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Journal of Sound and Vibration 280 (2005) 1132–1142
JOURNAL OF SOUND AND VIBRATION
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Short Communication

A comparative study of the Galerkin approximation utilized in the Timoshenko beam theory

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Received 19 September 2003; accepted 18 February 2004

Available online 17 September 2004

1. Motivation and problem statement

The time-function selection utilized in Galerkin approximation is investigated for the Timoshenko beam theory. Both identical and different time functions for transverse deflection and beam's cross-sectional rotation are considered. In order to explain the underlying concept here, we consider a general Timoshenko beam system with uniform cross-section, which is moving in the horizontal plane. To characterize the elastic deformations, we associate to each point on the undeformed neutral axis of the beam two quantities $v(x,t)$ and $\psi(x,t)$. The transverse elastic deflection is represented by v, while ψ denotes the orientation of the beam cross-section. x is the reference variable along the beam measured from one end of the support, and t is the time. Fig. 1 shows the kinematics of deformation of a beam element that undergoes a shear deformation in addition to a pure bending.

The equations of motion can be derived by applying Hamilton's principle. However, to facilitate the analysis, typically an assumed mode expansion (i.e., Galerkin approximation) is

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.02.028

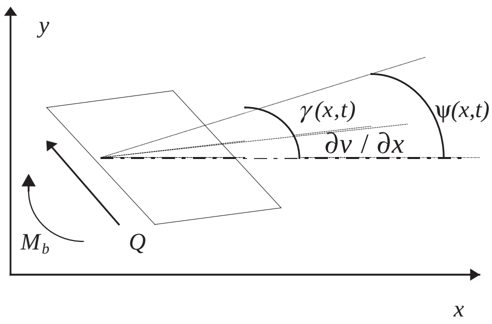

Fig. 1. Kinematics of deformation of a Timoshenko beam.

utilized. Specifically, it is assumed that v and ψ can be written as the finite sums:

$$v(x,t) = \sum_{i=1}^{n} \Phi_i(x) p_i(t), \qquad \psi(x,t) = \sum_{i=1}^{n} \Psi_i(x) p_i(t), \tag{1}$$

where $p_i(t)$ are the generalized coordinates (time functions) for the elastic deflection and orientation of the beam element. $\Phi_i(x)$ and $\Psi_i(x)$ are the respective transverse and rotational eigenfunctions (modal shapes) of a Timoshenko beam.

The use of same time functions, $p_i(t)$, in the Galerkin approximation is a common practice and the standard assumption that has been utilized by many researchers in other areas of mechanics [1] (for example, elastic rods [2], laminated composite plates [3,4], etc.). Such assumption ensures the synchronized nature of the motion of the transverse displacement $v(x,t)$ of the beam and the orientation of its cross-section $\psi(x,t)$. This assumption is also in total agreement with the limiting case for the Timoshenko beam theory, where by assuming $\psi = \partial v/\partial x$ it will lead to the Euler–Bernoulli formulation. However, by assuming a different set of generalized coordinates it will make this simplified form rather unreachable.

Although one may argue that the transverse displacement and the angle of rotation of the cross-section are independent variables in the Timoshenko beam theory, assuming the same modal amplitudes will not necessarily introduce any contradictions since the spatial functions are independent ($\Phi_i(x)$ and $\Psi_i(x)$). In fact, in a different publication by the authors, this assumption has been taken into consideration and instead, a more general formulation for *different modal amplitudes* has been obtained [5]. The results show that the relative error associated with the same time-function assumption is rather small (less than 0.25%) and being almost negligible for many practical applications (see Fig. 2).

2. Generalized formulation

We consider a Timoshenko beam with a total thickness h, width b in the lateral direction, and length L in the longitudinal (x-) direction. The beam is subjected to a general force $f(x,t)$ in the transverse direction. To demonstrate the effect of time-function selection for generalized

