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A theoretical formulation of the dynamical response of a master structure coupled with elastic continuous fuzzy subsystems with discrete attachments

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Abstract

We present the formulation of the dynamic response of a master structure coupled with a locally homogeneous and orthotropic structural fuzzy, with discrete attachment, composed of elastic continuous fuzzy subsystems. As introduced by Soize, the master structure is the part of the coupled system which is accessible by classical modeling, whereas the structural fuzzy represents systems connected to the master structure, whose characteristics are imprecisely known. A deterministic formulation of the boundary impedance of a general continuous structural fuzzy, which models its action on the master structure, is derived: it is shown that the formulation is different from the solution proposed by Soize in the context of the type I fuzzy law, established from the deterministic model of a linear oscillator excited by its support. Finally, the general boundary impedance is applied to the special situation of a structural fuzzy composed of elastic bars whose geometrical parameters are randomly defined, and numerical results are presented.

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1. Introduction

The problem under investigation in this paper is the dynamics of a ‘‘master’’ structure coupled to elastic continuous attachments. A master structure is defined as a structure with known geometrical and material characteristics, as well as boundary conditions and excitations. On the other hand, it is assumed a priori that the attachments are imprecisely known and cannot be described by a deterministic model. In the theory of Soize [1–3], the term ‘‘structural fuzzy’’ or simply ‘‘fuzzy’’ refers to these complex subsystems, and a master structure coupled with a structural fuzzy is then called a ‘‘fuzzy structure’’. Precisely, the study of fuzzy structures requires to formulate the dynamic response of a mechanical system containing a part, the master structure, which can be modeled using a deterministic model, and a part, the structural fuzzy, which is not accessible to deterministic modeling due to its complexity and whose parameters are described with uncertainties.

Soize initially suggested from experimental observations that when the fuzzy does not resonate (low-frequency domain), its action on the master structure results in an added-mass effect; however, when the fuzzy resonates (medium-frequency domain), the response of the coupled master structure becomes considerably damped. As explained by Soize, this apparent damping is due to the fact that mechanical energy is absorbed through the resonant fuzzy. Fundamentally, Soize proposed to describe the effect of a structural fuzzy on a master structure from a probabilistic boundary impedance, and suggested 2 types of fuzzy descriptions: type I fuzzy law corresponds to a probabilistic model of the boundary impedance of a locally homogeneous and orthotropic structural fuzzy and is constructed, at a given frequency, from the deterministic model of the boundary impedance of a linear oscillator excited by its support [1–3]. Type II fuzzy law corresponds to a probabilistic model of the boundary impedance of an orthotropic structural fuzzy with continuous attachment effect and is constructed, at a given frequency, from the deterministic model of the boundary impedance of a linear oscillator excited by its support, whose stiffness and damping spatially vary [2,3]. A modeling of this type of fuzzy from a locally homogeneous fuzzy has been adopted by Soize [2,3].

In the two cases, the parameters of the fuzzy (resonant mass, damping and eigenfrequency) are randomly defined from mean parameters (mean resonant mass, mean damping and mean modal density). The problem outlined by Soize is to identify these mean parameters, particularly the mean resonant mass (that is, the dynamic contribution of the fuzzy), in the case of a fuzzy which is not composed of linear oscillators, but generally of elastic and continuous substructures. An identification method based on a statistical energy analysis (SEA) approach [4] has been proposed by Soize in Ref. [2,5]: this approach assumes that the power flow from the master structure to the structural fuzzy can be estimated from an SEA model.

The structural fuzzy theory of Soize has been validated numerically, first in the case of a locally homogeneous fuzzy with discrete boundary (i.e. the connections of the fuzzy to the master structure are discrete) composed of a large number of linear oscillators [3,5–7], and second in the case of a fuzzy with continuous boundary (i.e. the connections of the fuzzy to the master structure are continuous) composed of plates coupled with oscillators [8]. The dynamic response of a master structure coupled with a homogeneous structural fuzzy, composed of a large number of linear oscillators has also been studied by other investigators using other types of descriptions: The study of a homogeneous fuzzy which is locally composed of a large number of linear oscillators, whose eigenfrequencies are different, has been proposed by Pierce et al. [9,10]. Weaver [11] has proposed a formulation of the vibratory behavior of a rigid mass (master structure) coupled with a large number of linear oscillators whose eigenfrequencies are randomly defined. A deterministic model of the action of linear oscillators on a rigid mass has also been proposed by Strasberg et al. [12].

Therefore, in the context of type I fuzzy law of the theory of Soize (the spatial memory effect inside the fuzzy is not included), as well as in other work [9–12], it is supposed that the dynamics of the structural fuzzy can be described in a deterministic sense from the model of a linear oscillator excited by its support; however, the application of this description to the case of a fuzzy which is not composed of linear oscillators remains difficult in the sense that the identification of the mean parameters of the fuzzy is not trivial and requires an SEA model of the fuzzy.

In this paper, we propose a formulation of the dynamical response of a master structure coupled with a locally homogeneous and orthotropic structural fuzzy, with discrete boundary (i.e. with discrete attachments), composed of elastic and continuous fuzzy subsystems. The aim is to extend the Soize's framework to the case of a fuzzy which is composed, in the general case, of *elastic* and *continuous* subsystems. A general description of the dynamics of the fuzzy, which takes its modal behavior into account, is derived. It appears that the formulation proposed in this paper is slightly different from the deterministic solution of Soize developed in the context of type I fuzzy law. The general formulation is applied to the special situation of a fuzzy which is composed of elastic bars, whose lengths and cross-sectional areas are randomly defined.

2. Problem description

The objective of this work is to evaluate the dynamic response of a master structure, subject to harmonic excitations under steady state conditions, coupled with a structural fuzzy (see Fig. 1). We recall that the master structure represents the part of the fuzzy structure (coupled system) whose characteristics are exactly known, while the structural fuzzy represents the part of the fuzzy structure which is described with uncertainties.

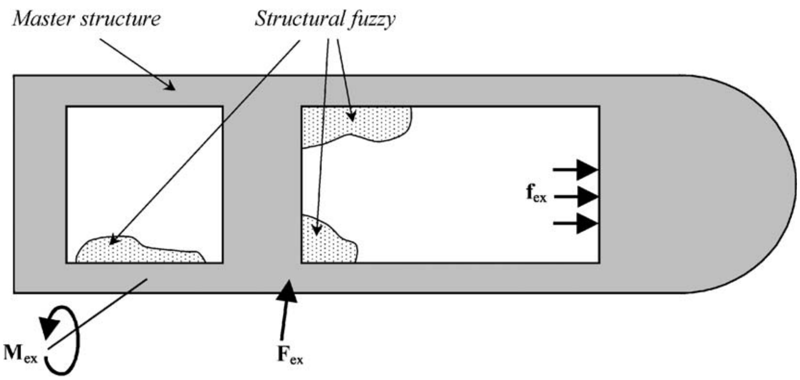

Fig. 1. Illustration of a master structure, coupled with a structural fuzzy, excited by a distributed force $\mathbf{f}_{ex}$, a point force $\mathbf{F}_{ex}$ and a point moment $\mathbf{M}_{ex}$.

As introduced by Soize [1], the following mechanical assumptions are used: (1) the fuzzy is elastic and has a linear behavior; (2) there are no excitation sources inside the fuzzy, i.e. it is only excited at the coupling surface; (3) the fuzzy is composed of weakly damped mechanical systems; (4) the mass of the fuzzy is small compared to the mass of the master structure. Furthermore, it is supposed that the master structure is elastic, has a linear behavior and is weakly damped.

We suppose that the fuzzy is coupled to the master structure on a coupling surface Γ. In the general case, it is assumed that the fuzzy is composed of independent fuzzy substructures, continuous over coupling surfaces $\Gamma_j \subset \Gamma$, such that $\Gamma = \bigcup_j \Gamma_j$. The action of the fuzzy on the master structure is then described for each coupling surface Γ_j: at the frequency $\omega/2\pi$, this action is locally modeled from a boundary impedance $\mathbf{Z}$ [1,3]:

$$\mathbf{f}(\mathbf{x},\omega) = \int_{\Gamma_j} \mathrm{i}\omega \mathbf{Z}(\mathbf{x},\mathbf{x}',\omega)\mathbf{u}(\mathbf{x}',\omega)\,\mathrm{d}s(\mathbf{x}') \quad \text{on } \Gamma_j, \tag{1}$$

where $\mathbf{f}$ represents the surface force applied to the fuzzy (that is, $-\mathbf{f}$ is the surface force applied by the fuzzy to the master structure), $\mathbf{u}$ is the displacement on Γ_j, and ds is the surface area element. In the general case, the vectors $\mathbf{f}(\mathbf{x},\omega)$ and $\mathbf{u}(\mathbf{x},\omega)$ at a point $\mathbf{x} \in \Gamma_j$ are described in a local orthonormal basis $\{\mathbf{w_1},\mathbf{w_2},\mathbf{w_3}\}$ related to Γ_j (Fig. 2).

