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Free vibration analysis of multi-span beams with intermediate flexible constraints

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Abstract

This paper deals with the free vibration analysis of a multi-span beam with an arbitrary number of flexible constraints. Each span of the continuous beam is assumed to obey Timoshenko beam theory. Considering the compatibility requirements on each constraint point, the relationships between two adjacent spans can be obtained. By using a transfer matrix method, eigensolutions of the entire system can be determined. Some numerical results are shown to present the effects of support stiffness and locations.

1. Introduction

The dynamic response of beam structures subjected to moving loads or masses has been studied extensively. There are numerous references available in the monographs of Fryba [1,2]. In Refs. [1,2], most of the cases treat a uniform simply supported beam of a single span and a continuous railway bridge was also modeled in Ref. [1] as a multi-span beam. The earliest work on the behavior of a single span beam subjected to a constant moving load was reported by Timoshenko [3]. Subsequent studies considering the effects of an elastic foundation, moving masses, etc. Cai et al. [4] investigated the dynamic interactions between the vehicle and guideway of magnetically levitated vehicles by modeling the vehicle as a moving force and as a two-degree-of-freedom

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model. Lee [5] studied the effects of an accelerating mass traveling on a Timoshenko beam and investigated the possible separation of the moving mass from the beam.

There are not so many studies on the dynamic analysis of a multi-span continuous beam subjected to moving loads or masses. Lee [6] analyzed the transverse vibration of a beam with intermediate point constraints subjected to a moving load by assumed mode method. Wang [7] investigated the desponse of multi-span Timoshenko beams. Yang et al. [8] presented impact formulas for vehicles over continuous beams. Chatterjee et al. [9] investigated the dynamic behavior of multi-span continuous bridges under a moving vehicular load which was modeled as a sprung mass. Ichikawa et al. [10] also studied the dynamic behavior of multi-span continuous beam traversed by a moving mass. In most of the previous studies, the model of Euler–Bernoulli beam theory by deriving the differential equation and the associated boundary conditions for a basic uniform Euler–Bernoulli beam are often used and discussed. This model is simpler; however, it has some restrictions in the applications, especially, in cases of short beams [11]. Some researches also study the different results between the models of Euler–Bernoulli beam theory and Timoshenko beam theory. Finally, it is possible to evaluate natural frequencies simply by finding roots of the *high-order determinant* of the coefficient matrix of the linear system if the accuracy of the eigensolutions is required.

This investigation presents hybrid analytical/numerical method that permits an efficient computation of the eigensolutions for an arbitrary number of flexible supports of a beam with various boundary conditions. The method is based on the use Timoshenko beam theory in each subsection, and by the compatibility conditions across each support, the relationships of the four integration constants of the eigenfunctions between adjacent subsections can be determined [12,13]. By using the transfer matrix, as a consequence, the entire system has only four unknown constants which can be solved through the satisfaction of four boundary conditions. The novelty of this approach is that the order of the determinant to obtain the eigenvalues does not increase as the number of intermediate supports of the system increases. There are only four unknown constants to be determined no matter how many intermediate supports exist. An analytical form of eigenvalue problem is introduced which is solved using closed form, transfer matrix methods in this article.

2. Theoretical model

A Timoshenko beam of length L and with k intermediate flexible supports is considered as in Fig. 1. It is assumed that the supports are located at points $X_1, X_2, \ldots, X_k$ such that $0 < X_1 < X_2 < \cdots < X_k < L$ and with stiffness $S_1, S_2, \ldots, S_k$, respectively. The entire beam is now divided into $(k+1)$ segments with lengths $L_1, L_2, \ldots, L_{k+1}$ respectively which are separated by k supports. The free vibration amplitudes of the transverse displacements and the slopes, due to bending, of each segment are denoted by $Y_{(j)}(X, T)$ and $\phi_{(j)}(X, T)$ on the interval $X_{j-1} < X < X_j$, where the sub-index j in the parentheses represents the jth segment and $j = 1, 2, \ldots, k+1$ (Fig. 1). By using the Timoshenko beam theory [11,14] and under the following assumptions: (i) under small displacements and strains (ii) with linear elastic material (iii) with constant cross-section and density (iv) no applied external loads, the equations of motion for each