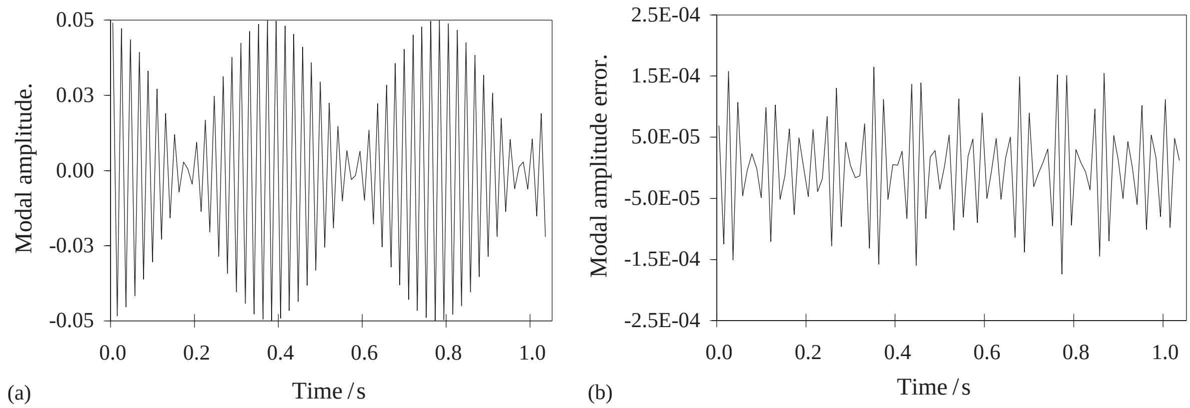

Fig. 2. (a) An example of the fundamental time function $p_1(t)$, and (b) the error associated with the use of identical functions, $e_1(t) = q_1(t) - p_1(t)$, where $v(x,t) = \sum_{i=1}^{n} \Phi_i(x) p_i(t), \psi(x,t) = \sum_{i=1}^{n} \Psi_i(x) q_i(t)$ [5].

coordinates, two cases are considered here: (i) different time functions and (ii) identical time functions as discussed next.

2.1. Different time-function

In this case, we assume different time functions for v and ψ as the finite sums

$$v(x,t) = \sum_{i=1}^{n} \Phi_i(x) p_i(t), \qquad \psi = (x,t) = \sum_{i=1}^{n} \Psi_i(x) q_i(t), \tag{2}$$

where $p_i(t)$ and $q_i(t)$ are the modal amplitudes for the deflection and rotation of the beam element, respectively. The kinetic, T, and potential, V, energies of the beam can be written as

$$\begin{aligned} T &= \tfrac{1}{2}\int_0^L \rho A \dot{v}^2 \, \mathrm{d}x + \tfrac{1}{2}\int_0^L \rho I \dot{\psi}^2 \, \mathrm{d}x, \\ V &= \tfrac{1}{2}\int_0^L EI(\psi')^2 \mathrm{d}x + \tfrac{1}{2}\int_0^L kAG(\psi - v')^2 \, \mathrm{d}x, \end{aligned} \tag{3}$$

where ρ is the beam volumetric density, I is the cross-sectional moment of inertia, A is the cross-sectional area, E is Young's modulus of elasticity, k is the shear correction factor in Timoshenko beam theory, and G is the shear modulus.

Substituting Eqs. (2) into Eqs. (3) and applying the Lagrange's equations for the forced vibration would result in the following governing equations of motion:

$$\begin{aligned} \rho A \sum_{j=1}^{n} C_{ij} \ddot{p}_j + kAG \sum_{j=1}^{n} \{(B_{ij} p_j - H_{ji} q_j)\} = f_i, \quad i = 1, 2, \ldots, n, \\ \rho I \sum_{j=1}^{n} D_{ij} \ddot{q}_j + \sum_{j=1}^{n} \{(EI\, F_{ij} + kAG\, D_{ij}) q_j - kAG\, H_{ij}\, p_j\} = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{4}$$

where

$$B_{ij} = \int_0^L \Phi_i'(x)\Phi_j'(x)\,\mathrm{d}x, \quad C_{ij} = \int_0^L \Phi_i(x)\Phi_j(x)\,\mathrm{d}x,$$
$$D_{ij} = \int_0^L \Psi_i(x)\Psi_j(x)\,\mathrm{d}x, \quad F_{ij} = \int_0^L \Psi_i'(x)\Psi_j'(x)\,\mathrm{d}x,$$
$$H_{ij} = \int_0^L \Psi_i(x)\Phi_j'(x)\,\mathrm{d}x, \quad f_i = \int_0^L \Phi_i(x)f(x,t)\,\mathrm{d}x. \tag{5}$$

2.2. Identical time-function

For this case, we now consider identical modal amplitudes for the deflection and rotation as

$$v(x,t) = \sum_{i=1}^{n} \Phi_i(x)p_i(t), \qquad \psi(x,t) = \sum_{i=1}^{n} \Psi_i(x)p_i(t). \tag{6}$$