Following the approach proposed by Soize, it is assumed that the structural fuzzy is homogeneous on Γ_j ($\mathbf{Z}(\mathbf{x},\mathbf{x}',\omega) = \mathbf{Z}(\omega)\,\forall(\mathbf{x},\mathbf{x}') \in \Gamma_j \times \Gamma_j$), and orthotropic on Γ_j ($\mathrm{Z}_{ik}(\mathbf{x},\mathbf{x}',\omega) = \delta_{ik} Z_i(\mathbf{x},\mathbf{x}',\omega)\,\forall(\mathbf{x},\mathbf{x}') \in \Gamma_j \times \Gamma_j$) relatively to the local basis $\{\mathbf{w_1},\mathbf{w_2},\mathbf{w_3}\}$. It is also assumed that the fuzzy is locally homogeneous and orthotropic on Γ, that is, it is homogeneous and orthotropic on each coupling surface Γ_j ($\Gamma = \bigcup_j \Gamma_j$).

Fundamentally, these conditions represent a fuzzy which is continuous on a coupling surface Γ_j. However, we can further define a homogeneous and orthotropic fuzzy on Γ_j with discrete

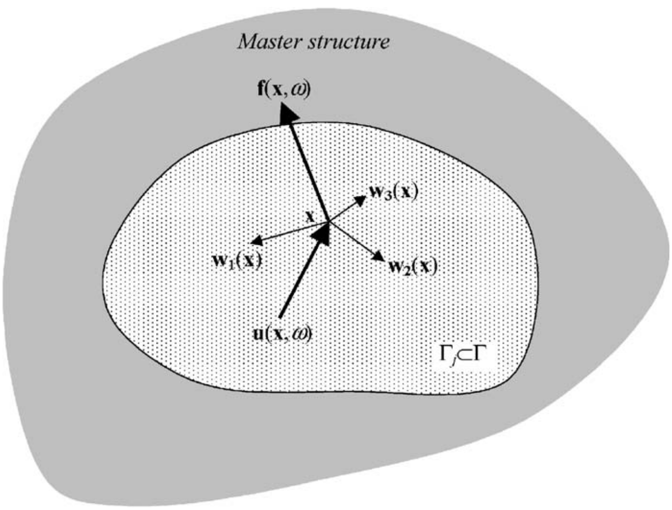

Fig. 2. General description of a coupling surface $\Gamma_j \subset \Gamma$ between a master structure and a continuous fuzzy substructure.

boundary (in this case, the connections of the fuzzy to the master structure are supposed to be discrete) is if we consider the following assumptions:

(H1) On Γ_j, the fuzzy is composed of a large number of identical and independent fuzzy subsystems, uniformly distributed and coupled in a similar manner over Γ_j.
(H2) The surface Γ_j can be discretized into coupling subsurfaces S_k, i.e. $\Gamma_j = \bigcup_k S_k$, of identical area S, such that, on each coupling subsurface S_k, the master structure is coupled with one single fuzzy subsystem (fuzzy subsystem k).
(H3) The displacement $\mathbf{u}$ is constant on each subsurface S_k.
(H4) Relatively to each subsurface S_k, each fuzzy subsystem k is excited in a same direction.
(H5) The surface force $\mathbf{f}$ and the surface displacement $\mathbf{u}$ are collinear at each coupling point.

A structural fuzzy with discrete attachment, which verifies assumptions (H1)–(H5), is illustrated in Fig. 3.

Assumptions (H1)–(H4) imply that the fuzzy is homogenous on Γ_j "relatively" to subsurfaces S_k, and assumptions (H1), (H2) and (H5) imply that the fuzzy is orthotropic on Γ_j relatively to subsurfaces S_k. From assumptions (H1)–(H5), the action of the fuzzy on the master structure is then easily modeled as

$$\mathbf{f}_k(\omega) = \mathrm{i}\omega Z(\omega)\mathbf{u}_k(\omega) \quad \text{on } \Gamma_j \ (k = 1, \ldots, N), \tag{2}$$

where $\mathbf{f}_k$ represents the resulting mean surface force applied on the fuzzy subsystem k, $\mathbf{u}_k$ represents the motion of the coupling subsurface S_k, N is the number of fuzzy subsystems on Γ_j. In the following, the fuzzy is referred to as homogeneous and orthotropic on Γ_j, which implicitly assumes that assumptions (H1)–(H5) are verified.

Finally, the dynamics of the master structure (defined by the domain Ω) is obtained by solving the classical linear elasticity equations. The response of the master structure coupled to a locally homogeneous and orthotropic fuzzy, with discrete boundary, is found by the principle of virtual works of the external forces applied to the master structure and the coupling forces over the

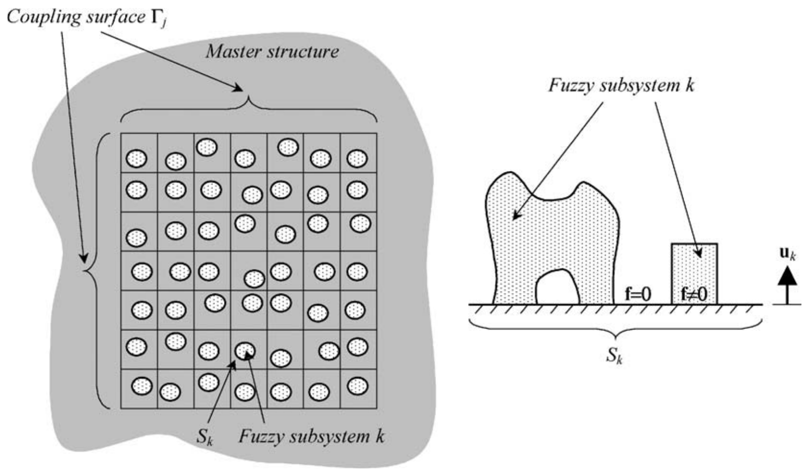

Fig. 3. Illustration of a structural fuzzy with discrete attachment, which is composed of identical and independent fuzzy subsystems over the coupling surface Γ_j.

coupling surface. The principle of virtual works for a virtual displacement $\boldsymbol{\delta}\mathbf{u}$ results in

$$\begin{aligned}&\int_{\Omega}(-\omega^2\rho(\mathbf{x})\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)+\boldsymbol{\varepsilon}^{\mathrm{T}}(\boldsymbol{\delta}\mathbf{u})\mathbf{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathrm{d}\mathbf{x}\\&\quad=\int_{\Gamma}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet(-\mathbf{f}(\mathbf{x},\omega))\,\mathrm{d}s(\mathbf{x})+\int_{\partial\Omega\backslash\Gamma}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{f}_{\mathrm{ex}}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x})\\&\quad+\sum_{r}\boldsymbol{\delta}\mathbf{u}(\mathbf{x}_r,\omega)\bullet\mathbf{F}_{\mathrm{ex}}(\mathbf{x}_r,\omega)+\sum_{s}(\nabla\times\boldsymbol{\delta}\mathbf{u}(\mathbf{x}_s,\omega))\bullet\mathbf{M}_{\mathrm{ex}}(\mathbf{x}_s,\omega).\end{aligned}\tag{3}$$

Here, $\partial\Omega$ is the boundary of the domain Ω, ρ is the density, $\boldsymbol{\varepsilon}$ and $\mathbf{C}$ are the strain tensor and the tensor of the elastic constants, respectively, $\mathbf{f}_{\mathrm{ex}}$ represents a distributed external force, $\mathbf{F}_{\mathrm{ex}}$ and $\mathbf{M}_{\mathrm{ex}}$ are external point forces and point moments (see Fig. 1), ds and d$\mathbf{x}$ are the surface area element and the volume element, respectively. Using Eq. (2), Eq. (3) becomes

$$\begin{aligned}&\int_{\Omega}(-\omega^2\rho(\mathbf{x})\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)+\boldsymbol{\varepsilon}^{\mathrm{T}}(\boldsymbol{\delta}\mathbf{u})\mathbf{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathrm{d}\mathbf{x}\\&\quad=-\sum_{j}\mathrm{i}\omega Z_j(\omega)\int_{\Gamma_j}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x})+\int_{\partial\Omega\backslash\Gamma}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{f}_{\mathrm{ex}}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x})\\&\quad+\sum_{r}\boldsymbol{\delta}\mathbf{u}(\mathbf{x}_r,\omega)\bullet\mathbf{F}_{\mathrm{ex}}(\mathbf{x}_r,\omega)+\sum_{s}(\nabla\times\boldsymbol{\delta}\mathbf{u}(\mathbf{x}_s,\omega))\bullet\mathbf{M}_{\mathrm{ex}}(\mathbf{x}_s,\omega).\end{aligned}\tag{4}$$

The subscript j has been introduced to emphasize the fact that the boundary impedance can vary between coupling surfaces Γ_j.