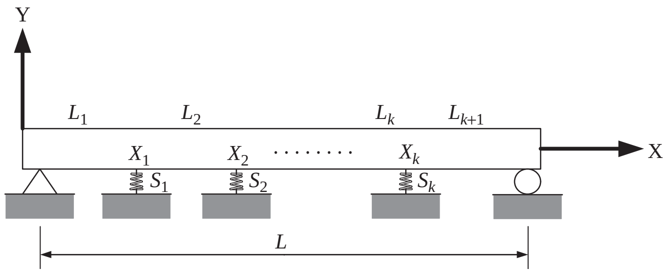

Fig. 1. A beam with k flexible constraints located at positions $X_1, X_2, \ldots, X_k$ respectively and the lengths of sub-sections are $L_1, L_2, \ldots, L_{k+1}$ where $L_1 + L_2 + \cdots L_k + L_{k+1} = L$.

segment are [11]

$$EI\frac{\partial^4 Y_{(i)}(X,T)}{\partial X^4} + \rho A\frac{\partial^2 Y_{(i)}(X,T)}{\partial T^2} - \rho I(1+\frac{E}{\kappa G})\frac{\partial^4 Y_{(i)}(X,T)}{\partial T^2 \partial X^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 Y_{(i)}(X,T)}{\partial T^4} = 0,$$

$$EI\frac{\partial^4 \phi_{(i)}(X,T)}{\partial X^4} + \rho A\frac{\partial^4 \phi_{(i)}(X,T)}{\partial T^2} - \rho I(1+\frac{E}{\kappa G})\frac{\partial^4 \phi_{(i)}(X,T)}{\partial T^2 \partial X^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 \phi_{(i)}(X,T)}{\partial T^4} = 0,$$

$$X_{i-1} < X < X_i, \quad i = 1,\ 2, \ldots, k+1, \tag{1a, b}$$

where E is Young's modulus of the material, I is the moment of inertia of the beam cross-section, ρ is the density of material, A is the cross-sectional area of the beam, G is the shear modulus of the material, κ is the Timoshenko shear coefficient which is a function of the cross-section and Poisson's ratio v [11], and T is time.

The boundary conditions of the beam for a simply supported case are

$$Y(0,T) = Y(L,T) = 0, \tag{2a}$$

$$\phi'(0,T) = \phi'(L,T) = 0. \tag{2b}$$

The "compatibility conditions" enforce continuities of the displacement field, the slope and the bending moment, respectively, across each support and can be expressed

$$Y_{(i)}(X_i^-,T) = Y_{(i+1)}(X_i^+,T), \tag{3a}$$

$$Y'_{(i)}(X_i^-,T) = Y'_{(i+1)}(X_i^+,T), \tag{3b}$$

$$EI\phi'_{(i)}(X_i^-,T) = EI\phi'_{(i+1)}(X_i^+,T), \tag{3c}$$

where the symbols X_i^+ and X_i^- denote the locations immediately above and below the position X_i and $i = 1, 2, \ldots, k$. Moreover, a discontinuity into the shear force of the beam across each support exists and can be expressed

$$\kappa GA[Y'_{(i)}(X_i^-,T) - \phi_{(i)}(X_i^-,T)] = \kappa GA[Y'_{(i+1)}(X_i^+,T) - \phi_{(i+1)}(X_i^+,T)] - S_i Y(X_i,T),$$

$$i = 1, 2, \ldots, k. \tag{3d}$$

In the above, the following quantities are introduced:

$$y_{(i)} = \frac{Y_{(i)}}{L}, \quad x = \frac{X}{L}, \quad x_{(i)} = \frac{X_{(i)}}{L} \quad t = \frac{T}{\sqrt{L}}, \quad l_i = \frac{L_i}{L}. \tag{4a–4d}$$

