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Short Communication

Closed-form solutions for axially graded beam-columns

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1. Introduction

Functionally graded materials (FGM) constitute an important area of materials science research, with potentially many practical applications. In the structures made of FGM, the material properties vary smoothly in the thickness dimension. Reviews of current FGM research may be found in the articles by Hirai [1] and Markworth et al. [2], and the book by Suresh and Mortensen [3]. In an FGM, the composition and structure gradually change over volume, resulting in corresponding changes in the properties of the material. It is envisioned that the functional grading will also be performed in the axial direction. The seeds of such researches, although only in the analytical setting, were planted some time ago. Apparently, Dinnik [4], was the first to study beams with variable material density for the vibrations of strings. Recently non-homogeneity in the material density was investigated by Masad [5], and Pronsato et al. [6], amongst others. Specifically, they considered a piece-wise constant variation of the density. Continuously varying material density but with constant elastic modulus has been investigated by Gutierez et al. [7]. A quite general formulation on free vibrations of non-uniform beams on elastic foundation, containing the characteristics of both the Winkler and the Pasternak foundation is due to Eisenberger [8] and Eisenberger and Clastornik [9]. They dealt with beams with constant modulus of elasticity and constant material density, but variable cross-section. In recent studies the present authors [10,11] investigated a special class of an inverse vibration problem for

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inhomogeneous beams on elastic foundation considering beams with variable elastic modulus and material density. Here the objective is to widen the class of closed-form solution for such axially graded beams in which the material characteristics vary smoothly in the axial direction. Specifically two cases of harmonically varying vibration modes are postulated, corresponding to beam-columns with elastically guided end conditions, and appropriate semi-inverse problems are solved.

2. Formulation of the problem

Let us consider an axially graded beam resting on elastic foundation subjected to an axial loading. The beam's length is L, cross-sectional area A is constant, flexural rigidity is denoted by $D(x)$, and varying material density by $\rho(x)$. The governing differential equation of the dynamic behavior of such an inhomogeneous beam on an elastic foundation reads

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\frac{\partial}{\partial x}\left[N(x)\frac{\partial w(x,t)}{\partial x}-k_P(x)\frac{\partial w(x,t)}{\partial x}\right]+k_W(x)w(x,t)-\rho(x)A\frac{\partial^2 w(x,t)}{\partial t^2}=0, \tag{1}$$

where x is the axial coordinate, t is the time, $w(x,t)$ is the transverse displacement, $N(x)$ is the axial compressive load distribution, $k_W(x)$ is the variable coefficient of the Winkler foundation, and $k_P(x)$ is the variable coefficient of the Pasternak foundation. The coefficient $k_P(x)$ was introduced by Pasternak [12] and Vlasov and Leontiev [13].

In this study the differential equation (1) will be solved in a closed form for two sets of boundary conditions corresponding to the following beams: (a) simply supported at one end and elastically guided at the other, (b) elastically guided at both its ends. For simplicity, the non-dimensional co-ordinate $\xi = x/L$ is introduced. Harmonic vibration is studied so that the displacement $w(x,t)$ is represented as follows:

$$w(\xi,t)=W(\xi)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where $W(\xi)$ is the postulated mode shape and ω is the corresponding natural frequency that should be determined. Upon substitution of Eq. (2) into Eq. (1), the latter becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[D(\xi)\frac{\mathrm{d}^2W(\xi)}{\mathrm{d}\xi^2}\right]+L^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left[N(\xi)\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}-k_P(\xi)\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi}\right]+L^4\left[k_W(\xi)-\rho(\xi)A\omega^2\right]W(\xi)=0. \tag{3}$$

The semi-inverse eigenvalue problem is posed as follows: find an axially graded beam with a specified harmonic mode, $W(\xi)$, that satisfies the boundary conditions and the governing dynamic differential equation. This semi-inverse problem requires the determination of the distribution of flexural rigidity, $D(\xi)$, that together with a specific law of material density, $\rho(\xi)$, for particular variability of soil properties, defined by the Winkler and Pasternak coefficients, and according to a specific axial load distribution, $N(\xi)$, satisfies the governing eigenvalue problem.