3. Deterministic formulation of the boundary impedance of the structural fuzzy

In this section, a new deterministic formulation of the boundary impedance of the structural fuzzy is proposed for a continuous, elastic fuzzy. The proposed formulation is not constructed from the model of a linear oscillator excited by its support, as suggested by Soize in the context of type I fuzzy law, but is rather the dynamic equilibrium of a continuous elastic fuzzy. The formulation however remains subject to assumptions (H1)–(H5) in Section 2.

We consider a coupling surface Γ_j on which the fuzzy is homogeneous and orthotropic: according to assumption (H2), the coupling surface Γ_j can be discretized into subsurfaces S_k ($k=1,\ldots,N$) of identical area S, such that, on each coupling subsurface S_k, the master structure is coupled with one single fuzzy subsystem (fuzzy subsystem k). At the frequency $\omega/2\pi$, the principle of virtual works applied to a given subsystem k, coupled on Γ_j, through a virtual displacement $\boldsymbol{\delta}\mathbf{u}$ results in

$$\begin{aligned}&\int_{V_k}(-\omega^2\rho(\mathbf{x})\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)+\boldsymbol{\varepsilon}^{\mathrm{T}}(\boldsymbol{\delta}\mathbf{u})\mathbf{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathrm{d}\mathbf{x}\\&\quad=\int_{S_k}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{f}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x}),\end{aligned}\tag{5}$$

which is associated with the following homogeneous boundary condition:

$$\boldsymbol{\varepsilon}(\mathbf{u}_k) = \mathbf{0} \quad \text{in } V_k. \tag{6}$$

Here, V_k is the domain occupied by the subsystem k and $\mathbf{u}_k$ is the rigid-body motion ($\mathbf{u}_k$ represents the constant displacement of the subsurface S_k). Finally, $\mathbf{f}$ is the surface force applied to the fuzzy on S_k. We recall that the function $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \omega)$ is defined at any point of S_k and at $\mathbf{x} \in S_k$, $-\mathbf{f}(\mathbf{x},\omega)$ is the surface force applied by the fuzzy to the master structure (Fig. 3).

By choosing $\boldsymbol{\delta}\mathbf{u}(\mathbf{x}, \omega) = \mathbf{u}_k(\omega)\, \forall \mathbf{x} \in V_k$, Eq. (5) becomes

$$\int_{S_k} \mathbf{u}_k(\omega) \bullet \mathbf{f}(\mathbf{x}, \omega)\, \mathrm{d}s(\mathbf{x}) = -\omega^2 \int_{V_k} \rho(\mathbf{x}) \mathbf{u}_k(\omega) \bullet \mathbf{u}(\mathbf{x}, \omega) \mathrm{d}\mathbf{x}, \tag{7}$$

which reduces to

$$\mathbf{u}_k(\omega) \bullet \mathbf{f}_k(\omega) S = -\omega^2 \int_{V_k} \rho(\mathbf{x}) \mathbf{u}_k(\omega) \bullet \mathbf{u}(\mathbf{x}, \omega)\, \mathrm{d}\mathbf{x}, \tag{8}$$

where S is the area of S_k, $\mathbf{f}_k$ is the resulting mean surface force applied to the fuzzy subsystem k on S_k (we recall that according to (H5), the vectors $\mathbf{f}(\mathbf{x},\omega)$ and $\mathbf{u}_k(\omega)$ are collinear at any point $\mathbf{x} \in S_k$):

$$\mathbf{f}_k(\omega) = \frac{\mathbf{u}_k(\omega)}{\left|\left|\mathbf{u}_k(\omega)\right|\right|^2 S} \int_{S_k} \mathbf{u}_k(\omega) \bullet \mathbf{f}(\mathbf{x}, \omega)\, \mathrm{d}s(\mathbf{x}), \tag{9}$$

where $\|\mathbf{u}_{k(\omega)\|^2 = \mathbf{u}k}(\omega) \bullet \mathbf{u}_k(\omega)$.

Eq. (5) is expanded over the vibration modes of the fuzzy subsystem k clamped on S_k.

Since the displacement $\mathbf{u}_k$ is constant on S_k (assumption (H3), which means that the quasi-static displacement field [13], i.e. the static response of the system to the displacement of the subsurface S_k, is supposed to be constant in the domain V_k), the displacement $\mathbf{u}$ of the subsystem k can be approximately expressed from the displacement $\mathbf{u}^*$ of the subsystem k clamped on S_k ($\mathbf{u}^*(\mathbf{x},\omega) = \mathbf{0}$ on S_k):

$$\mathbf{u}(\mathbf{x}, \omega) = \mathbf{u}_k(\omega) + \mathbf{u}^*(\mathbf{x}, \omega) \quad \text{in } V_k. \tag{10}$$

The displacement $\mathbf{u}^*$ can be expanded over eigenfunctions $\{X_p\}_{p \geqslant 1}$ associated with the clamped subsystem S_k ($\mathbf{X}_p(\mathbf{x}) = 0$ on S_k, $\forall p$),

$$\mathbf{u}^*(\mathbf{x}, \omega) = \sum_{p=1}^{\infty} \phi_p(\omega) \mathbf{X}_p(\mathbf{x}) \quad \text{in } V_k, \tag{11}$$

where $\{\phi_p\}$ are the complex modal displacements. By choosing a virtual displacement $\boldsymbol{\delta}\mathbf{u}(\mathbf{x}, \omega) = \mathbf{X}_p(\mathbf{x})\, (\mathbf{x} \in V_k)$ in Eq. (5) and using orthogonality properties of the modes $\mathbf{X}_p(\mathbf{x})$, the dynamic response of the subsystem k for a specific mode p is given by

$$\begin{aligned} &\phi_p(\omega) \int_{V_k} (-\omega^2 \rho(\mathbf{x}) \left|\left|\mathbf{X}_p(\mathbf{x})\right|\right|^2 + \boldsymbol{\varepsilon}^{\mathrm{T}}(\mathbf{X}_p) \mathbf{C}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{X}_p))\, \mathrm{d}\mathbf{x} \\ &\quad = \omega^2 \int_{V_k} \rho(\mathbf{x}) \mathbf{X}_p(\mathbf{x}) \bullet \mathbf{u}_k(\omega)\, \mathrm{d}\mathbf{x}. \end{aligned} \tag{12}$$

We introduce the modal mass M_p and the modal stiffness K_p:

$$M_p = \int_{V_k} \rho(\mathbf{x})\left|\left|\mathbf{X}_p(\mathbf{x})\right|\right|^2 \mathrm{d}\mathbf{x} \quad \forall k \in \{1, \ldots, N\}, \tag{13}$$

$$K_p = \int_{V_k} \boldsymbol{\varepsilon}^{\mathrm{T}}(\mathbf{X}_p)\mathbf{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{X}_p)\,\mathrm{d}\mathbf{x} \quad \forall k \in \{1, \ldots, N\}, \tag{14}$$

which are independent on the considered subsystem according to assumption (H1). Eq. (12) therefore takes the form

$$-\omega^2 M_p(1 - (\Omega_p/\omega)^2(1 + \mathrm{i}\eta_p))\phi_p(\omega) = \omega^2 \int_{V_k} \rho(\mathbf{x})\mathbf{X}_p(\mathbf{x})\bullet\mathbf{u}_k(\omega)\,\mathrm{d}\mathbf{x}, \tag{15}$$

where Ω_p is the angular natural frequency of the mode p, $\Omega_p^2 = K_p/M_p$. In Eq. (15), dissipation phenomena are taken into account via a modal structural damping η_p. Finally, from Eqs. (10), (11) and (15), the right-hand side term of Eq. (8) takes the following form:

$$\begin{aligned}\mathbf{u}_k(\omega)\bullet\mathbf{f}_k(\omega)S = & -\omega^2 \int_{V_k} \rho(\mathbf{x})\left|\left|\mathbf{u}_k(\omega)\right|\right|^2 \mathrm{d}\mathbf{x} \\ & - \sum_{p=1}^{\infty} \frac{\left(\omega^2 \int_{V_k} \rho(\mathbf{x})\mathbf{u}_k(\omega)\bullet\mathbf{X}_p(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^2}{-\omega^2 M_p(1 - (\Omega_p/\omega)^2(1 + \mathrm{i}\eta_p))}.\end{aligned} \tag{16}$$