The equations of motion for Timoshenko beam then become

$$\rho A \frac{\partial^2 v}{\partial t^2} - kAG\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = f(x,t),$$
$$EI\frac{\partial^2 \psi}{\partial x^2} + kAG\left(\frac{\partial v}{\partial x} - \psi\right) - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0. \tag{7}$$

Substituting Eqs. (6) into Eqs. (7), yields

$$\sum_{i=1}^{n}\left\{\rho A \Phi_i \ddot{p}_i - kAG\left(\Phi_i'' - \Psi_i''\right)p_i\right\} = f(x,t)$$
$$\sum_{i=1}^{n}\left\{EI\Psi_i'' p_i + kAG\left(\Phi_i' - \Psi_i\right)p_i - \rho I \Psi_i \ddot{p}_i\right\} = 0. \tag{8}$$

On the other hand, from the free vibration analysis we have

$$-kAG\left(\Phi_i'' - \Psi_i'\right) = \rho A \omega_i^2 \Phi_i,$$
$$EI\Psi_i'' + kAG\left(\Phi_i' - \Psi_i\right) = -\rho I \omega_i^2 \Psi_i, \tag{9}$$

where ω_i is the ith natural frequency of the beam. Substituting Eqs. (9) into Eqs. (8) results in

$$\rho A \sum_{i=1}^{n} \Phi_i(x)\left\{\ddot{p}_i(t) + \omega_i^2 p_i(t)\right\} = f(x,t), \tag{10a}$$

$$\rho I \sum_{i=1}^{n} \Psi_i(x)\left\{\ddot{p}_i(t) + \omega_i^2 p_i(t)\right\} = 0. \tag{10b}$$

In order to solve for $p_i(t)$ from Eqs. (8) and (9), the orthogonality conditions [6,7] of

$$\int_0^L \left(\rho A \Phi_i(x)\Phi_j(x) + \rho I \Psi_i(x)\Psi_j(x)\right) \mathrm{d}x = N_i \delta_{ij} \tag{11}$$

are utilized, where $i,j = 1, 2, \ldots, n$, δ_{ij} is the Kronecker delta, and N_i is defined by setting $i=j$ in Eq. (11). Multiplying Eq. (10a) by $\Phi_j(x)$ and Eq. (10b) by $\Psi_j(x)$, adding each side together, integrating over the entire length of the beam, and with the use of the orthogonality relationship (11), we get

$$N_i\{\ddot{p}_i(t) + \omega_i^2 p_i(t)\} = \int_0^L f(x,t)\Phi_i(x)\,\mathrm{d}x \tag{12}$$

or

$$\ddot{p}_i(t) + \omega_i^2 p_i(t) = S_i(t), \tag{13}$$

where

$$\begin{aligned} N_i &= \textstyle\int_0^L \{\rho A \Phi_i^2(x) + \rho I \Psi_i^2(x)\}\,\mathrm{d}x, \\ S_i(t) &= \frac{1}{N_i}\int_0^L f(x,t)\Phi_i(x)\,\mathrm{d}x. \end{aligned} \tag{14}$$

It is obvious that for $f(x,t) = g(t)\delta(x-l)$, the generalized force $S_i(t)$ becomes $S_i(t) = \Phi_i(l)g(t)/N_i$. Without the loss of generality, one may assume zero initial conditions for the beam and hence obtain the analytical solution of Eq. (13) as

$$p_i(t) = \frac{1}{\omega_i}\int_0^t S_i(\tau)\sin\omega_i(t-\tau)\,\mathrm{d}\tau. \tag{15}$$

3. An example case study

In order to better demonstrate the derivations obtained in the preceding sections, a beam with a simply supported boundary condition at either ends is assumed with a sinusoidal distributed loading as

$$f(x,t) = f_0 \sin\omega t \sin\frac{r\pi x}{L}, \tag{16}$$

where r is a constant positive integer. The eigenfunctions for this boundary condition are

$$\Phi_i(x) = \sin(\beta_i x), \qquad \Psi_i(x) = R_i \cos(\beta_i x), \tag{17}$$

where

$$\beta_i = i\pi/L \tag{18}$$

and R_i is a function of the natural frequency of the beam (ω_i). ω_i and R_i can be easily found by substituting Eqs. (17) into the equations of motion. Substituting Eqs. (16) and (17) into