The flexural rigidity, $D(\xi)$, and the axial force, $N(\xi)$, are represented as follows:

$$D(\xi) = A_0 + A_1 \sin(\varphi\pi\xi) + A_2 \cos(\varphi\pi\xi), \tag{4}$$

$$N(\xi) = B_0 + B_1 \sin(\gamma\pi\xi) + B_2 \cos(\gamma\pi\xi), \tag{5}$$

where A_0, A_1, A_2, B_0, B_1, B_2 are constants, while φ and γ are real numbers. The posed semi-inverse problem may have no solution or it may possess multiple solutions or a unique solution. It will be shown that for a specified distribution of material density the solution turns out to be a unique one.

3. Beam that is simply supported at one end and elastically guided at the other

For a beam that is simply supported at one end and guided, with an elastic spring, at the other, the boundary conditions read

$$\begin{aligned} &W(0) = 0, \quad W''(0) = 0, \\ &W'(1) = 0, \quad [D(\xi)W''(\xi)]' + kL^3 W(\xi)|_1 = 0. \end{aligned} \tag{6a–d}$$

The unknown vibration mode, represented in Fig. 1a, is taken as

$$W(\xi) = \psi(\xi) = \sin\left(\frac{\pi}{2}\xi\right). \tag{7}$$

At this stage a non-trivial question arises: does the function in Eq. (7) satisfy the condition in Eq. (6d)? We will postpone replying to this question until later on.

Furthermore, the following trigonometric representation of the stiffness is considered:

$$D(\xi) = A_0\left[1 + \alpha \cos\left(\frac{\pi}{2}\xi\right)\right], \tag{8}$$

in conjunction with the harmonic distribution of axial load

$$N(\xi) = \lambda\left[1 + \beta \cos\left(\frac{\pi}{2}\xi\right)\right], \tag{9}$$

and the following distribution of the Pasternak coefficient function:

$$k_P(\xi) = \hat{k}_P\left[1 + \beta \cos\left(\frac{\pi}{2}\xi\right)\right]. \tag{10}$$

By considering the assumed distributions given by Eqs. (8)–(10) together with the postulated vibration mode the first two terms of the differential equation (3) can be rewritten as

$$R(\xi) = \tfrac{1}{16}\pi^2\left\{\pi^2 A_0\left[1 + 4\alpha \cos\left(\frac{\pi}{2}\xi\right)\right] - 4L^2(\lambda - \hat{k}_P)\left[1 + 2\beta \cos\left(\frac{\pi}{2}\xi\right)\right]\right\}\sin\left(\frac{\pi}{2}\xi\right). \tag{11}$$

In order to obtain a multiplicative representation it is sufficient to fix β at 2α, to obtain

$$R(\xi) = \tfrac{1}{16}\pi^2\left[\pi^2 A_0 - 4L^2\left(\lambda - \hat{k}_P\right)\right]\left[1 + 4\alpha \cos\left(\frac{\pi}{2}\xi\right)\right]\sin\left(\frac{\pi}{2}\xi\right). \tag{12}$$