The boundary impedance Z_k of the fuzzy subsystem k is defined by

$$\mathbf{f}_k(\omega) = \mathrm{i}\omega Z_k(\omega)\mathbf{u}_k(\omega). \tag{17}$$

By introducing the rigid-body mode $(\mathbf{X}_0)_k$, such that $(\mathbf{X}_0)_k(\omega) = \mathbf{u}_k(\omega)/\left|\left|\mathbf{u}_k(\omega)\right|\right|$, Eq. (16) takes the form:

$$\begin{aligned}\mathrm{i}\omega Z_k(\omega) = & -\frac{\omega^2}{S}\int_{V_k} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ & - \frac{1}{S}\sum_{p=1}^{\infty} \frac{\left(\omega^2 \int_{V_k} \rho(\mathbf{x})(\mathbf{X}_0)_k(\omega)\bullet\mathbf{X}_p(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^2}{-\omega^2 M_p(1 - (\Omega_p/\omega)^2(1 + \mathrm{i}\eta_p))}.\end{aligned} \tag{18}$$

In Eq. (18), the boundary impedance of the fuzzy subsystem k is the sum of a static component (defined from the mass per unit area of the subsystem), and a dynamic component.

An extension of Eq. (18) from the coupling surface S_k to the coupling surface Γ_j requires that $Z_k(\omega) = Z(\omega)$ for all S_k on Γ_j. This is clearly verified since the first term of the right hand side of Eq. (18) is defined from the mass per unit area of the fuzzy subsystem k, which is independent on the considered fuzzy subsystem k (assumption (H1)). Moreover, the term $(\mathbf{X}_0)_k(\omega)\bullet\mathbf{X}_p(\mathbf{x})\,(\mathbf{x} \in V_k)$, representing the projection of mode p on the rigid-body mode, is also independent on the considered coupling subsurface S_k according to assumption (H4).

We define the mass per unit area M_{p0} from the projection of mode p of any fuzzy subsystem k on the rigid-body mode,

$$M_{p0}(\omega) = \int_{V_k} \rho(\mathbf{x})(\mathbf{X}_0)_k(\omega) \bullet \mathbf{X}_p(\mathbf{x})\,\mathrm{d}\mathbf{x} \quad \forall k \in \{1, \ldots, N\}, \tag{19}$$

and we also define the resonant mass per unit area μ_p of the mode p,

$$\mu_p(\omega) = \frac{\alpha_p(\omega) M_{p0}(\omega)}{S}, \tag{20}$$

where $\alpha_p(\omega) = M_{p0}(\omega)/M_p$.

Eq. (18) is written for the surface Γ_j in the final form

$$\mathrm{i}\omega Z(\omega) = -\omega^2 \mu_0 + \omega^2 \sum_{p=1}^{\infty} \frac{\mu_p(\omega)}{1 - (\Omega_p/\omega)^2(1 + \mathrm{i}\eta_p)} \quad \text{on } \Gamma_j, \tag{21}$$

or,

$$\mathrm{i}\omega Z(\omega) = -\omega^2 \mu_0 + \sum_{p=1}^{\infty} (-\omega^2 R_p(\omega) + \mathrm{i}\omega I_p(\omega)) \quad \text{on } \Gamma_j, \tag{22}$$

where R_p represents the apparent mass per unit area of the fuzzy on Γ_j due to mode p:

$$R_p(\omega) = \frac{\mu_p(\omega)((\Omega_p/\omega)^2 - 1)}{((\Omega_p/\omega)^2 - 1)^2 + \eta_p^2(\Omega_p/\omega)^4}, \tag{23}$$

I_p represents the apparent damping per unit area of the fuzzy on Γ_j due to mode p:

$$I_p(\omega) = \frac{\omega \mu_p(\omega)(\Omega_p/\omega)^2 \eta_p}{((\Omega_p/\omega)^2 - 1)^2 + \eta_p^2(\Omega_p/\omega)^4}, \tag{24}$$

and μ_0 is the mass per unit area of the fuzzy on Γ_j,

$$\mu_0 = \frac{1}{S} \int_{V_k} \rho(\mathbf{x})\mathrm{d}\mathbf{x} \quad \text{on } \Gamma_j \quad \forall k \in \{1, \ldots, N\}, \tag{25}$$

Eq. (21) suggests that, for a given structural fuzzy, there exists a cutoff frequency $\Omega_c/2\pi$, such that for $\omega < \Omega_c$, the dynamic component of the boundary impedance of the structural fuzzy on Γ_j can be neglected compared to the static component, $\mathrm{i}\omega Z(\omega) \approx -\omega^2 \mu_0$. In other words, for $\omega < \Omega_c$, the action of the fuzzy on the master structure on Γ_j results in an added-mass effect. This concept was observed experimentally by Soize [1,2]. However, the boundary impedance of a fuzzy, with discrete attachment, composed of elastic continuous fuzzy subsystems, leads to a difference with the deterministic solution proposed by Soize (case of the type I fuzzy law) for a single linear oscillator excited by its support [1,2] (Appendix).

This section has presented a formulation of the boundary impedance of a locally homogeneous and orthotropic structural fuzzy with discrete boundary. The formulation is deterministic as it is based on the knowledge of the modal parameters of any fuzzy subsystem k coupled on a specific coupling surface Γ_j: resonant masses per unit area $\{\mu_p\}$, angular natural frequencies $\{\Omega_p\}$ and modal structural damping $\{\eta_p\}$. However, an exact determination of these modal parameters remains impossible due to the complexity of the fuzzy. It is necessary to take uncertainties into account in the description of the parameters of the mechanical system. The next section presents a probabilistic model of the boundary impedance of a simple locally homogeneous and orthotropic fuzzy which is composed of elastic bars.

4. Probabilistic model of the boundary impedance of a structural fuzzy composed of elastic bars

In this section, we consider a special situation of a structural fuzzy composed of elastic bars whose geometrical parameters, lengths and cross-sectional areas, are imprecisely known and are randomly defined. We show that a probabilistic model of the boundary impedance of this fuzzy can be easily derived from the deterministic formulation of the boundary impedance presented in the previous section. This approach differs from previous work based on a fuzzy composed of linear oscillators [9–12].

Let us consider a specific coupling surface Γ_j on which the fuzzy is homogeneous and orthotropic (Section 2): on Γ_j, the fuzzy is composed of N identical and independent fuzzy subsystems. On Γ_j, it is supposed that each fuzzy subsystem k is composed of M clamped–free elastic bars perpendicular to Γ_j and that it is excited by a support displacement $\mathbf{u}_k$ constant over S_k and perpendicular to Γ_j (Fig. 4). For the sake of simplicity, we assume that the density, the modulus of elasticity and the modal structural damping are constants over V_k : $\rho(\mathbf{x}) = \rho$ and $E_0(\mathbf{x}) = E_0 \, \forall \mathbf{x} \in V_k \, (k = 1, \ldots, N)$; $\eta_p = \eta \, \forall p$.

