & -r_b(k_b^2 EI - \mathrm{i}k_b T_{z_0}) \end{pmatrix}$	$\begin{pmatrix} [(r_a+1)K_{z_0} - \mathrm{i}k_a KGA] & 0 \\ 0 & [(r_b+1)K_{z_0} - \mathrm{i}k_b KGA] \end{pmatrix}$	$\begin{pmatrix} (r_a+1) & 0 \\ 0 & (r_b+1) \end{pmatrix}$	$\begin{pmatrix} \mathrm{i}k_a r_a & 0 \\ 0 & \mathrm{i}k_b r_b \end{pmatrix}$
$[_{-}\mathbf{R}] = [_{+}\mathbf{R}]^{*}$	$\begin{pmatrix} [(r_a+1)K_{z_0} + \mathrm{i}k_a KGA] & [(r_b+1)K_{z_0} + \mathrm{i}k_b KGA] \\ -r_a(k_a^2 EI + \mathrm{i}k_a T_{z_0}) & -r_b(k_b^2 EI + \mathrm{i}k_b T_{z_0}) \end{pmatrix}$	$\begin{pmatrix} [(r_a+1)K_{z_0} + \mathrm{i}k_a KGA] & 0 \\ 0 & [(r_b+1)K_{z_0} + \mathrm{i}k_b KGA] \end{pmatrix}$	$\begin{pmatrix} (r_a+1) & 0 \\ 0 & (r_b+1) \end{pmatrix}$	$\begin{pmatrix} -\mathrm{i}k_a r_a & 0 \\ 0 & -\mathrm{i}k_b r_b \end{pmatrix}$
Right-side $\mathbf{R}_R$	$-\{[_{-}\mathbf{R}]\}^{-1}[_{+}\mathbf{R}]$	$\begin{pmatrix} -\frac{[(r_a+1)K_{z_0} - \mathrm{i}k_a KGA]}{[(r_a+1)K_{z_0} + \mathrm{i}k_a KGA]} & 0 \\ 0 & -\frac{[(r_b+1)K_{z_0} - \mathrm{i}k_b KGA]}{[(r_b+1)K_{z_0} + \mathrm{i}k_b KGA]} \end{pmatrix} = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\theta^{in}} & 0 \\ 0 & -\mathrm{e}^{-\mathrm{i}\theta^{out}} \end{pmatrix}$	**-I**	**I**
Left-side $\mathbf{R}_L$	$-\{[_{+}\mathbf{R}]\}^{-1}[_{-}\mathbf{R}]$	$\begin{pmatrix} -\frac{[(r_a+1)K_{z_0} + \mathrm{i}k_a KGA]}{[(r_a+1)K_{z_0} - \mathrm{i}k_a KGA]} & 0 \\ 0 & -\frac{[(r_b+1)K_{z_0} + \mathrm{i}k_b KGA]}{[(r_b+1)K_{z_0} - \mathrm{i}k_b KGA]} \end{pmatrix} = \begin{pmatrix} -\mathrm{e}^{\mathrm{i}\theta^{in}} & 0 \\ 0 & -\mathrm{e}^{\mathrm{i}\theta^{out}} \end{pmatrix}$	**-I**	**I**

The subscripts 'R' and 'L' represent wave reflection at the right- and the left-side boundaries of a beam, respectively.

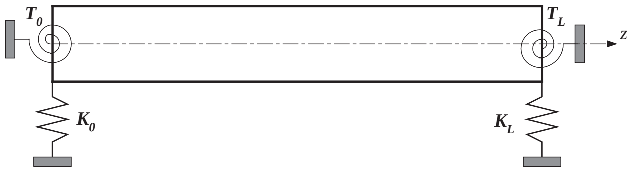


Fig. 3. Illustration of an elastically supported finite-length beam.

The wave propagation matrices are diagonal because the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves do not interfere with each other during the propagation along the beam, despite the fact that the waves could be superposed at the boundary.