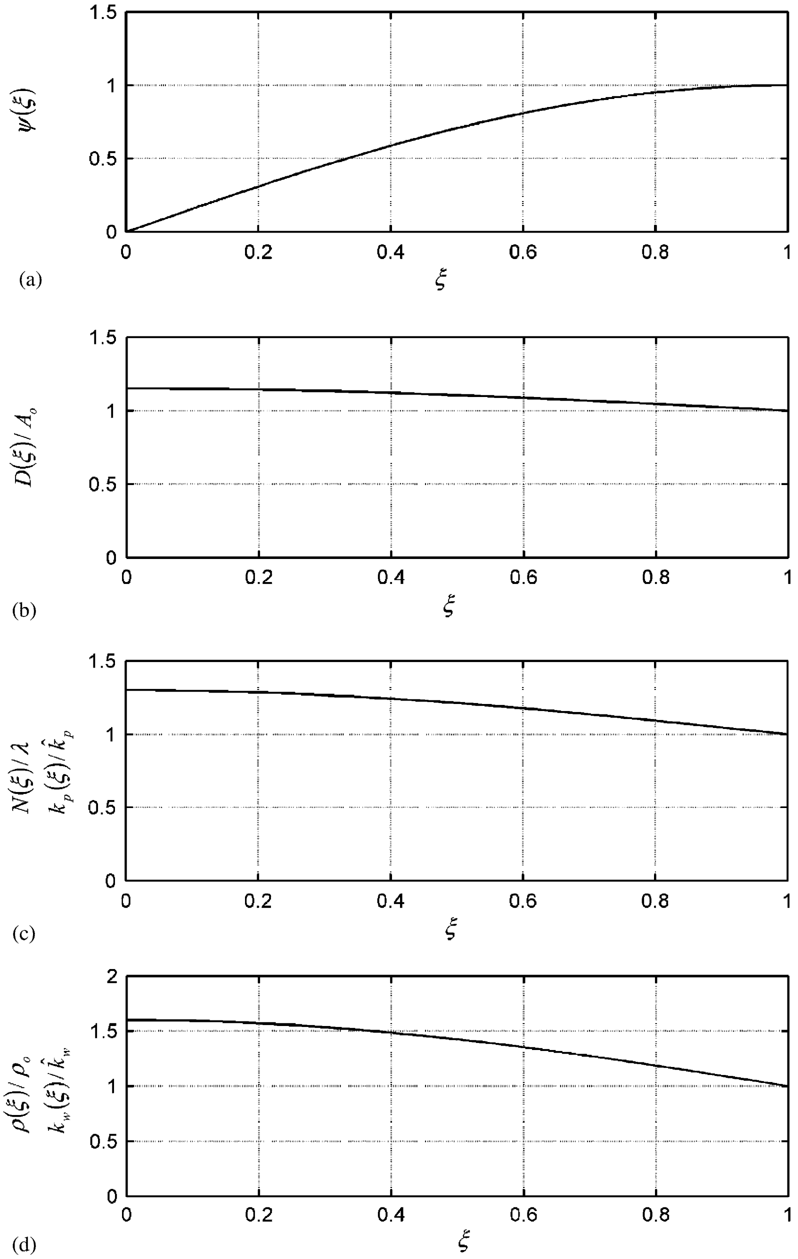

Fig. 1. Beam on elastic foundation that is simply supported at one end and elastically guided at the other ($\alpha = 0.1$). (a) Vibration mode; (b) distribution of the flexural rigidity; (c) axial load distribution, and Pasternak coefficient function; (d) material density distribution and Winkler coefficient function.

Bearing in mind the postulated vibration mode, Eq. (7), the differential equation (3) takes the form

$$\left\{\tfrac{1}{16}\pi^2\left[\pi^2 A_0 - 4L^2(\lambda - k_P)\right]\left[1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right)\right] + L^4\left[k_W(\xi) - \rho(\xi)A\omega^2\right]\right\}\sin\left(\frac{\pi}{2}\xi\right) = 0. \tag{13}$$

Eq. (13) is satisfied for all the distributions of material density $\rho(\xi)$ and Winkler coefficient function $k_W(\xi)$ that are proportional to the following quantity:

$$f(\xi) \propto 1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right). \tag{14}$$

Therefore we assume

$$\rho(\xi) = \rho_0\left[1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right)\right], \tag{15}$$

$$k_W(\xi) = \hat{k}_W\left[1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right)\right]. \tag{16}$$

In order to obtain positive values of density distribution and Winkler coefficient distribution the inequality $\alpha > -\frac{1}{4}$ must be imposed. We consider the following particular cases:

Case 1*: homogeneous beam*: $\alpha = 0$: This case corresponds to a beam with constant stiffness $D(\xi) = A_0$ subjected to a constant axial loading $N(\xi) = \lambda = P$. Then, for $\rho(\xi) = \rho_0 = \text{const}$, $k_W(\xi) = \hat{k}_W = \text{const}$ and $k_p(\xi) = \hat{k}_p = \text{const}$ the natural frequency of the homogeneous beam on uniform elastic foundation under constant axial load is obtained

$$\omega^2 = \frac{\pi^2}{16}\frac{\left(\pi^2 A_0 + 4\hat{k}_P L^2 - 4PL^2\right)}{\rho_0 A L^4} + \frac{\hat{k}_W}{\rho_0 A} = \frac{(\pi^2/16)\left(\pi^2 A_0 + 4\hat{k}_P L^2 - 4PL^2\right) + \hat{k}_W L^4}{\rho_0 A L^4}. \tag{17}$$