For a particular fuzzy subsystem k coupled on Γ_j , the eigenfunction $\mathbf{X}_p$ of a mode p of this fuzzy subsystem is expressed from the modal solution of a clamped-free bar of length Λ_p and cross-sectional area Σ_p [14],

$$X_p(x) = \sin\left(\frac{\Omega_p x}{c}\right), \quad x \in [0, \Lambda_p] \quad \text{and} \quad \frac{\Omega_p \Lambda_p}{c} = (q - 1/2)\pi. \tag{26}$$

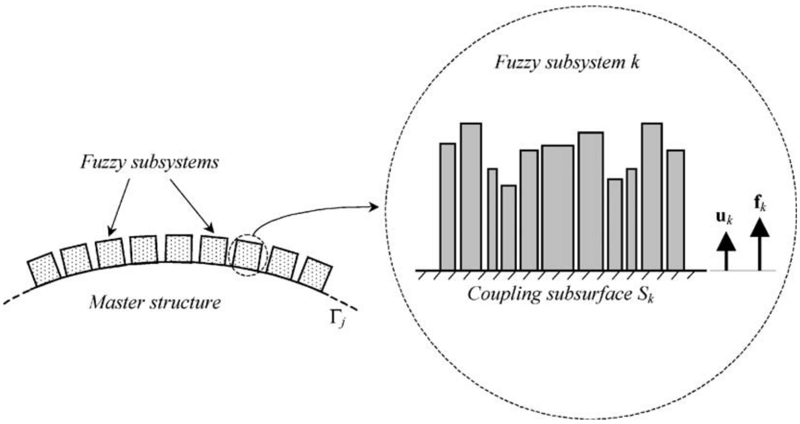

Fig. 4. Illustration of a master structure coupled with a structural fuzzy, homogeneous and orthotropic on a coupling surface Γ_j, composed of elastic bars described with random geometrical parameters.

In this equation, the subscript q designates a longitudinal mode of a specific individual bar, whereas the subscript p designates a mode of the whole subsystem k ($q \leqslant p$), c is the phase velocity, $c^2 = E_0/\rho$. We emphasize the fact that both the length Λ_p and the cross-sectional area Σ_p depend on the mode p, which simply means that two modes p are in general associated to two different bars of the subsystem k. From Eqs. (20) and (26), the resonant mass per unit area μ_p of the mode p is given by

$$\mu_p = \frac{2\rho\Sigma_p c^2}{\Omega_p^2 \Lambda_p S}. \tag{27}$$

The length Λ_p and the cross-sectional area Σ_p of the bar, associated with the mode p, are randomly defined from known mean values, $\underline{\Lambda}$ and $\underline{\Sigma}$, respectively:

$$\Lambda_p = \underline{\Lambda}\left(1 + \frac{\lambda_1}{\sqrt{3}} Y_1\right), \tag{28}$$

$$\Sigma_p = \underline{\Sigma}\left(1 + \frac{\lambda_2}{\sqrt{3}} Y_2\right). \tag{29}$$

In these equations, Y_1 and Y_2 are two continuous independent normalized random variables [1–2,15], associated with probability laws $P(\mathrm{d}y_k)\,(k = 1, 2)$,

$$P(\mathrm{d}y_k) = p(y_k)\,\mathrm{d}y_k, \tag{30}$$

where p is a probability density,

$$p(y_k) = \frac{1}{2\sqrt{3}} \mathbf{1}_{[-\sqrt{3},\sqrt{3}]}(y_k). \tag{31}$$

In this equation $\mathbf{1}_H(y) = 1$ if $y \in H$ and $\mathbf{1}_H(y) = 0$ if $y \notin H$. To adjust the variance of the geometrical parameters, we have introduced in Eqs. (28) and (29) two dispersion parameters, λ_1 and λ_2: $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$. The mathematical expectation (or mean value) $\underline{h}$ of a function h of n continuous independent normalized random variables Y_k is defined by [15]

$$\underline{h} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(y_1, y_2, \ldots, y_n) p(y_1) p(y_2) \ldots p(y_n)\,\mathrm{d}y_1\,\mathrm{d}y_2 \ldots \mathrm{d}y_n. \tag{32}$$

Thus, the mathematical expectation $\underline{\mu}_p$ of the resonant mass per unit area of the mode p results from Eqs. (27)–(31):

$$\underline{\mu}_p = \frac{\rho\,\underline{\Sigma}\,c^2}{\Omega_p^2\,\underline{\Lambda}\,S\lambda_1} \ln\left\{\frac{1 + \lambda_1}{1 - \lambda_1}\right\}. \tag{33}$$

The mathematical expectation $\underline{Z}$ of the boundary impedance of the fuzzy on Γ_j is then obtained from Eqs. (21)–(24) by replacing $\mu_p(\omega)$ by $\underline{\mu}_p$. In the following, the expression of $\underline{Z}$ is further simplified.

Let us consider a natural angular frequency Ω of the fuzzy subsystem k and let us consider the frequency band $[\Omega - \Delta\Omega/2, \Omega + \Delta\Omega/2]$. In this frequency band, there are $n\Delta\Omega$ modes, where n represents the modal density of subsystem k coupled on Γ_j. Assuming that subsystem k is composed of M identical bars of length $\underline{\Lambda}$,

$$n = \frac{M\underline{\Lambda}}{\pi c} \quad \text{on } \Gamma_j. \tag{34}$$

Thus, in the limit $\Delta\Omega \to 0$ and $n \to \infty$, the mathematical expectation $\underline{Z}$ of the boundary impedance of the fuzzy on Γ_j, Eqs. (21)–(24) takes the form of a Riemann integral,

$$\mathrm{i}\omega\,\underline{Z}(\omega) = -\omega^2\mu_0 + \omega^2 \sum_{j=0}^{\infty} \frac{n\underline{\mu}_{\Omega_1+j\Delta\Omega}}{1-((\Omega_1+j\Delta\Omega)/\omega)^2(1+\mathrm{i}\eta)}\Delta\Omega \quad \text{on } \Gamma_j, \tag{35}$$

or,

$$\mathrm{i}\omega\,\underline{Z}(\omega) = -\omega^2(\mu_0 + \underline{R}(\omega)) + \mathrm{i}\omega\,\underline{I}(\omega) \quad \text{on } \Gamma_j, \tag{36}$$

where $\mu_0 + \underline{R}$ represents the mean apparent mass per unit area of the fuzzy on Γ_j, expressed from the dynamic mass $\underline{R}$:

$$\underline{R}(\omega) = \frac{M\rho\,\underline{\Sigma}\,c}{\pi S\lambda_1}\ln\left\{\frac{1+\lambda_1}{1-\lambda_1}\right\}\int_{\Omega_1}^{\infty} \frac{1/\omega^2 - 1/\Omega^2}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4}\,\mathrm{d}\Omega, \tag{37}$$

and $\underline{I}$ represents the mean apparent damping per unit area of the fuzzy on Γ_j:

$$\underline{I}(\omega) = \frac{M\rho\,\underline{\Sigma}\,c}{\pi S\lambda_1}\ln\left\{\frac{1+\lambda_1}{1-\lambda_1}\right\}\int_{\Omega_1}^{\infty} \frac{\eta/\omega}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4}\,\mathrm{d}\Omega. \tag{38}$$

In Eqs. (37) and (38), Ω_1 represents the angular frequency of the fundamental mode of subsystem k coupled on Γ_j: it is expressed approximately from the case of M identical bars of length $\underline{\Lambda}$,

$$\Omega_1 \approx \frac{\pi c}{2\,\underline{\Lambda}(1+\lambda_1)}, \tag{39}$$

and μ_0 is the mass per unit area of the fuzzy on Γ_j, expressed approximately from the same case,

$$\mu_0 \approx \frac{M\rho\,\underline{\Lambda}\,\underline{\Sigma}}{S} \quad \text{on } \Gamma_j. \tag{40}$$

Finally, the dynamic response of the master structure coupled with the fuzzy is derived from Eq. (4) by replacing, on each coupling surface Γ_j, the boundary impedance of the fuzzy by its mathematical expectation, that is,

$$Z_j(\omega) \approx \underline{Z}_j(\omega) \quad \forall j. \tag{41}$$

This assumption is verified if, on each coupling surface Γ_j, each fuzzy subsystem is composed of a large number of different bars, the geometrical parameters of these bars being evaluated from Eqs. (28) and (29): in other words, it is supposed that the resonant mass per unit area of the modes contained in the frequency band $[\Omega - \Delta\Omega/2, \Omega + \Delta\Omega/2]$ is approximately $n\underline{\mu}_{\Omega}\Delta\Omega$.