Thus, in each segment, Eq. (1a) and (1b) can then be expressed in the non-dimensional form as

$$\frac{EI}{L^3}\frac{\partial^4 y_{(i)}(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y_{(i)}(x,t)}{\partial t^2} - \frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 y_{(i)}(x,t)}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{\kappa G L}\frac{\partial^4 y_{(i)}(x,t)}{\partial t^4} = 0,$$
$$\frac{EI}{L^3}\frac{\partial^4 \phi_{(i)}(x,t)}{\partial x^4} + \rho A\frac{\partial^2 \phi_{(i)}(x,t)}{\partial t^2} - \frac{\rho I}{L^2}\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 \phi_{(i)}(x,t)}{\partial t^2 \partial x^2} + \frac{\rho^2 I}{\kappa G L}\frac{\partial^4 \phi_{(i)}(x,t)}{\partial t^4} = 0,$$
$$x_{i-1} < x < x_i, \quad i = 1, 2, ..., k+1. \tag{5a, b}$$

The non-dimensional "compatibility conditions" from Eqs. (3a) to (3d) are

$$y_{(i)}(x_i^-, t) = y_{(i+1)}(x_i^+, t), \tag{6a}$$

$$y'_{(i)}(x_i^-, t) = y'_{(i+1)}(x_i^+, t), \tag{6b}$$

$$\phi'_{(i)}(x_i^-, t) = \phi'_{(i+1)}(x_1^+, t), \tag{6c}$$

$$y'_{(i)}(x_i^-, t) - \phi_{(i)}(x_i^-, t) = y'_{(i+1)}(x_i^+, t) - \phi_{(i+1)}(x_i^+, t) - s_i y(x_i, t), \quad i = 1, 2, \ldots, k, \tag{6d}$$

where

$$s_i = \frac{S_i L}{\kappa G A}.$$

3. Method to find eigensolutions

The eigensolutions for the cases of commonly used different boundary conditions are derived. The solutions of the other boundary conditions can also be obtained easily through the similar procedure. Using the separable solutions: $y_{(i)}(x,t) = w_{(i)}(x)e^{j\omega t}$ and $\phi_{(i)}(x,t) = \varphi_{(i)}(x)e^{j\omega t}$ in Eqs. (5a) and (5b) leads to an associated eigenvalue problem,

$$w''''_{(i)}(x) + (\sigma+\tau)w''_{(i)}(x) - (\alpha - \sigma\tau)w_{(i)}(x) = 0, \;\; x_{i-1} < x < x_i, \; i = 1, 2, \ldots, k+1, \tag{7a}$$

$$\varphi''''_{(i)}(x) + (\sigma+\tau)\varphi''_{(i)}(x) - (\alpha - \sigma\tau)\varphi_{(i)}(x) = 0, \;\; x_{i-1} < x < x_i, \; i = 1, 2, \ldots, k+1, \tag{7b}$$

where

$$\frac{\sigma = \rho L\omega^2}{E}, \quad \tau = \frac{\rho L\omega^2}{\kappa G}, \quad \alpha = \frac{A\rho L^3\omega^2}{EI}. \tag{7c–e}$$

From Eqs. (6a)–(6d), the corresponding compatibility conditions across each flexible support lead to

$$w_{(i)}(x_i^-) = w_{(i+1)}(x_i^+), \tag{8a}$$

$$w'_{(i)}(x_i^-) = w'_{(i+1)}(x_i^+), \tag{8b}$$

$$\varphi'_{(i)}(x_i^-) = \varphi'_{(i+1)}(x_i^+), \tag{8c}$$

$$w'_{(i)}(x_i^-) - \varphi_{(i)}(x_i^-) = w'_{(i+1)}(x_i^+) - \varphi_{(i+1)}(x_i^+) - s_i w(x_i), \tag{8d}$$

for $i = 1, 2, \ldots, k$. A closed form solution to this eigenvalue problem can be obtained by employing transfer matrix methods [12,13]. The general solutions of Eqs. (7a) and (7b), for each segment, are [14]