In this case the natural frequency vanishes if the load P equals the critical buckling value that is given by

$$P_{\text{cr}} = \hat{k}_P + 4\frac{\hat{k}_W L^2}{\pi^2} + \frac{\pi^2}{4}\frac{A_0}{L^2} \tag{18}$$

It appears that expressions (17) and (18) have never previously been reported in the literature.

Case 2: *Axial graded beam*: $\alpha \neq 0$: Consider the non-uniform beam with the material density and axial load distribution, respectively,

$$\rho(\xi) = \rho_0\left[1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right)\right], \quad N(\xi) = \lambda\left[1 + \beta\cos\left(\frac{\pi}{2}\xi\right)\right]. \tag{19}$$

The elastic foundation possesses the following Winkler and Pasternak coefficient functions:

$$k_W(\xi) = \hat{k}_W\left[1 + 4\alpha\cos\left(\frac{\pi}{2}\xi\right)\right], \quad k_P(\xi) = \hat{k}_P\left[1 + \beta\cos\left(\frac{\pi}{2}\xi\right)\right]. \tag{20}$$

The following natural frequency is derived:

$$\omega^2 = \frac{\pi^2}{16}\frac{\left(\pi^2 A_0 + 4\hat{k}_P L^2 - 4\lambda L^2\right)}{\rho_0 A L^4} + \frac{\hat{k}_W}{\rho_0 A} = \frac{(\pi^2/16)\left(\pi^2 A_0 + 4\hat{k}_P L^2 - 4\lambda L^2\right) + \hat{k}_W L^4}{\rho_0 A L^4}. \tag{21}$$

It is remarkable that this expression formally coincides with its counterpart, in Eq. (17), that is valid for the homogeneous beam on uniform elastic foundation under constant axial load. The natural frequency does not depend on the parameter α; it depends only on A_0, ρ_0 and on the foundation coefficients $\hat{k}_W$ and $\hat{k}_P$. The natural frequency vanishes if the distribution of load attains its critical value

$$\lambda_{\mathrm{cr}} = \hat{k}_P + 4\frac{\hat{k}_W L^2}{\pi^2} + \frac{\pi^2}{4}\frac{A_0}{L^2}. \tag{22}$$

Now we recall the important question posed earlier whether or not the boundary condition (6d) is satisfied. Obviously, for an arbitrary value of the stiffness k, the boundary condition will not be satisfied. The *special* value of the stiffness of the linear elastic spring at the guided end can be evaluated by imposing the satisfaction of the natural boundary condition (6d).

By substituting expressions (7) and (6d) and (8) we obtain

$$\tfrac{1}{8}A_0\pi^3\left\{\alpha \sin\left(\frac{\pi\xi}{2}\right)^2 - \cos\left(\frac{\pi\xi}{2}\right)\left[1 + \alpha\cos\left(\frac{\pi\xi}{2}\right)\right]\right\} - kL^3 \sin\left(\frac{\pi\xi}{2}\right)\bigg|_1 = 0 \tag{23}$$

from which the sought value of the elastic spring at the guided end is derived as