Therefore, Eq. (4) reduces to

$$\int_{\Omega}(-\omega^2\rho(\mathbf{x})\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)+\boldsymbol{\varepsilon}^{\mathrm{T}}(\boldsymbol{\delta}\mathbf{u})\mathbf{C}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathrm{d}\mathbf{x}$$
$$= -\sum_j \mathrm{i}\omega\underline{Z}_j(\omega)\int_{\Gamma_j}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{u}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x}) + \int_{\partial\Omega\backslash\Gamma}\boldsymbol{\delta}\mathbf{u}(\mathbf{x},\omega)\bullet\mathbf{f}_{\mathrm{ex}}(\mathbf{x},\omega)\,\mathrm{d}s(\mathbf{x})$$
$$+\sum_r \boldsymbol{\delta}\mathbf{u}(\mathbf{x}_r,\omega)\bullet\mathbf{F}_{\mathrm{ex}}(\mathbf{x}_r,\omega)+\sum_s(\nabla\times\boldsymbol{\delta}\mathbf{u}(\mathbf{x}_s,\omega))\bullet\mathbf{M}_{\mathrm{ex}}(\mathbf{x}_s,\omega). \tag{42}$$

5. Numerical application

5.1. Master structure coupled with 1 fuzzy substructure

The probabilistic model of the boundary impedance of a locally homogeneous and orthotropic fuzzy composed of elastic bars, presented in Section 4, is numerically applied to the case illustrated in Fig. 5: we consider the vibrations of a simply supported Euler–Bernoulli beam (master structure) uniformly coupled over its length with a homogeneous and orthotropic fuzzy composed of $N = 12$ identical fuzzy subsystems. Each fuzzy subsystem is composed of $M = 60$ elastic clamped–free bars whose lengths and cross-sectional areas are randomly defined from Eqs. (28) and (29).

The characteristics of the beam are: density $\rho' = 7800\,\mathrm{kg/m^3}$, width $l = 0.1\,\mathrm{m}$, cross-sectional area $l^2 = 10^{-2}\,\mathrm{m}^2$, length $L = 10\,\mathrm{m}$, bending stiffness $E_0'I = 1.75\times10^6\,\mathrm{N\,m^2}$, structural damping $\eta' = 5\times10^{-3}$. The structural fuzzy is homogenous and orthotropic on the coupling surface Γ of area $|\Gamma| = l\times L$, relatively to coupling subsurfaces S_k of identical area $S = |\Gamma|/N$ (see Section 2). The length and the cross-sectional area of the bars which form each fuzzy subsystem are randomly defined from the following mean values, $\underline{\Lambda} = 1.5\,\mathrm{m}$ and $\underline{\Sigma} = 2\times10^{-6}\,\mathrm{m}^2$, associated, respectively, with the following dispersion parameters: $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$.

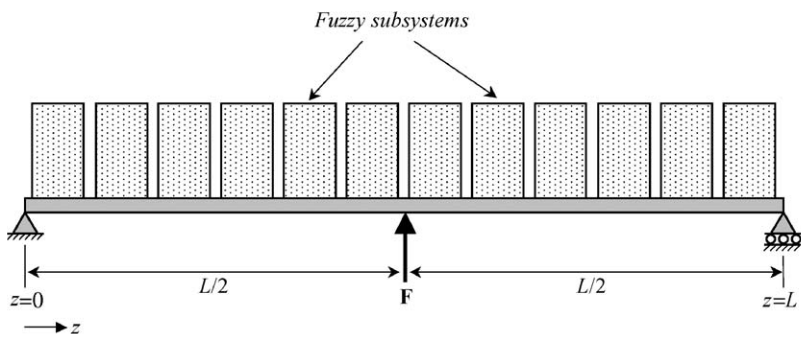

Fig. 5. Euler–Bernoulli beam (master structure) coupled over its length with a homogeneous structural fuzzy.

The bars have the following material characteristics: density $\rho = 31{,}400\,\mathrm{kg/m^3}$, Young's modulus $E_0 = 2.1 \times 10^{11}\,\mathrm{Pa}$, structural damping $\eta = 5 \times 10^{-2}$. The other characteristics of the structural fuzzy are: modal density $n \approx 10^{-2}(\mathrm{rad/s})^{-1}$; fundamental frequency according to Eq. (39), $\Omega_1/2\pi \approx 269\,\mathrm{Hz}$; mass per unit area according to Eq. (40), $\mu_0 \approx 68\,\mathrm{kg/m^2}$. The ratio between the mass of the structural fuzzy and the mass of the master structure is $\mu_0/(\rho' l) \approx 9\%$. The master structure is excited at its center ($z = L/2$) by a harmonic force $\mathbf{F}$ of modulus $\|\mathbf{F}\| = 1000\,\mathrm{N}$ on the frequency range [100 Hz, 1200 Hz]. At 1200 Hz, the uncoupled beam contains almost 11 flexural wavelengths.

The exact response of the uncoupled beam is numerically compared with the exact response of the beam coupled with the fuzzy and the response as obtained from the proposed probabilistic model. The exact response of the coupled beam is numerically calculated by solving the equation of motion of the homogeneous uncoupled beam [14] associated with the boundary conditions at the coupling points (transverse displacement and shear force compatibility). The response of the coupled beam obtained from the probabilistic model is numerically evaluated by solving the equation of motion resulting from the principle of virtual works, Eq. (42),

$$\frac{\partial^4 u(z,\omega)}{\partial z^4} - \left(\frac{\omega^2 \rho' l^2 - \mathrm{i}\omega\,\underline{Z}(\omega) l}{E' I}\right) u(z,\omega) = 0 \quad z \in]0, L/2[\cup]L/2, L[. \tag{43}$$

In Eq. (43), $E' = E'_0(1 + \mathrm{i}\eta')$ represents the complex modulus of elasticity of the coupled beam.

The mean apparent mass per unit area $\mu_0 + \underline{R}$ and the mean apparent damping per unit area I of the fuzzy on Γ have been computed according to Eqs. (37) and (38) on the frequency range [100 Hz, 1200 Hz] from numerical integration over an interval $[\Omega_1/2\pi, 2400\,\mathrm{Hz}]$ (it was verified that the modes of the fuzzy above 2400 Hz do not modify the computed response below 1200 Hz). The functions $\omega \mapsto \mu_0 + \underline{R}(\omega)$ and $\omega \mapsto \underline{I}(\omega)$ are plotted in Fig. 6.

The function $\omega \mapsto \mu_0 + \underline{R}(\omega)$ has its maximum at $\Omega_1/2\pi$. Moreover, when $\omega \to \infty$, the mass introduced by the fuzzy becomes negligible compared to the mass of the beam. For $\omega < \Omega_1/2\pi$, $\underline{I}(\omega) \approx 0$ and for $\omega > \Omega_1/2\pi$, $\underline{I}(\omega)$ rapidly reaches an asymptotic value. Thus, below the fundamental mode of the continuous fuzzy, the fuzzy essentially acts as an added mass, and

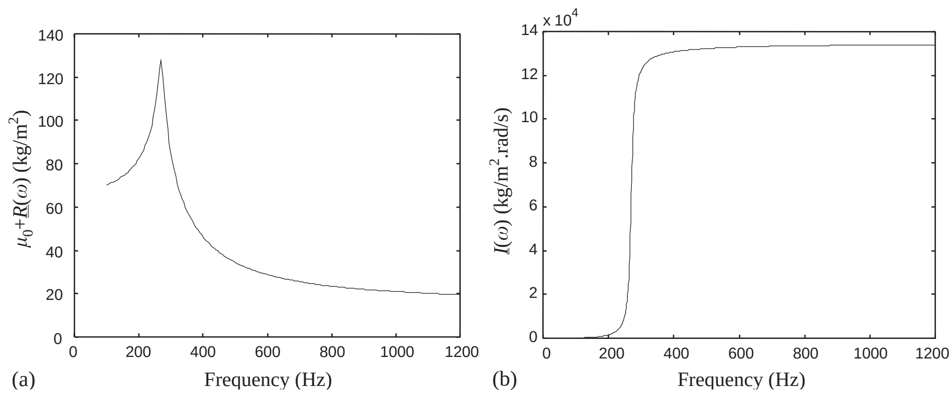

Fig. 6. (a) Mean apparent mass per unit area of the fuzzy on Γ and (b) mean apparent damping per unit area of the fuzzy on Γ.