Eq. (14) gives

$$S_i(t) = \frac{f_0 \sin\omega t}{\rho(A + IR_i)} \delta_{ri}, \tag{19}$$

which means that for $i \neq r$, $S_i(t)$ and ultimately $p_r(t)$ would become zero. Consequently, with zero initial conditions we get

$$p_r(t) = \frac{f_0}{\rho(A + IR_r)} \left\{ \frac{\omega/\omega_r}{\omega^2 - \omega_r^2} \sin \omega_r t - \frac{1}{\omega^2 - \omega_r^2} \sin \omega t \right\}. \tag{20}$$

One can obtain the solutions to the deflection v and the sectional rotation ψ of the beam as

$$\begin{aligned} v(x,t) &= \frac{f_0}{\rho} \sin \frac{r\pi x}{L} \sum_{k=1}^{2} \left(\frac{(\omega/\omega_{rk}) \sin \omega_{rk} t - \sin \omega t}{(\omega^2 - \omega_{rk}^2)(A + IR_k)} \right), \\ \psi(x,t) &= \frac{f_0}{\rho} \cos \frac{r\pi x}{L} \sum_{k=1}^{2} \left(R_k \frac{(\omega/\omega_{rk}) \sin \omega_{rk} t - \sin \omega t}{(\omega^2 - \omega_{rk}^2)(A + IR_k)} \right). \end{aligned} \tag{21}$$

4. Numerical results and discussions

In order to compare the effect of time-function selection, a beam with the following geometrical and material properties is considered:

$$\begin{aligned} &h = 0.035\,\text{m}, \quad L = 0.5\,\text{m}, \quad I = 8.9323\text{e-}8\,\text{m}^4, \quad A = 8.75\text{e}{-}4\,\text{m}^2, \\ &E = 207\,\text{GPa}, \quad G = 79\,\text{GPa}, \quad \rho = 7850\,\text{kg/m}^3, \quad k = 5/6. \end{aligned} \tag{22}$$

4.1. Example 1: simply supported boundary conditions at either ends

For the first example, we consider a simply supported boundary condition at either ends of the beam

$$v = \psi' = 0 \text{ at } x = 0, x = L. \tag{23}$$

The beam is subjected to a sinusoidal concentrated force applied at $x = L/5$:

$$f(x,t) = 10 \sin(100t)\delta(x - L/5), \tag{24}$$

where $\delta(x)$ denotes the delta function. The eigenfunctions for this boundary condition can be found as [8]:

$$\Phi_i(x) = \sin(\beta_i x), \qquad \Psi_i(x) = R_i \cos(\beta_i x), \tag{25}$$

where R_i's are functions of natural frequency of the beam and

$$\beta_i = i\pi/L. \tag{26}$$

To solve this system with different time functions, we substitute Eqs. (25) into Eqs. (5) to get

$$B_{ij} = \frac{i^2\pi^2}{2L}\delta_{ij}, \quad C_{ij} = \delta_{ij}L/2,$$
$$D_{ij} = LR_i^2\delta_{ij}/2, \quad F_{ij} = \frac{i^2\pi^2}{2L}R_i^2\delta_{ij}, \quad H_{ij} = \frac{i\pi}{2}R_i\delta_{ij}. \tag{27}$$

Substituting Eqs. (27) into Eqs. (4) results in the following equations:

$$\rho AC_{ii}\ddot{p}_i + kAG(B_{ii}p_i - H_{ii}q_i) = 10\sin(100t)\sin(i\pi/5), \quad i = 1, 2, ..., n,$$
$$\rho ID_{ii}\ddot{q}_i + (EIF_{ii} + kAGD_{ii})q_i - kAGH_{ii}p_i = 0, \quad i = 1, 2, ..., n. \tag{28}$$

Consequently, $p_i(t)$ and $q_i(t)$ can be found, for each i, by solving these two sets of coupled differential equations. The modal amplitudes for $i = 1$ and 2 are calculated and compared in Fig. 3, which shows that $p_i(t)$ and $q_i(t)$ are nearly the same.