$$k = \alpha A_0\pi^3/8L^3. \tag{24}$$

For any fixed value of the parameter A_0 the stiffness of the elastic spring varies linearly with the parameter α that identifies the particular distribution of stiffness and material density. It is worth noting that in the limiting case of the homogeneous beam, $\alpha = 0$, the boundary condition (6d) is satisfied for $k = 0$, which corresponds to the guided end. Furthermore, by considering that the stiffness k must be positive, the more stringent inequality, namely $\alpha \geqslant 0$, is derived compared to the previously obtained one, $\alpha > -\frac{1}{4}$. With reference to axially graded beams it is important to emphasize the remarkable and unanticipated independence of the eigenvalue, given by expression (21), from the coefficient α. Hence *different* beams characterized by the *same* values of A_0 and ρ_0 but *different* values of α have the *same* eigenvalues and mode shape. For example, with reference to a vibration problem without axial load distribution, this means that if we consider two beams with different values of α (corresponding to different distributions of flexural rigidity, material density and different elastic foundation stiffness distributions) but with springs characterized by the corresponding values of $k(\alpha)$, given by Eq. (24), these two beams will *share* the same frequency and vibration mode. Hence the difference in the distribution of the stiffness, material density and elastic foundation properties is somewhat compensated by the different values of the elastic spring.

For example, two beams characterized by

$$D(\xi) = \left[1 + \varepsilon\cos\left(\frac{\pi}{2}\xi\right)\right], \quad \rho(\xi) = \rho_0\left[1 + 4\varepsilon\cos\left(\frac{\pi\xi}{2}\right)\right],$$

(I) $k = \varepsilon A_0 \pi^3/8L^3$ with $\varepsilon \geqslant 0$

$$k_P(\xi) = \hat{k}_P\left[1 + \varepsilon \cos\left(\frac{\pi}{2}\xi\right)\right], \quad k_W(\xi) = \hat{k}_W\left[1 + 4\varepsilon \cos\left(\frac{\pi\xi}{2}\right)\right],$$
$$D(\xi) = \left[1 + \gamma \cos\left(\frac{\pi}{2}\xi\right)\right], \quad \rho(\xi) = \rho_0\left[1 + 4\gamma \cos\left(\frac{\pi\xi}{2}\right)\right],$$

(II) $k = \gamma A_0 \pi^3/8L^3$ with $\gamma \geqslant 0$

$$k_P(\xi) = \hat{k}_P\left[1 + \gamma \cos\left(\frac{\pi}{2}\xi\right)\right], \quad k_W(\xi) = \hat{k}_W\left[1 + 4\gamma \cos\left(\frac{\pi\xi}{2}\right)\right]$$

share the same vibration mode, given by Eq. (7), and the same frequency, expressed by Eq. (21). This is because coefficient ε or γ does not appear in Eq. (21).

In Fig. 1 the normalized distributions of flexural rigidity, axial load, material density and foundation coefficient functions, corresponding to the case $\alpha = 0.1$, are reported.

4. Beam that is elastically guided at both ends

For a beam on elastic foundation that is elastically guided at both its ends the boundary conditions, which take into account the presence of identical elastic springs at the ends of beam, are

$$W'(0) = 0, \quad [D(\xi)W''(\xi)]' - kL^3 W(\xi)|_0 = 0,$$
$$W'(1) = 0, \quad [D(\xi)W''(\xi)]' + kL^3 W(\xi)|_1 = 0. \qquad (25\text{a}-\text{d})$$

The vibration mode, reported in Fig. 2a, is postulated as

$$W(\xi) = \psi(\xi) = \cos(\pi\xi). \qquad (26)$$

In addition, the following representation of the flexural rigidity is considered:

$$D(\xi) = A_0[1 + \alpha \sin(\pi\xi)]. \qquad (27)$$

Likewise, similar distributions of axial load and Pasternak coefficient functions are postulated

$$N(\xi) = \lambda[1 + \beta \sin(\pi\xi)], \qquad (28)$$

$$k_P(\xi) = \hat{k}_P[1 + \beta \sin(\pi\xi)]. \qquad (29)$$

By considering expressions (26)–(29) the first two terms of the differential equation (3) become

$$R(\xi) = \pi^2\left\{\pi^2 A_0[1 + 4\alpha \sin(\pi\xi)] - L^2\left(\lambda - \hat{k}_P\right)[1 + 2\beta \sin(\pi\xi)]\right\}\cos(\pi\xi). \qquad (30)$$

By setting $\beta = 2\alpha$, Eq. (30) assumes the simpler, multiplicative, expression

$$R(\xi) = \pi^2\left[\pi^2 A_0 - \lambda L^2 + k_P L^2\right](1 + 4\alpha \sin(\pi\xi))\cos(\pi\xi). \qquad (31)$$