$$\begin{aligned} w_{(i)}(x) &= A_i \cosh \lambda_1 (x - x_{i-1}) + B_i \sinh \lambda_1 (x - x_{i-1}) \\ &\quad + C_i \cos \lambda_2 (x - x_{i-1}) + D_i \sin \lambda_2 (x - x_{i-1}), \\ \varphi_{(i)}(x) &= B_i q_1 \cosh \lambda_1 (x - x_{i-1}) + A_i q_1 \sinh \lambda_1 (x - x_{i-1}) \\ &\quad - D_i q_2 \cos \lambda_2 (x - x_{i-1}) + C_i q_2 \sin \lambda_2 (x - x_{i-1}), \\ &\quad x_{i-1} < x < x_i, \quad i = 1, 2, \ldots, k+1 \end{aligned} \tag{9a, b}$$

where

$$\lambda_1 = \left(\sqrt{\left(\frac{\sigma - \tau}{2}\right)^2 + \alpha} - \frac{\sigma + \tau}{2} \right)^{1/2}, \quad \lambda_2 = \left(\sqrt{\left(\frac{\sigma - \tau}{2}\right)^2 + \alpha} + \frac{\sigma + \tau}{2} \right)^{1/2}, \tag{9c, d}$$

$$\lambda_3 = (\rho L \omega^2 / \kappa G)^{1/2}, \quad q_1 = (\lambda_3^2 + \lambda_1^2)/\lambda_1, \quad q_2 = (\lambda_3^2 - \lambda_2^2)/\lambda_2, \tag{9e–g}$$

and A_i, B_i, C_i and D_i are constants associated with the ith segment ($i = 1, 2, \ldots, k+1$). These constants in the (i+1)th segment (A_{i+1}, B_{i+1}, C_{i+1} and D_{i+1}) are related to those in the ith segment (A_i, B_i, C_i and D_i) through the compatibility conditions in Eqs. (8a)–(8d) and can be expressed as

$$\begin{Bmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{Bmatrix} = \begin{bmatrix} t_{11} \; t_{12} \; t_{13} \; t_{14} \\ \vdots \\ \ldots \; \ldots \; \ldots \; t_{44} \end{bmatrix}^{(i)} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix} = \mathbf{T}^{(i)}_{4\times 4} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix}, \quad i = 1, 2, \ldots, k, \tag{10}$$

where $\mathbf{T}^{(i)}_{4\times4}$ is the 4×4 transfer matrix which depends on the values λ_1, λ_2, (thus, eigenvalue ω) and the elements are derived in Appendix A and are rewritten here

$$
\begin{aligned}
t_{11} &= \cosh \lambda_1 l_i,\\
t_{12} &= \sinh \lambda_1 l_i,\\
t_{13} &= 0,\\
t_{14} &= 0,\\
t_{21} &= \sinh \lambda_1 l_i - \frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1}\cosh \lambda_1 l_i,\\
t_{22} &= \cosh \lambda_1 l_i - \frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1}\sinh \lambda_1 l_i,\\
t_{23} &= -\frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1}\cos \lambda_2 l_i,\\
t_{24} &= -\frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1}\sin \lambda_2 l_i,\\
t_{31} &= 0,\\
t_{32} &= 0,\\
t_{33} &= \cos \lambda_2 l_i,\\
t_{34} &= \sin \lambda_2 l_i,\\
t_{41} &= \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2}\cosh \lambda_1 l_i,\\
t_{42} &= \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2}\sinh \lambda_1 l_i,\\
t_{43} &= -\sin \lambda_2 l_i + \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2}\cos \lambda_2 l_i,\\
t_{44} &= \cos \lambda_2 l_i + \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2}\sin \lambda_2 l_i.
\end{aligned}
\tag{10a–p}
$$

Through repeated application of Eq. (10), the four constants in the first segment (A_1, B_1, C_1 and D_1) can be mapped into those of the last segment, reducing the number of independent constants to four.