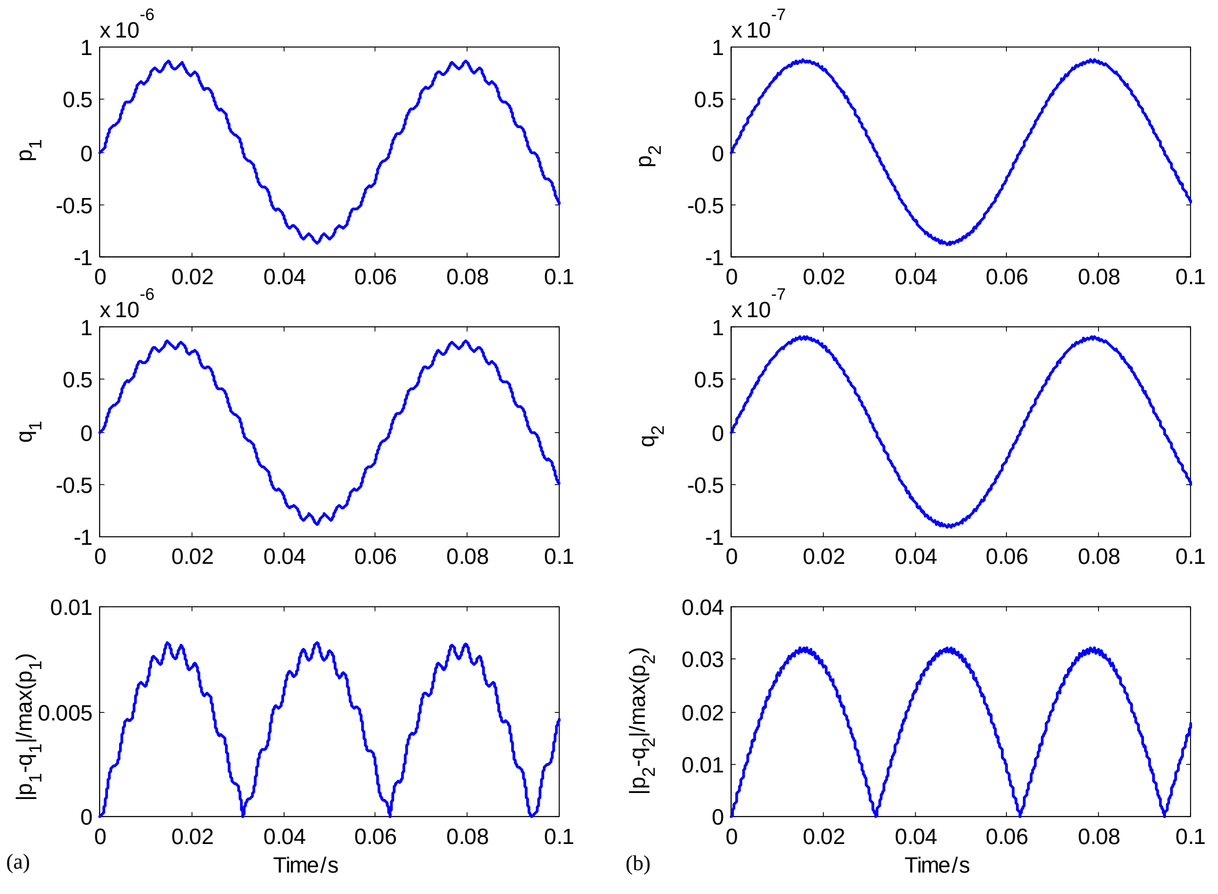

Fig. 3. Modal amplitudes for different time functions: (a) first mode and (b) second mode.