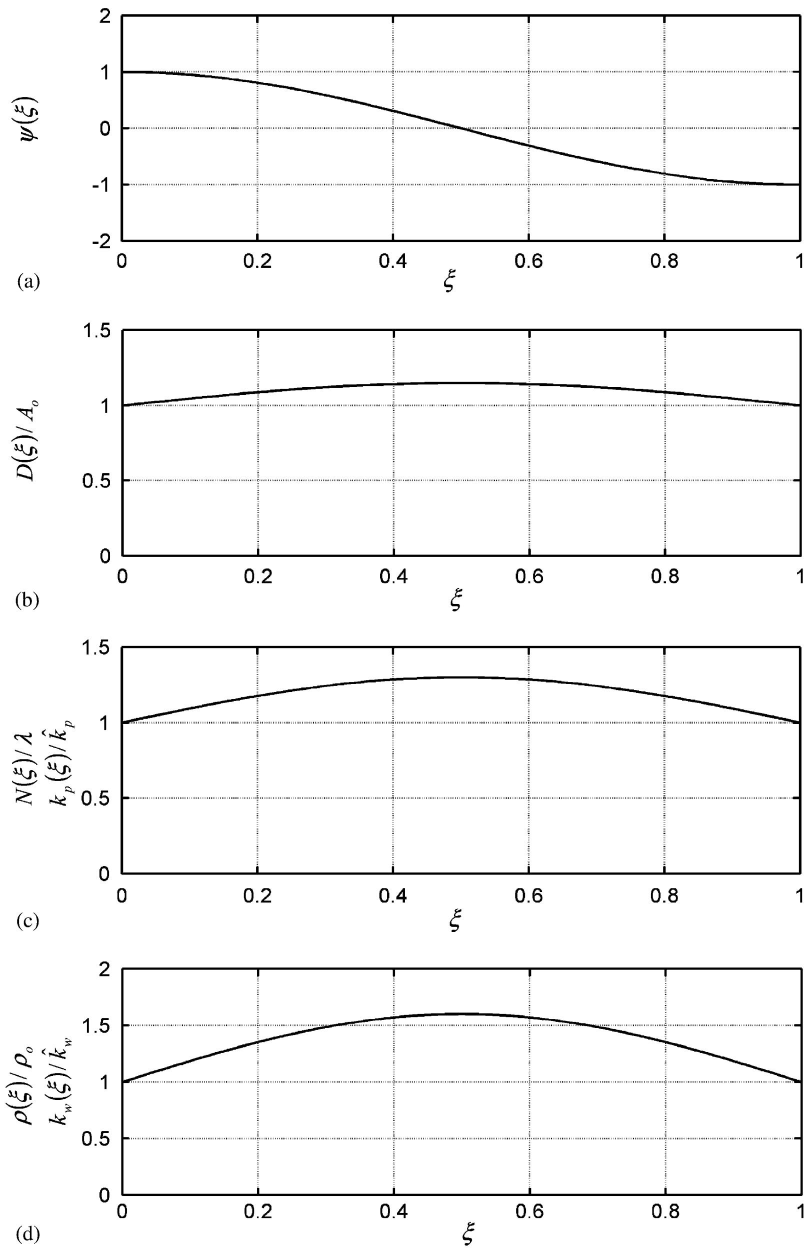

Fig. 2. Beam on elastic foundation that is elastically guided at both ends ($\alpha = 0.2$). (a) Vibration mode; (b) distribution of the flexural rigidity; (c) axial load distribution, and Pasternak coefficient function; (d) material density distribution and Winkler coefficient function.