As a wave propagates along the whole beam, is reflected at each end, and finally returns to its starting point to finish one circuit, the returning and starting wave amplitudes are found to be related by

$$({}_{+}\mathbf{w}|_{z=Z}) = [\mathbf{T}_{0Z}][\mathbf{R}_0][\mathbf{T}_{L0}][\mathbf{R}_L][\mathbf{T}_{ZL}]({}_{+}\mathbf{w}|_{z=Z}), \tag{35}$$

where $[\mathbf{R}_0]$ is the reflection matrix at the left-side boundary $z = 0$ and $[\mathbf{R}_L]$ is the reflection matrix at the right-side boundary $z = L$. Expressions for these reflection matrices are given in Table 2. Eq. (35) can be written in a simplified form as

$$\{\mathbf{I} - [\mathbf{RT}]\}({}_{+}\mathbf{w}|_{z=Z}) = \mathbf{0}, \tag{36}$$

where $\mathbf{I}$ is the unit diagonal matrix, and $[\mathbf{RT}] = [\mathbf{T}_{0Z}][\mathbf{R}_0][\mathbf{T}_{L0}][\mathbf{R}_L][\mathbf{T}_{ZL}]$.

In Eq. (36), it can be seen that while the propagation matrices are the same for degenerate and superposed waves, expressions of reflection matrices are different. Consequently, the resulting standing waves will have different forms as demonstrated in the following.

4.2. Degenerate and single standing waves

The existence of degenerate standing waves in a finite-length Timoshenko beam is examined first. Upon reflection at a boundary, it has been shown that propagating waves remain degenerate only at the specific frequency

$$\omega_{spe} = \sqrt{\frac{K_{z0}T_{z0}}{KGA \cdot EI}}\, c_2'^2.$$

For a finite-length beam, wave reflection occur at two boundaries: the wave reflection at $z = 0$ is a left-side reflection, where

$$\omega_{spe}|_{z=0} = \sqrt{\frac{K_0 T_0}{KGA \cdot EI}}\, c_2'^2$$

and the wave reflection at $z = L$ is a right-side reflection, where

$$\omega_{spe}|_{z=L} = \sqrt{\frac{K_L T_L}{KGA \cdot EI}}\, c_2'^2.$$

If a standing wave is to be generated by a single degenerate wave, these two frequencies must be equal. Hence,

$$K_0 T_0 = K_L T_L. \tag{37}$$

The natural frequency of such a degenerate standing wave is pre-determined as

$$\omega^2 = \frac{K_0 T_0}{KGA \cdot EI} c_2'^2 = \frac{K_L T_L}{KGA \cdot EI} c_2'^2. \tag{38}$$

However, this is not the only condition for the existence of degenerate standing waves. At $z = 0$ and L, the wave reflection matrices are given by

$$\mathbf{R}_0 = -\begin{pmatrix} \dfrac{[(r_a+1)K_0 + \mathrm{i}k_a KGA]}{[(r_a+1)K_0 - \mathrm{i}k_a KGA]} & 0 \\ 0 & \dfrac{[(r_b+1)K_0 + \mathrm{i}k_b KGA]}{[(r_b+1)K_0 - \mathrm{i}k_b KGA]} \end{pmatrix} \tag{39a}$$

and

$$\mathbf{R}_L = -\begin{pmatrix} \dfrac{[(r_a+1)K_L - \mathrm{i}k_a KGA]}{[(r_a+1)K_L + \mathrm{i}k_a KGA]} & 0 \\ 0 & \dfrac{[(r_b+1)K_L - \mathrm{i}k_b KGA]}{[(r_b+1)K_L + \mathrm{i}k_b KGA]} \end{pmatrix}, \tag{39b}$$

respectively. The wave-train closure principle yields,

$$\{\mathbf{I} - [\mathbf{RT}]\} \left\{ \begin{matrix} {}_{+}\mathrm{w}_\gamma^{in} \\ {}_{+}\mathrm{w}_\gamma^{out} \end{matrix} \right\} \Bigg|_{z=Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{40}$$

where $\mathbf{I} - [\mathbf{RT}] = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}2k_a L} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}2k_b L} \end{pmatrix}$.