$$
\begin{Bmatrix} A_{k+1}\\ B_{k+1}\\ C_{k+1}\\ D_{k+1} \end{Bmatrix} = \mathbf{T}^{(k)}_{4\times4}\begin{Bmatrix} A_{k}\\ B_{k}\\ C_{k}\\ D_{k} \end{Bmatrix} = \mathbf{T}^{(k)}_{4\times4}\,\mathbf{T}^{(k-1)}_{4\times4}\begin{Bmatrix} A_{k-1}\\ B_{k-1}\\ C_{k-1}\\ D_{k-1} \end{Bmatrix} = \mathbf{T}^{(k)}_{4\times4}\,\mathbf{T}^{(k-1)}_{4\times4}\ldots\mathbf{T}^{(1)}_{4\times4}\begin{Bmatrix} A_{1}\\ B_{1}\\ C_{1}\\ D_{1} \end{Bmatrix}. \tag{11}
$$

These four remaining constants (A_1, B_1, C_1 and D_1) can be found through the satisfaction of the boundary conditions.

For the case of a simply supported beam, the corresponding boundary conditions of Eqs. (2a) and (2b) can thus be expressed

$$Y(0,T)=0 \rightarrow w(0)=0, \tag{12a}$$

$$Y(L,T)=0 \rightarrow w(1)=0, \tag{12b}$$

$$\phi'(0,T)=0 \rightarrow \varphi'(0)=0, \tag{12c}$$

$$\phi'(L,T)=0 \rightarrow \varphi'(1)=0. \tag{12d}$$

Beginning with those at the left support, Eqs. (9a), (9b), (12a) and (12c), leads to

$$A_1 = 0 \quad \text{and} \quad C_1 = 0. \tag{13a, b}$$

Satisfaction of the boundary conditions of Eqs. (9a,b) at the right supports, Eqs. (12b) and (12d) require

$$A_{k+1}\cosh\lambda_1 l_{k+1} + B_{k+1}\sinh\lambda_1 l_{k+1} + C_{k+1}\cos\lambda_2 l_{k+1} + D_{k+1}\sin\lambda_2 l_{k+1} = 0, \tag{14a}$$

$$A_{k+1}q_1\lambda_1\cosh\lambda_1 l_{k+1} + B_{k+1}q_1\lambda_1\sinh\lambda_1 l_{k+1} + C_{k+1}q_2\lambda_2\cos\lambda_2 l_{k+1} + D_{k+1}q_2\lambda_2\sin\lambda_2 l_{k+1} = 0, \tag{14b}$$

which can be expressed in matrix form as

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \cosh\lambda_1 l_{k+1} & \sinh\lambda_1 l_{k+1} & \cos\lambda_2 l_{k+1} & \sin\lambda_2 l_{k+1} \\ q_1\lambda_1\cosh\lambda_1 l_{k+1} & q_1\lambda_1\sinh\lambda_1 l_{k+1} & q_2\lambda_2\cos\lambda_2 l_{k+1} & q_2\lambda_2\sin\lambda_2 l_{k+1} \end{bmatrix} \times \begin{Bmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \\ D_{k+1} \end{Bmatrix} = \mathbf{B}_{2\times 4}\begin{Bmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \\ D_{k+1} \end{Bmatrix}, \tag{15a}$$

where

$$\mathbf{B}_{2\times 4} = \begin{bmatrix} \cosh\lambda_1 l_{k+1} & \sinh\lambda_1 l_{k+1} & \cos\lambda_2 l_{k+1} & \sin\lambda_2 l_{k+1} \\ q_1\lambda_1\cosh\lambda_1 l_{k+1} & q_1\lambda_1\sinh\lambda_1 l_{k+1} & q_2\lambda_2\cos\lambda_2 l_{k+1} & q_2\lambda_2\sin\lambda_2 l_{k+1} \end{bmatrix}. \tag{15b}$$