Bearing in mind the postulated vibration mode in Eq. (26), the differential equation (3) takes the following final form:

$$\left\{\pi^2\left[\pi^2 A_0 - \lambda L^2 + k_P L^2\right](1 + 4\alpha\,\sin(\pi\xi)) + L^4\left[k_W(\xi) - \rho(\xi)A\omega^2\right]\right\}\cos(\pi\xi) = 0. \tag{32}$$

It is easy to recognize that Eq. (32) is satisfied for the following distributions of material density $\rho(\xi)$ and Winkler coefficient function $k_W(\xi)$:

$$\rho(\xi) = \rho_0[1 + 4\alpha\sin(\pi\xi)], \tag{33}$$

$$k_W(\xi) = \hat{k}_W[1 + 4\alpha\,\sin(\pi\xi)]. \tag{34}$$

Furthermore, the inequality $\alpha > -\frac{1}{4}$ must be introduced in order to obtain a positive material density distribution. We consider the following particular cases:

Case 1: *Homogeneous beam*: $\alpha = 0$: This first case corresponds to the homogeneous beam, $D(\xi) = A_0$, subjected to a constant axial load, $N(\xi) = \lambda = P$. Assuming $\rho(\xi) = \rho_0 = \text{const}$, the natural frequency is of the homogeneous beam-column on elastic foundation guided at both its ends is obtained

$$\omega^2 = \frac{\pi^2\left(\pi^2 A_0 + \hat{k}_P L^2 - PL^2\right)}{\rho A L^4} + \frac{\hat{k}_W}{\rho_0 A} = \frac{\pi^2\left(\pi^2 A_0 + \hat{k}_P L^2 - PL^2\right) + \hat{k}_W L^4}{\rho_0 A L^4}. \tag{35}$$

The natural frequency vanishes if the load P equals the critical buckling value that is given by

$$P_{\text{cr}} = \hat{k}_P + \frac{\hat{k}_W L^2}{\pi^2} + \pi^2\frac{A_0}{L^2}. \tag{36}$$

It is interesting to recognize that expressions (35) and (36) are formally identical to the corresponding expressions of the homogeneous simply supported beam-column on uniform elastic foundation reported in Ref.[11].

Case 2: *Axial graded beam*: $\alpha \neq 0$: According to the expressions (28) and (33) the following distributions of axial force and material density are introduced:

$$N(\xi) = \lambda[1 + 2\alpha\,\sin(\pi\xi)]; \quad \rho(\xi) = \rho_0[1 + 4\alpha\,\sin(\pi\xi)] \tag{37}$$

while the Winkler and the Pasternak coefficient functions are taken as

$$k_W(\xi) = \hat{k}_W(1 + 4\alpha\,\sin(\pi\xi)), \quad k_P(\xi) = \hat{k}_P(1 + 2\alpha\,\sin(\pi\xi)). \tag{38}$$

Therefore the natural frequency reads

$$\omega^2 = \frac{\pi^2\left(\pi^2 A_0 + \hat{k}_P L^2 - \lambda L^2\right)}{\rho_0 A L^4} + \frac{\hat{k}_W}{\rho_0 A} = \frac{\pi^2\left(\pi^2 A_0 + \hat{k}_P L^2 - \lambda L^2\right) + \hat{k}_W L^4}{\rho_0 A L^4} \tag{39}$$

which vanishes if the distribution of load reaches its critical value

$$\lambda_{\text{cr}} = \hat{k}_p + \frac{k_W L^2}{\pi^2} + \pi^2\frac{A_0}{L^2}. \tag{40}$$

Expressions (39) and (40) are formally equivalent to the corresponding expressions of homogeneous beam under constant axial load (35) and (36). It must be noted that the natural frequencies and the buckling load obtained for the axially graded beam guided at both its ends have the same analytical expressions as those obtained in Ref. [11] for the simply supported axially graded beam but refer to different distributions of flexural rigidity, axial load, material density and elastic foundation coefficient functions.