If degenerate standing waves exist, $|\mathbf{I} - [\mathbf{RT}]| = 0$. This leads to two equations for $\mathbf{w}^{in}$- and $\mathbf{w}^{out}$-standing waves, respectively. These equations are

$$\begin{aligned} k_a L &= n_a \pi, \quad n_a = 1, 2, \ldots, \\ k_b L &= n_b \pi, \quad n_b = 1, 2, \ldots\ . \end{aligned} \tag{41}$$

Since natural frequencies of degenerate standing waves have been determined by Eq. (38), k_a and k_b in Eq. (41) are accordingly determined. Therefore, Eq. (41) is not just a frequency equation but gives an additional condition for the existence of standing waves. If the standing waves remain degenerate, the two equations in (41) must be satisfied simultaneously, yielding

$$k_a / k_b = n_a / n_b, \quad n_a, n_b = 1, 2, \ldots\ . \tag{42}$$

Eqs. (37) and (42) combined together give the conditions for the existence of degenerate standing waves in an elastically supported beam. The natural frequency of the degenerate standing waves is given by Eq. (38).

If the two equations in Eq. (41) are not satisfied simultaneously, standing waves are still possible but they belong to a new type. They originate either from $\mathbf{w}^{in}$-waves alone or from $\mathbf{w}^{out}$-waves alone, thus can be termed as *single* standing waves. For this type of standing waves, it would be instructive to write the wave-train closure principle as

$$\left\{[\mathbf{T}_{0Z}][\mathbf{R}_0][\mathbf{T}_{L0}][\mathbf{R}_L][\mathbf{T}_{ZL}] - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} {}_+\mathrm{w}_\gamma^{in} \\ 0 \end{pmatrix}\Bigg|_{z=Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{43a}$$

for $\mathbf{w}^{in}$-standing waves, and

$$\left\{[\mathbf{T}_{0Z}][\mathbf{R}_0][\mathbf{T}_{L0}][\mathbf{R}_L][\mathbf{T}_{ZL}] - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} 0 \\ {}_+\mathrm{w}_\gamma^{out} \end{pmatrix}\Bigg|_{z=Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{43b}$$

for $\mathbf{w}^{out}$-standing waves.

Now consider the case of a beam with classical boundary conditions and the two waves remain degenerate upon reflection over the whole frequency range. According to the previous analysis of wave reflection, this case includes the simply supported beam, the sliding–sliding beam, and the beam with one end sliding and the other simply supported. In this case, the wave-train closure principle is still represented by

$$\{\mathbf{I} - [\mathbf{RT}]\}\left\{\begin{matrix} {}_+\mathrm{w}_\gamma^{in} \\ {}_+\mathrm{w}_\gamma^{out} \end{matrix}\right\}\Bigg|_{z=Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{44}$$

but with different wave reflection matrices.

For a simply supported beam, the reflection matrices are given by $\mathbf{R}_0 = -\mathbf{I}$ and $\mathbf{R}_L = -\mathbf{I}$, then $\mathbf{I} - [\mathbf{RT}] = \begin{pmatrix} 1-\mathrm{e}^{-\mathrm{i}2k_aL} & 0 \\ 0 & 1-\mathrm{e}^{-\mathrm{i}2k_bL} \end{pmatrix}$. If the standing waves exist, then $|\mathbf{I} - [\mathbf{RT}]| = 0$ and leads to

$$\begin{aligned} k_aL &= n_a\pi, \quad n_a = 1, 2, \ldots, \\ k_bL &= n_b\pi, \quad n_b = 1, 2, \ldots\ . \end{aligned} \tag{45}$$

Eq. (45) is exactly the frequency equation of a simply supported beam obtained by using the usual modal analysis method. It appears to be identical to Eq. (41) for an elastically supported beam. However, it should be noted that in Eq. (41) the wave numbers have been determined at a specific frequency. In the present case of a simply supported beam, however, Eq. (45) is a frequency equation. The wave numbers are determined first, and the natural frequencies are calculated from the following relations:

$$\omega_a^2 = \frac{1}{2}\left\{\left[\frac{KGA}{J} + \left(\frac{KGA}{m} + \frac{EI}{J}\right)k_a^2\right] + \sqrt{\left[\frac{KGA}{J} + \left(\frac{KGA}{m} + \frac{EI}{J}\right)k_a^2\right]^2 - \frac{4EI\cdot KGA}{mJ}}\right\}, \tag{46a}$$

$$\omega_b^2 = \frac{1}{2}\left\{\left[\frac{KGA}{J} + \left(\frac{KGA}{m} + \frac{EI}{J}\right)k_b^2\right] + \sqrt{\left[\frac{KGA}{J} + \left(\frac{KGA}{m} + \frac{EI}{J}\right)k_b^2\right]^2 - \frac{4EI\cdot KGA}{mJ}}\right\}. \tag{46b}$$