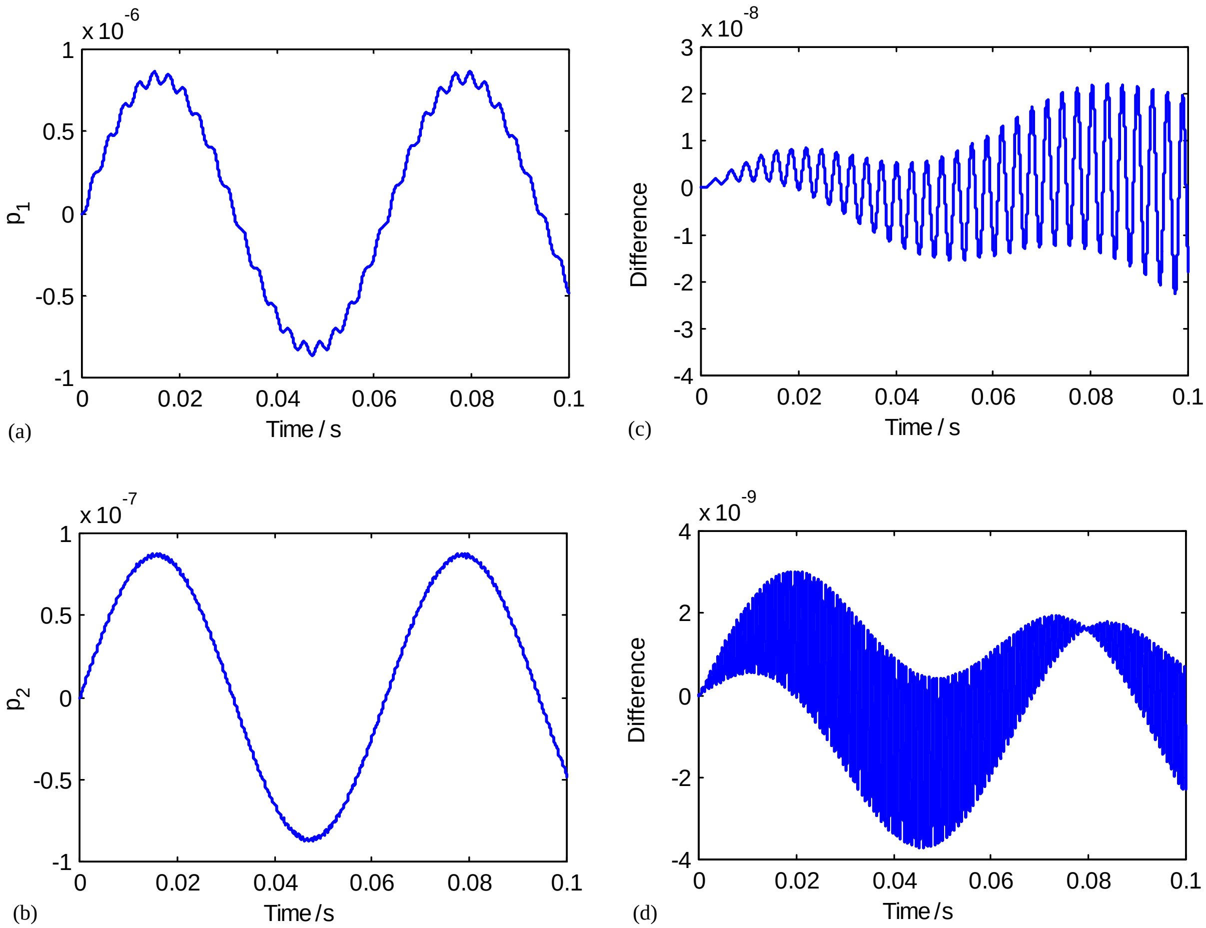

Fig. 4. Modal amplitudes with identical time function for (a) the first mode and (b) the second mode; and the difference associated with the use of different time functions for (c) the first mode and (d) the second mode.

We now consider the identical time-function case for this example. From Eqs. (14), (15) and utilizing Eq. (19), $p_i(t)$ can be calculated for each i. The results for $i = 1$ and 2 are shown in Fig. 4. The differences between the related time functions in two cases (identical and different time function) are also plotted in Fig. 4.

4.2. *Example 2: clamped–simply supported boundary conditions*

As the second example, the clamped–simply supported boundary condition is now assumed for the Timoshenko beam:

$$v = \psi = 0 \text{ at } x = 0; \qquad v = \psi' = 0 \text{ at } x = L. \tag{29}$$