The values of the stiffness of the elastic supports, corresponding to the reported solution, are derived hereinafter by imposing the fulfillment of the natural boundary conditions at both ends. By substituting the postulated mode and the distribution of the stiffness in to the expressions (25b) and (25d) the value of the elastic spring, equal at both ends, is derived

$$k = \alpha A_0 I \pi^3 / L^3. \tag{41}$$

It is worth noting that guided axially graded beams on elastic foundation, with elastic distributions specified by Eqs. (38), for which the stiffness of the springs at the ends are specified by Eq. (41) and have stiffness and density distribution given by Eqs. (27) and (33) and are eventually subjected to an axial load distribution, given by Eq. (37a), are all characterized by the same mode, Eq. (26), and the same fundamental frequency, given by expression (39), independent of the particular value of the parameter α that characterizes the particular distributions of stiffness, density, axial load and foundation coefficients. Bearing in mind that the stiffness k must be positive, in this case also a more stringent inequality, $\alpha \geqslant 0$, must be considered than the one derived before $\alpha > -\frac{1}{4}$. The distributions of flexural rigidity, axial load, material density and foundation coefficient functions are reported in Fig. 2 for the case $\alpha = 0.2$.

5. Comparison with the results obtained via the Rayleigh quotient

Since all the solutions reported above are written in the closed form, they do not need an additional corroboration. Still, it is of some interest to compare them with the results derived by any approximate method in order to possibly gain additional insight. Therefore, the natural frequencies were also calculated by using the Rayleigh quotient

$$\omega^2 = \left(\int_0^L D(x)\left[\phi''(x)\right]^2 \mathrm{d}x - \int_0^L N(x)\left[\phi'(x)\right]^2 \mathrm{d}x + \int_0^L k_W(x)[\phi(x)]^2 \mathrm{d}x + \int_0^L k_P(x)\left[\phi'(x)\right]^2 \mathrm{d}x + k[\phi(0)]^2 + k[\phi(L)]^2 \right) \left(\int_0^L A\rho(x)[\phi(x)]^2 \mathrm{d}\xi \right)^{-1}. \tag{42}$$

By substituting into the Rayleigh quotient the *exact* mode shape of vibration the *exact* eigenvalues, expressed by the Eqs. (21) and (39), have been re-obtained. For the *approximate* evaluation, polynomial trial functions are used for simplicity. The results of the comparison are reported in Table 1.

The approximate results are slightly dependent on the parameter α, while the closed form solution does not exhibit such a dependence. The results reported in the table are referred to a value of $\alpha = 0.5$.

The solutions reported herein can be used as benchmark problems. Also, in the future when the technology will be available to produce any desired distribution of flexural rigidity along the axis of the beam on a variable elastic foundation and a given variation of material density $\rho(\xi)$, one

Table 1
Verification of the analytical results ($\alpha = 0.5$)

Boundary condition	Polynomial trial function	Results obtained via Rayleigh's quotient		Difference with closed-form solution (%)
S–G	$8\xi - 4\xi^3 + \xi^4$		$\omega^2 = \frac{\pi^4 A_0 I - \pi^2 \lambda L^2}{16\rho_0 A L^4}$	
		$\lambda = 0$	$\omega^2 = 6.0944\,\frac{A_0 I}{A L^4}$	0.10
		$\omega = 0$	$\lambda_{cr} = 2.469\,\frac{A_0 I}{L^2}$	0.09
G–G	$1 - 6\xi^2 + 4\xi^3$		$\omega^2 = \frac{98.339 A_0 I - 9.879 \lambda L^2}{\rho_0 A L^4}$	
		$\lambda = 0$	$\omega^2 = 98.339\,\frac{A_0 I}{\rho_0 A L^4}$	0.95
		$\omega = 0$	$\lambda_{cr} = 9.953\,\frac{A_0 I}{L^2}$	0.84

will be able to design axially graded beams with *pre-selected* natural frequency or buckling load values. Thus, the methodology of solving semi-inverse problems presented in this study may represent a valuable design tool for vibration and buckling problems within the trigonometric class of inhomogeneity.

6. Conclusion

Apparently for the first time in the literature closed solutions have been derived for the natural frequencies of axially graded beam-columns on elastic foundation with guided end conditions. As particular cases the frequencies and buckling loads of the corresponding homogeneous beams have been obtained. The distributions of the flexural rigidity, material density as well the variability of foundation coefficients are sought in terms of trigonometric functions. The trigonometric function was also postulated for the mode shape; conditions were established for which this postulate holds. This seemingly transparent approach yields a series of new non-trivial closed-form solutions.

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