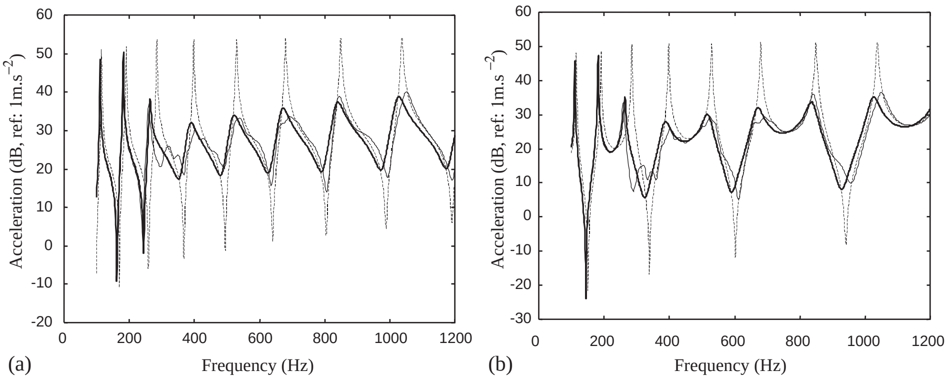

Fig. 7. Exact response of the uncoupled beam (- - - - - -), exact response of the beam coupled with the structural fuzzy (——) and predicted response of the beam coupled with the structural fuzzy (——) at (a) $z = L/2$ (excitation point) and at (b) $z = L/4$.

above the fundamental mode, the action of the fuzzy on the master structure results essentially in a dissipative effect.

The exact response of the uncoupled beam, the exact response of the beam coupled with the fuzzy and the response of the beam coupled with the fuzzy obtained from Eq. (43) are compared in Fig. 7.

The theoretical prediction of the response of the coupled beam, provided by Eq. (43), is almost similar to the exact response provided by the numerical simulation. These results clearly validate the probabilistic model proposed in this paper. Above the fundamental frequency $\Omega_1/2\pi$ of the fuzzy, the response of the coupled beam is considerably damped compared to the response of the uncoupled beam and below that frequency the mass loading is the dominant mechanism. It can be noticed that in this situation, assumption (H3) of a constant displacement over a given subsurface S_k may not be valid at higher frequency since at 1200 Hz, the uncoupled beam contains approximately 11 wavelengths whereas only $N = 12$ fuzzy subsystems are considered.

5.2. Master structure coupled with 2 distinct fuzzy substructures

In a second numerical simulation, we consider the vibrations of a simply supported Euler–Bernoulli beam (master structure) coupled with a locally homogeneous fuzzy composed of two homogeneous fuzzy substructures (fuzzy substructures 1 and 2), associated with two different fuzzy laws. This coupled structure is described in Fig. 8.

The characteristics of the beam and the material characteristics of the bars are similar to those in the previous section. The fuzzy substructure 1 is composed of $N_1 = 20$ identical fuzzy subsystems; each fuzzy subsystem contains $M_1 = 15$ elastic bars. The fuzzy substructure 1 is homogenous and orthotropic on the coupling surface Γ_1, $|\Gamma_1| = |\Gamma|/2$ (where the surface Γ has been defined in the previous section) relatively to the coupling subsurfaces S_k of identical area $S = |\Gamma_1|/N_1$. The length and the cross-sectional area of the bars which form each fuzzy subsystem are randomly defined from the following mean values, $\underline{\Lambda}_1 = 2\,\mathrm{m}$ and $\underline{\Sigma}_1 = 2 \times 10^{-6}\,\mathrm{m}^2$, associated, respectively, with dispersion parameters $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$. The other characteristics of the

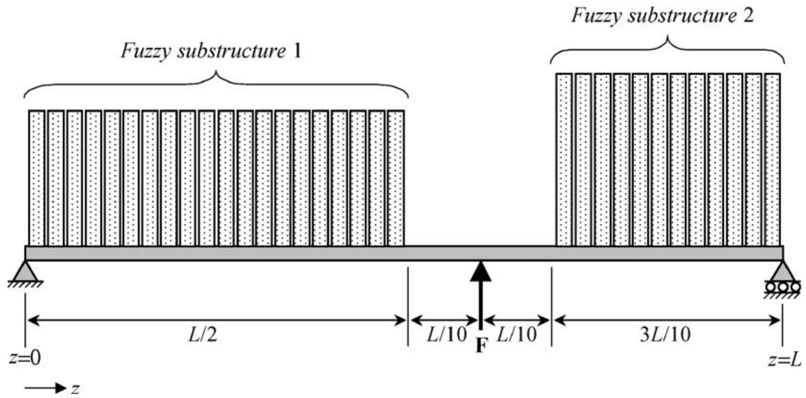

Fig. 8. Euler–Bernoulli beam (master structure) coupled with two homogeneous fuzzy substructures.

fuzzy substructure 1 are: modal density $n_1 \approx 4 \times 10^{-3}(\mathrm{rad/s})^{-1}$; frequency of the fundamental mode according to Eq. (39), $(\Omega_1)_1/2\pi \approx 202\,\mathrm{Hz}$; mass per unit area according to Eq. (40), $(\mu_0)_1 \approx 75\,\mathrm{kg/m^2}$.

The fuzzy substructure 2 is composed of $N_2 = 12$ identical fuzzy subsystems; each fuzzy subsystem contains $M_2 = 15$ elastic bars. The fuzzy substructure 2 is homogenous and orthotropic on the coupling surface Γ_2, $|\Gamma_2| = 3|\Gamma|/10$ relatively to the coupling subsurfaces S_k of identical area $S = |\Gamma_2|/N_2 = |\Gamma_1|/N_1$. The length and the cross-sectional area of the bars which form each fuzzy subsystem are randomly defined from the following mean values, $\underline{\Lambda}_2 = 2.5\,\mathrm{m}$ and $\underline{\Sigma}_2 = 3 \times 10^{-6}\,\mathrm{m^2}$, associated, respectively, with dispersion parameters $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$. The other characteristics of the fuzzy substructure 2 are: the modal density $n_2 \approx 5 \times 10^{-3}(\mathrm{rad/s})^{-1}$, frequency of the fundamental mode according to Eq. (39), $(\Omega_1)_2/2\pi \approx 162\,\mathrm{Hz}$, mass per unit area according to Eq. (40), $(\mu_0)_2 \approx 141\,\mathrm{kg/m^2}$.

The ratio between the mass of the structural fuzzy (fuzzy substructure 1 and 2) and the mass of the master structure is approximately 10%. The master structure is excited at $z = 3L/5$ by a harmonic force $\mathbf{F}$ ($||\mathbf{F}|| = 1000\,\mathrm{N}$) on the frequency range [100 Hz, 1200 Hz].

The response of the coupled beam is numerically evaluated by solving the following system of equation of motions:

$$
\begin{aligned}
&\frac{\partial^4 u(z,\omega)}{\partial z^4} - \left(\frac{\omega^2 \rho' l^2 - \mathrm{i}\omega \underline{Z}_1(\omega) l}{E'I}\right) u(z,\omega) = 0, \quad && z \in]0, L/2[,\\
&\frac{\partial^4 u(z,\omega)}{\partial z^4} - \left(\frac{\omega^2 \rho' l^2}{E'I}\right) u(z,\omega) = 0, && z \in]L/2, 3L/5[\cup]3L/5, 7L/10[,\\
&\frac{\partial^4 u(z,\omega)}{\partial z^4} - \left(\frac{\omega^2 \rho' l^2 - \mathrm{i}\omega \underline{Z}_2(\omega) l}{E'I}\right) u(z,\omega) = 0, && z \in]7L/10, L[,
\end{aligned}
\tag{44}
$$

where $\underline{Z}_1$ and $\underline{Z}_2$ are the mean boundary impedances of fuzzy 1 and 2, respectively.

The mean apparent mass per unit area $(\mu_0)_j + \underline{R}_j$ and the mean apparent damping per unit area $\underline{I}_j$ of the fuzzy on Γ_j ($j = 1, 2$) are computed according to Eqs. (37) and (38) on the frequency range

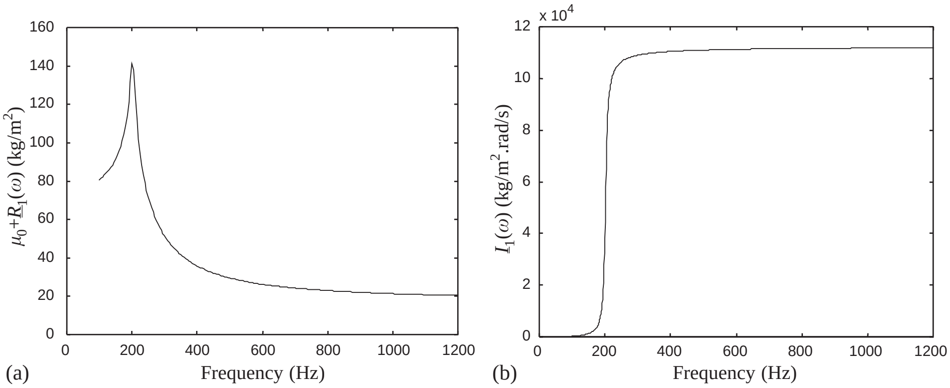

Fig. 9. (a) Mean apparent mass per unit area of the fuzzy on Γ_1 and (b) mean apparent damping per unit area of the fuzzy on Γ_1.