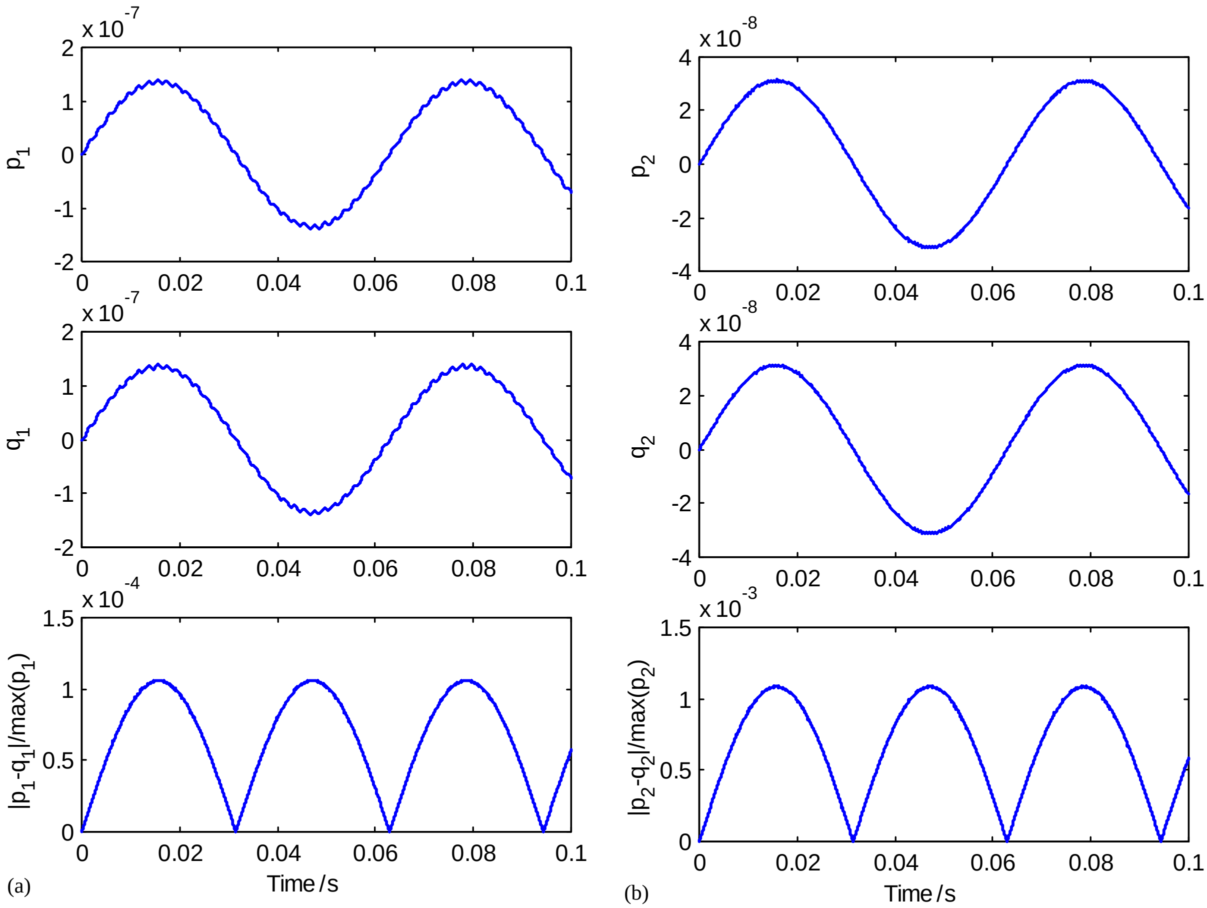

Fig. 5. Modal amplitudes for different time functions: (a) first mode and (b) second mode.

The beam is again subjected to the sinusoidal concentrated force (24). The eigenfunctions for this boundary condition are in the following forms [8]:

$$\begin{aligned}\Phi_i(x) &= \cosh(b_i\alpha_i\xi) - \coth(b_i\alpha_i)\sinh(b_i\alpha_i\xi) - \cos(b_i\beta_i\xi) + \cot(b_i\beta_i)\sin(b_i\beta_i\xi),\\ \Psi_i(x) &= R_i\left\{\cosh(b_i\alpha_i\xi) + \left(\frac{\theta_i}{\lambda_i\zeta_i}\right)\sinh(b_i\alpha_i\xi) - \cos(b_i\beta_i\xi) + \theta_i\sin(b_i\beta_i\xi)\right\},\end{aligned} \quad (30)$$

where $\xi = x/L$ and constants b_i, α_i, θ_i, λ_i, ζ_i, R_i depend on the natural frequencies of the beam. By solving the system of Eq. (4) which is a set of $2n$ coupled differential equations, one can find $p_i(t)$ and $q_i(t)$ for $i = 1, 2, \ldots, n$. The results for $n = 2$ are presented in Fig. 5, which again demonstrates that $p_i(t)$ and $q_i(t)$ are nearly the same.

By considering identical time functions for this example, $p_i(t)$ can be found from Eqs. (13) and (14) using shape modes in equation (30). The results for $i = 1$ and 2 are shown in Fig. 6. The differences between the related time functions in two cases (identical and different time-function) are also plotted in Fig. 6, which shows the insignificant nature of errors.