If the two standing waves are degenerate, then $\omega_a = \omega_b$, leading to,

$$r_g^2 = \frac{c_2'^2(c_0^2 + c_2'^2)(k_a^2 + k_b^2)}{(k_a^2c_0^2 - k_b^2c_2'^2)(k_a^2c_2'^2 - k_b^2c_0^2)} = \frac{c_2'^2(c_0^2 + c_2'^2)(n_a^2 + n_b^2)}{(n_a^2c_0^2 - n_b^2c_2'^2)(n_a^2c_2'^2 - n_b^2c_0^2)(\pi/L)^2}. \tag{47}$$

Therefore, Eqs. (45) and (47) give the conditions for a $\mathbf{w}^{in}$- and a $\mathbf{w}^{out}$-standing waves to be degenerate in a simply supported beam. The standing waves not satisfying the condition given by

Eq. (47) are not degenerate but belong to single standing waves. They are divided into two groups of standing waves, one originating from $\mathbf{w}^{in}$-waves and the other from $\mathbf{w}^{out}$-waves.

For the sliding–sliding beam, $\mathbf{R}_0 = \mathbf{I}$ and $\mathbf{R}_L = \mathbf{I}$, thus $\mathbf{I} - [\mathbf{RT}] = \begin{pmatrix} e^{-i2k_aL} & 0 \\ 0 & e^{-i2k_bL} \end{pmatrix}$. For the beam with one end sliding and the other simply supported, $\mathbf{R}_0 = \mathbf{I}$ and $\mathbf{R}_L = -\mathbf{I}$, thus $\mathbf{I} - [\mathbf{RT}] = \begin{pmatrix} -e^{-i2k_aL} & 0 \\ 0 & -e^{-i2k_bL} \end{pmatrix}$. Eventually, they also lead to Eq. (45), identical to the results for the simply supported beam. Therefore, the same conclusions can be drawn.

4.3. Superposed standing waves

When $K_0T_0 \neq K_LT_L$, a $\mathbf{w}^{in}$- and a $\mathbf{w}^{out}$-waves have to be superposed upon reflection at one or both boundaries. For the latter case, the reflection matrices are given by

$$\mathbf{R}_0 = -\{[_+\mathbf{R}_0]\}^{-1}[_-\mathbf{R}_0] \quad \text{at } z = 0, \tag{48a}$$

where $[_+\mathbf{R}_0] = \begin{pmatrix} [(r_a+1)K_0 - ik_aKGA] & [(r_b+1)K_0 - ik_bKGA] \\ -r_a(k_a^2EI - ik_aT_0) & -r_b(k_b^2EI - ik_bT_0) \end{pmatrix}$,

$$[_-\mathbf{R}_0] = \begin{pmatrix} [(r_a+1)K_0 + ik_aKGA] & [(r_b+1)K_0 + ik_bKGA] \\ -r_a(k_a^2EI + ik_aT_0) & -r_b(k_b^2EI + ik_bT_0) \end{pmatrix},$$

$$\mathbf{R}_L = -\{[_-\mathbf{R}_L]\}^{-1}[_+\mathbf{R}_L] \quad \text{at } z = L,$$

$$[_-\mathbf{R}_L] = \begin{pmatrix} [(r_a+1)K_L + ik_aKGA] & [(r_b+1)K_L + ik_bKGA] \\ -r_a(k_a^2EI + ik_aT_L) & -r_b(k_b^2EI + ik_bT_L) \end{pmatrix},$$

$$[_+\mathbf{R}_L] = \begin{pmatrix} [(r_a+1)K_L - ik_aKGA] & [(r_b+1)K_L - ik_bKGA] \\ -r_a(k_a^2EI - ik_aT_L) & -r_b(k_b^2EI - ik_bT_L) \end{pmatrix}. \tag{48b}$$

The wave-train closure principle yields

$$\{\mathbf{I} - [\mathbf{RT}]\}\begin{pmatrix} _+\mathrm{w}_\gamma^{in} \\ _+\mathrm{w}_\gamma^{out} \end{pmatrix}\Bigg|_{z=Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{49}$$