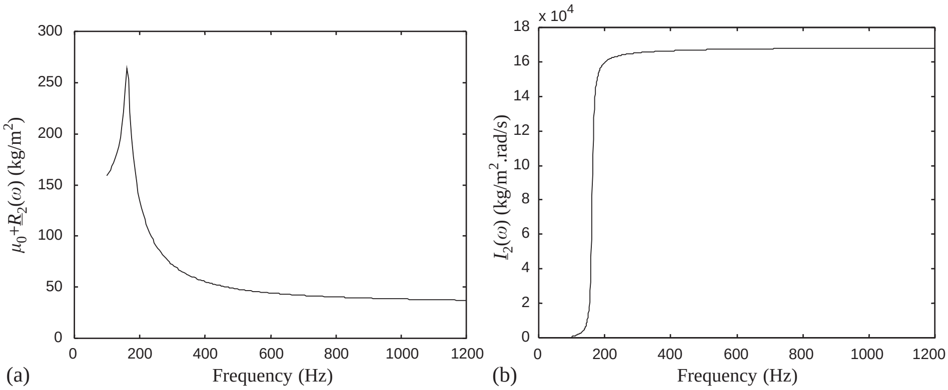

Fig. 10. (a) Mean apparent mass per unit area of the fuzzy on Γ_2 and (b) mean apparent damping per unit area of the fuzzy on Γ_2.

[100 Hz, 1200 Hz] from numerical integration over the interval $[(\Omega_1)_j/2\pi$, 2400 Hz]. The functions $\omega \mapsto (\mu_{0)_j+\underline{R}j}(\omega)$ and $\omega \mapsto \underline{I}_j(\omega)\,(j=1,2)$ are plotted in Figs. 9 and 10.

The functions $\omega \mapsto \underline{R}_1(\omega)$ and $\omega \mapsto \underline{R}_2(\omega)$ reach a maximum at the frequency of the fundamental modes, $(\Omega_1)_1/2\pi$ and $(\Omega_1)_2/2\pi$, respectively. Again, below the fundamental mode of the continuous fuzzy, the fuzzy essentially acts as an added mass, and above the fundamental mode, the action of the fuzzy on the master structure results essentially in a dissipative effect.

Finally, the exact response of the uncoupled beam, the exact response of the beam coupled with the fuzzy and the response of the beam coupled with the fuzzy obtained from Eq. (44) are compared in Fig. 11.

Here again, the results validate the theoretical solution developed in this paper, since both the exact and the probabilistic model show a transition between the mass-loading effect and the dissipative effect of the fuzzy in the region of the fundamental modes of the fuzzy.

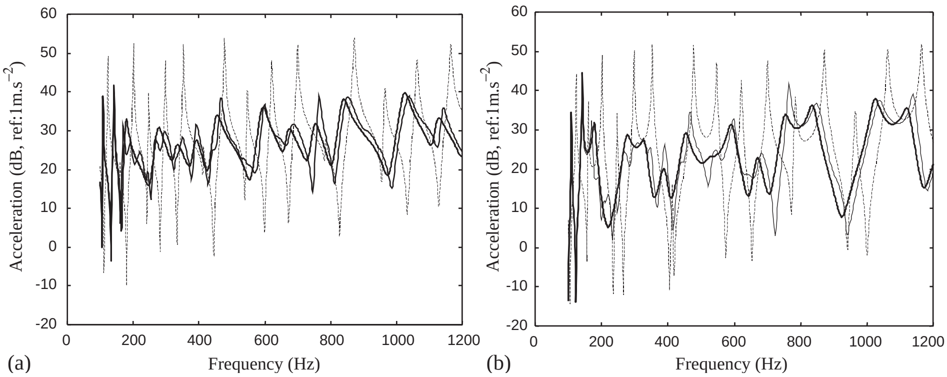

Fig. 11. Exact response of the uncoupled beam (- - - - - -), exact response of the beam coupled with the structural fuzzy (——) and predicted response of the beam coupled with the structural fuzzy (——) at (a) $z = 3L/5$ (excitation point) and at (b) $z = 3L/10$.

6. Conclusion

This paper has proposed a theoretical framework for the dynamic response of a master structure coupled to a locally homogeneous and orthotropic structural fuzzy, with discrete attachment, composed of elastic continuous fuzzy subsystems. The approach generalizes previous work based on the study of a structural fuzzy composed of linear oscillators excited by their supports. A deterministic model of the action of the continuous fuzzy on the master structure has been derived from a boundary impedance of the fuzzy, which is characterized by its modal parameters. We have presented a simple application of this model in the case of a structural fuzzy composed of elastic bars whose geometrical parameters are randomly defined. In this case, a probabilistic model of the boundary impedance can be simply derived from uncertainty on geometrical parameters of the fuzzy (length and cross-sectional area of the bars). The theoretical solution has been successfully validated through numerical applications performed on an Euler–Bernoulli beam coupled with such a continuous structural fuzzy. The results show that above the fundamental mode of the fuzzy, the response of the system is considerably damped compared to the response of the master structure and below that frequency the mass loading is the dominant mechanism.

Acknowledgments

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Appendix A. Formulation of the boundary impedance of a structural fuzzy composed of identical linear oscillators excited by their supports

We consider a master structure coupled with a locally homogeneous and orthotropic structural fuzzy and we consider a coupling surface Γ_j on which the fuzzy is homogeneous: on Γ_j, the fuzzy is composed of N identical linear oscillators excited by their supports; each oscillator is characterized by a mass M and a complex stiffness $K(1+i\eta)$. Assumptions (H1)–(H5) of Section 2 are verified on Γ_j. An expression of the boundary impedance of the fuzzy on Γ_j is straightforward from Eqs. (20)–(25): the mass density of an oscillator k is

$$\rho(\mathbf{x}) = M\delta(\mathbf{x}-\mathbf{x}_0), \quad (\mathbf{x},\mathbf{x}_0) \in V_k \times V_k, \tag{A.1}$$

where δ is the Dirac function,

$$h(\mathbf{x}_0) = \int_{V_k} h(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}\mathbf{x}. \tag{A.2}$$

The displacement of the oscillator are given by

$$\mathbf{u}^*(\omega) = u^*(\omega)\mathbf{X}, \quad \mathbf{X} = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}, \tag{A.3}$$

$$\mathbf{u}_k(\omega) = u_k(\omega)\,(\mathbf{X}_0)_k, \quad (\mathbf{X}_0)_k = [\,1 \quad 1\,]^{\mathrm{T}}. \tag{A.4}$$

According to Eqs. (A.3) and (A.4), Eq. (21) becomes

$$\mathrm{i}\omega Z(\omega) = -\omega^2 R_{\mathrm{oscil}}(\omega) + \mathrm{i}\omega I_{\mathrm{oscil}}(\omega) \quad \text{on } \Gamma_j, \tag{A.5}$$

where R_{oscil} represents the apparent mass per unit area of the oscillators:

$$R_{\mathrm{oscil}}(\omega) = \frac{M}{S}\frac{(\Omega/\omega)^2((\Omega/\omega)^2(1+\eta^2)-1)}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4}, \tag{A.6}$$

and I_{oscil} represents the apparent damping per unit area of the oscillators:

$$I_{\mathrm{oscil}}(\omega) = \frac{M}{S}\frac{\omega(\Omega/\omega)^2\eta}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4}. \tag{A.7}$$

In these equations, Ω is the natural angular frequency of the oscillator $k\,(k=1,\ldots,N)$, $\Omega^2 = K/M$, $S = |\Gamma_j|/N$ is the area of the coupling subsurface S_k. In the particular case where the fuzzy is composed of linear oscillators, the deterministic model developed by Soize (type I fuzzy law, [1,2]) is then obtained from Eqs. (A.5) to (A.7), which appears to be a special case of the general Eqs. (20)–(25) (with the distinction that Eqs. (20)–(25) assume a structural damping whereas Eqs. (A.5)–(A.7) assume a viscous damping).

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