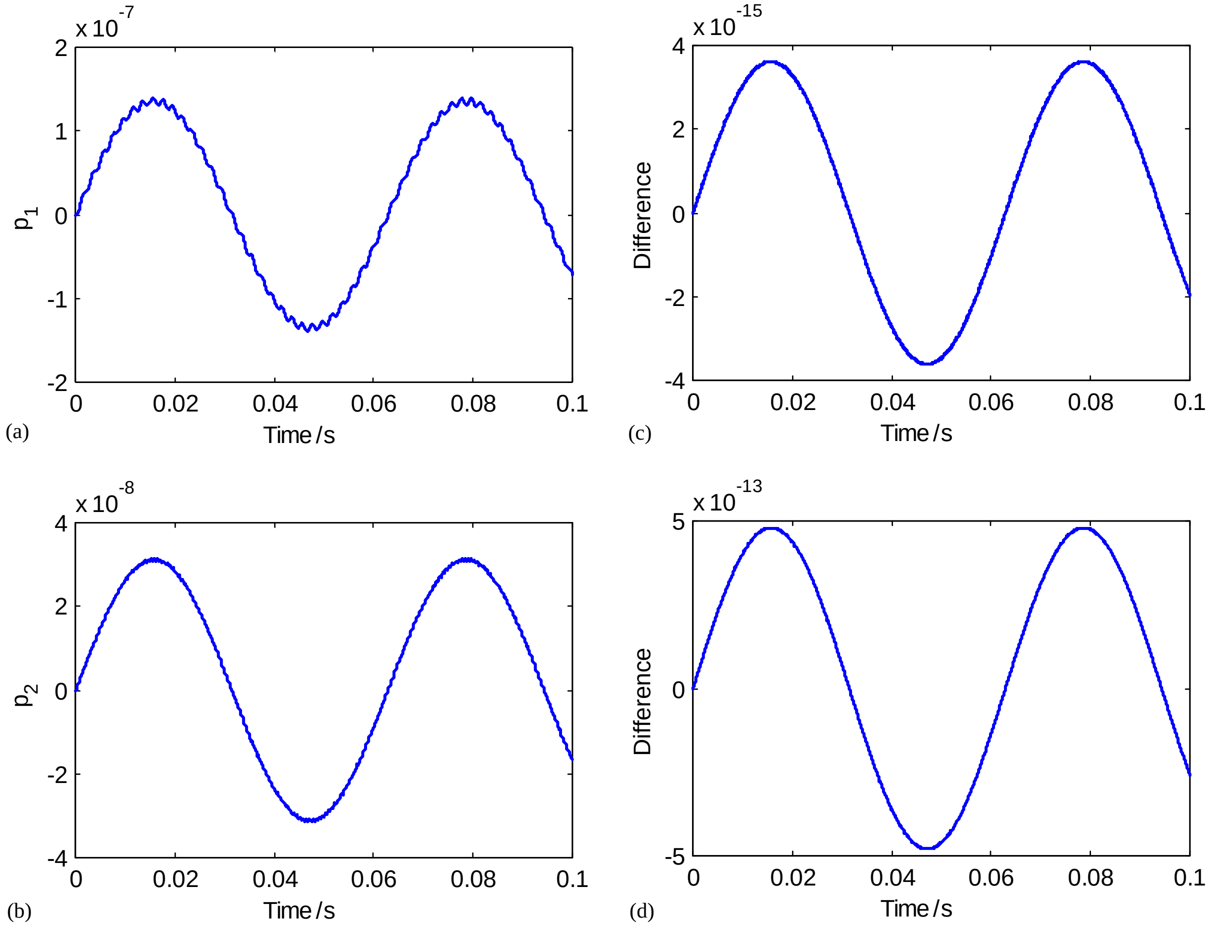

Fig. 6. Modal amplitudes with identical time function for (a) the first mode and (b) the second mode; and the difference associated with the use of different time functions for (c) the first mode and (d) the second mode.

The h/L ratio in the example given in the numerical results is small enough to violate the Euler–Bernoulli assumption ($\psi = \partial v/\partial x$). For the larger ratios for which the assumption is valid, the time function will give even smaller error values. Hence, different h/L ratio will result in a small error similar to the figures given in the paper. These details are not presented here for the sake of brevity [9].

5. Conclusion

The Galerkin approximation with identical and different time functions for transverse deflection and beam's cross-sectional rotation has been considered. The equations of motion for identical time functions were obtained using orthogonality conditions. For the different time functions in Galerkin approximation, the equations of motion were derived using Lagrangian approach. Two boundary conditions, simply supported–simply supported and clamped-simply supported, were considered for the numerical simulations. The numerical results demonstrated that the related time functions in each case are almost identical and the error resulting from identical time-function assumptions can be negligible.

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