For superposed standing waves to exist, the following equation should be satisfied:

$$|\mathbf{I} - [\mathbf{RT}]| = 0. \tag{50}$$

Since the matrix $\mathbf{I} - [\mathbf{RT}]$ is no longer diagonal, Eq. (50) is a single equation that determines the natural frequencies of the superposed standing waves. In reality, Eq. (49) represents a compatibility condition of wave motion, i.e., the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves have to fit together in order to formulate a standing wave. As a result, Eq. (50) is different from the expression of wave-train closure principle for a Euler–Bernoulli beam as presented by Mead [2], in which the wave-train closure principle could be applied to either propagating waves (originated from the present $\mathbf{w}^{in}$-wave) or evanescent waves (originated from the present $\mathbf{w}^{out}$-wave) separately, leading to the same frequency equation.

It should be noted that the Euler–Bernoulli beam model is an approximation to the Timoshenko beam model by neglecting shear deformation and rotary inertia. Consequently, a one-degree-of-freedom model, such as bending displacement motion only, is enough to describe beam vibration. The present $\mathbf{w}^{out}$-wave is no longer a different type of wave, but reduced, along with the present $\mathbf{w}^{in}$-wave, to a single wave. The wave then becomes a one-dimensional entity. The compatibility condition of wave motion at the boundary is unnecessary since there is only one single wave, thus there is no need for further consideration like that given in the present paper for superposed standing waves. This is similar to the present case of single and degenerate standing waves, for which the compatibility condition of wave motion is also unnecessary.

A case remaining to be discussed is the beam with one end elastically supported and the other sliding or simply supported. For such a beam, at the specific frequency determined by the spring constants of the elastic support, the waves are reflected at both ends in the degenerate state. Degenerate standing waves are formulated, and the wave-train closure principle gives a condition for the existence of such degenerate standing waves. At the frequencies other than the specific frequency, the waves are reflected at the elastically supported boundary in the superposed state, thus superposed standing waves are formulated. The wave-train closure principle gives natural frequencies and mode shapes of such superposed standing waves.

5. Conclusions

Various vibration modes of a finite-length Timoshenko beam are studied using the wave-train closure principle, with emphasis on the mechanism of the formulation of various vibration modes in a beam, particularly, on the existence of degenerate modes. Flexural waves accommodated in an infinite beam are investigated first. The role of shear deformation is studied by introducing a novel wave entity, i.e., a displacement vector of bending and shear for these waves. It is demonstrated that the shear displacement is either in-phase or out-of-phase with respect to the bending displacement, leading to two types of flexural waves in an infinite beam, respectively. They are the in-phase flexural wave ($\mathbf{w}^{in}$-wave) and the out-of-phase flexural wave ($\mathbf{w}^{out}$-wave). The two flexural waves are degenerate in an infinite beam.

The behavior of wave reflection at an elastically supported boundary is then studied. It is demonstrated that the introduction of a boundary removes the degeneracy of the $\mathbf{w}^{in}$- and the $\mathbf{w}^{out}$-waves in general; the two waves have to be superposed upon reflection at the boundary. Wave reflection at classical free or clamped boundaries belongs to this case also. Under exceptional circumstances, either of these waves may be reflected at a boundary without inducing the other. In this paper, they are then said to "remain degenerate". One is that these two waves are at a specific frequency related to the spring constants of the elastic support, and the other is classical simply supported or sliding boundary condition.

The wave-train closure principle is extended to a vector form for the finite-length Timoshenko beam. Various wave reflection behavior results in three types of standing waves, namely, *superposed*, *degenerate*, and *single* standing waves, in a finite-length Timoshenko beam. In general, vibration modes in a Timoshenko beam are superposed standing waves. A compatibility condition of wave motion should be satisfied in order to formulate superposed standing waves.

Degenerate standing waves can exist in several special cases. They include a beam with simply supported and/or sliding boundary conditions and an elastically supported beam at the specific frequency, provided that additional conditions are satisfied simultaneously. Apart from degenerate standing waves, standing waves of another type are possible. They are termed as *single* standing waves or *single* modes, in the sense that each of them originates from either a $\mathbf{w}^{in}$- or a $\mathbf{w}^{out}$-wave but two such standing waves do not have the same natural frequency as degenerate modes. For degenerate and single standing waves, the compatibility condition is satisfied naturally.

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