Substitution of Eq. (11) into Eq. (15a) and the use of Eq. (13a,b) leads to

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \mathbf{B}_{2\times 4}\begin{Bmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \\ D_{k+1} \end{Bmatrix} = \mathbf{B}_{2\times 4}\mathbf{T}^{(k)}_{4\times 4}\mathbf{T}^{(k-1)}_{4\times 4}\ldots\mathbf{T}^{(1)}_{4\times 4}\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \mathbf{R}_{2\times 4}\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix}\begin{Bmatrix} 0 \\ B_1 \\ 0 \\ D_1 \end{Bmatrix}, \tag{16}$$

where

$$\mathbf{R}_{2\times4} = \mathbf{B}_{2\times4}\,\mathbf{T}^{(k)}_{4\times4}\,\mathbf{T}^{(k-1)}_{4\times4}\ldots\mathbf{T}^{(1)}_{4\times4} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix}.$$

Thus, the existence of non-trivial solutions requires

$$\det\begin{bmatrix} r_{12}(\omega) & r_{14}(\omega) \\ r_{22}(\omega) & r_{24}(\omega) \end{bmatrix} = 0. \tag{17}$$

This determinant provides the single (characteristic) equation for the solution of the eigenvalue ω_n. The coefficients of the eigenfunctions, $w_n(x)$, are obtained by back substitution into Eqs. (16), (10) and then Eq. (9).

For the cases of other usually used boundary conditions, through the similar procedure, the following relations can be obtained:

(a) *Cantilever beam*: The existence of non-trivial solutions for the constants A_1, B_1, C_1 and D_1 requires

$$\det\begin{bmatrix} r_{11} - r_{13} & r_{12} - r_{14} \\ r_{21} - r_{23} & r_{22} - r_{24} \end{bmatrix} = 0. \tag{18}$$

The matrix $\underline{B}_{2\times4}$ in Eq. (15a) now becomes

$$\mathbf{B}_{2\times4} = \begin{bmatrix} q_1\lambda_1\cosh\lambda_1 l_{k+1} & q_1\lambda_1\sinh\lambda_1 l_{k+1} & q_2\lambda_2\cos\lambda_2 l_{k+1} & q_2\lambda_2\sin\lambda_2 l_{k+1} \\ (\lambda_1 - q_1)\sinh\lambda_1 l_{k+1} & (\lambda_1 - q_1)\cosh\lambda_1 l_{k+1} & -(\lambda_2 + q_2)\sin\lambda_2 l_{k+1} & (\lambda_2 + q_2)\cos\lambda_2 l_{k+1} \end{bmatrix}. \tag{19}$$

(b) *Fixed-fixed beam*: The existence of non-trivial solutions is the same as Eq. (18) but the matrix $\mathbf{B}_{2\times4}$ in Eq. (15a) now becomes

$$\mathbf{B}_{2\times4} = \begin{bmatrix} \cosh\lambda_1 l_{k+1} & \sinh\lambda_1 l_{k+1} & \cos\lambda_2 l_{k+1} & \sin\lambda_2 l_{k+1} \\ \lambda_1\sinh\lambda_1 l_{k+1} & \lambda_1\cosh\lambda_1 l_{k+1} & -\lambda_2\sin\lambda_2 l_{k+1} & \lambda_2\cos\lambda_2 l_{k+1} \end{bmatrix}. \tag{20}$$

(c) *Free-free beam*: The existence of non-trivial solutions now requires

$$\det\begin{bmatrix} r_{11} - \dfrac{\lambda_1 q_1}{\lambda_2 q_2} r_{13} & r_{12} - \dfrac{\lambda_1 - q_1}{\lambda_2 + q_2} r_{14} \\ r_{21} - \dfrac{\lambda_1 q_1}{\lambda_2 q_2} r_{23} & r_{22} - \dfrac{\lambda_1 - q_1}{\lambda_2 + q_2} r_{24} \end{bmatrix} = 0 \tag{21}$$

and the matrix $\mathbf{B}_{2\times4}$ in Eq. (15a) now is the same as Eq. (19).

4. Numerical results and discussion

The method for obtaining the eigenvalues (natural frequencies) proposed in this article is that of finding the non-trivial solutions of the determinants in Eqs. (17), (18) and (21) for various boundary conditions. These are nonlinear algebraic equations which can be solved by using the

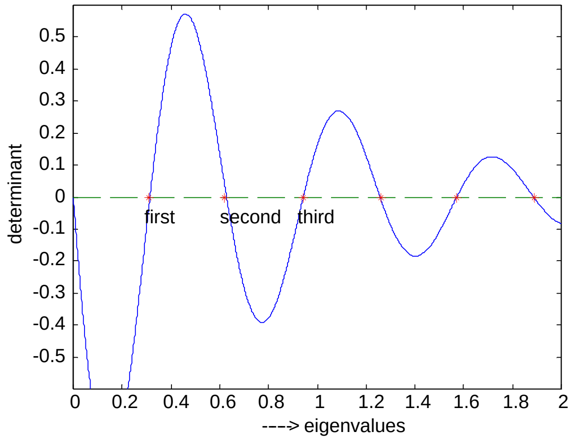

Fig. 2. Simple calculation of eigenvalues.

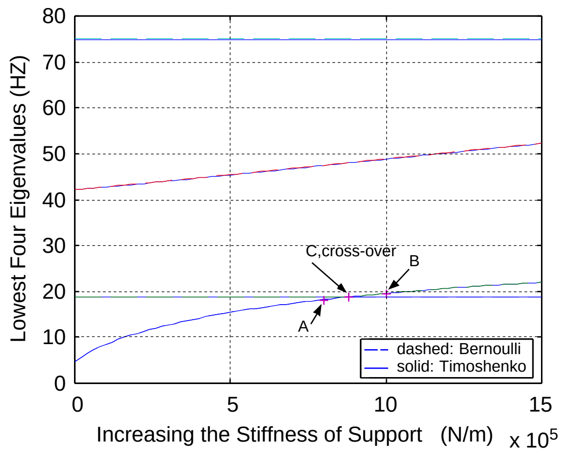

Fig. 3. Lowest four eigenvalues of a simply supported beam with one flexible constraint at mid-point as the stiffness of support S varies.

standard Newton–Raphson iterations or, for simplification, by using the method as shown in Fig. 2 to obtain the eigenvalues.

The Timoshenko shear coefficient κ in the governing equations (Eqs. (1a) and (1b)) is used to simplify the non-uniform shear stress distribution at a cross-section to retain the one-dimensional approach. There are virtually as many different definitions of κ as there are published papers on the Timoshenko beam. Here, Cowper's definition of κ, which is a function of a cross-section and Poisson's ratio v [11]. For the following numerical cases of the square cross-section used in this article, the value $\kappa = 10(1+v)/(12+11v)$ is used.

In order to show the method used in this article, some numerical examples are presented. First is the case of a simply supported beam structure with one flexible support at the mid-point. The beam is square cross-section with width $B = 0.05$ m, height $H = 0.05$ m, total length $L = 5.0$ m, span lengths $L_1 = L_2 = 2.5$ m, density $\rho = 7800\ \mathrm{g/m^3}$, Young's modulus $E = 2.06 \times 10^{11}\ \mathrm{N/m^2}$, shear modulus of elasticity $G = 79 \times 10^9\ \mathrm{N/m^2}$ and Poisson's ratio $\nu = 0.3$. Fig. 3 shows the lowest four natural frequencies of this system obtained by the method presented in this research as the stiffness S is increased. There are two sets of curves in Fig. 3 which represent the results by Bernoulli beam theory (dashed curves) and Timoshenko beam theory (solid curves), respectively. From Fig. 3, it is observed that these two sets of curves almost coincide in the lower modes. There is a little difference in the fourth mode and the variations will be more in higher modes. Also note that, because system is symmetric in this case (the flexible support is at the mid-point), the constraint point is a node to the second and the fourth modes (refer to Fig. 4), so the nature frequencies of these two modes are not affected by the stiffness of the support. The first and the

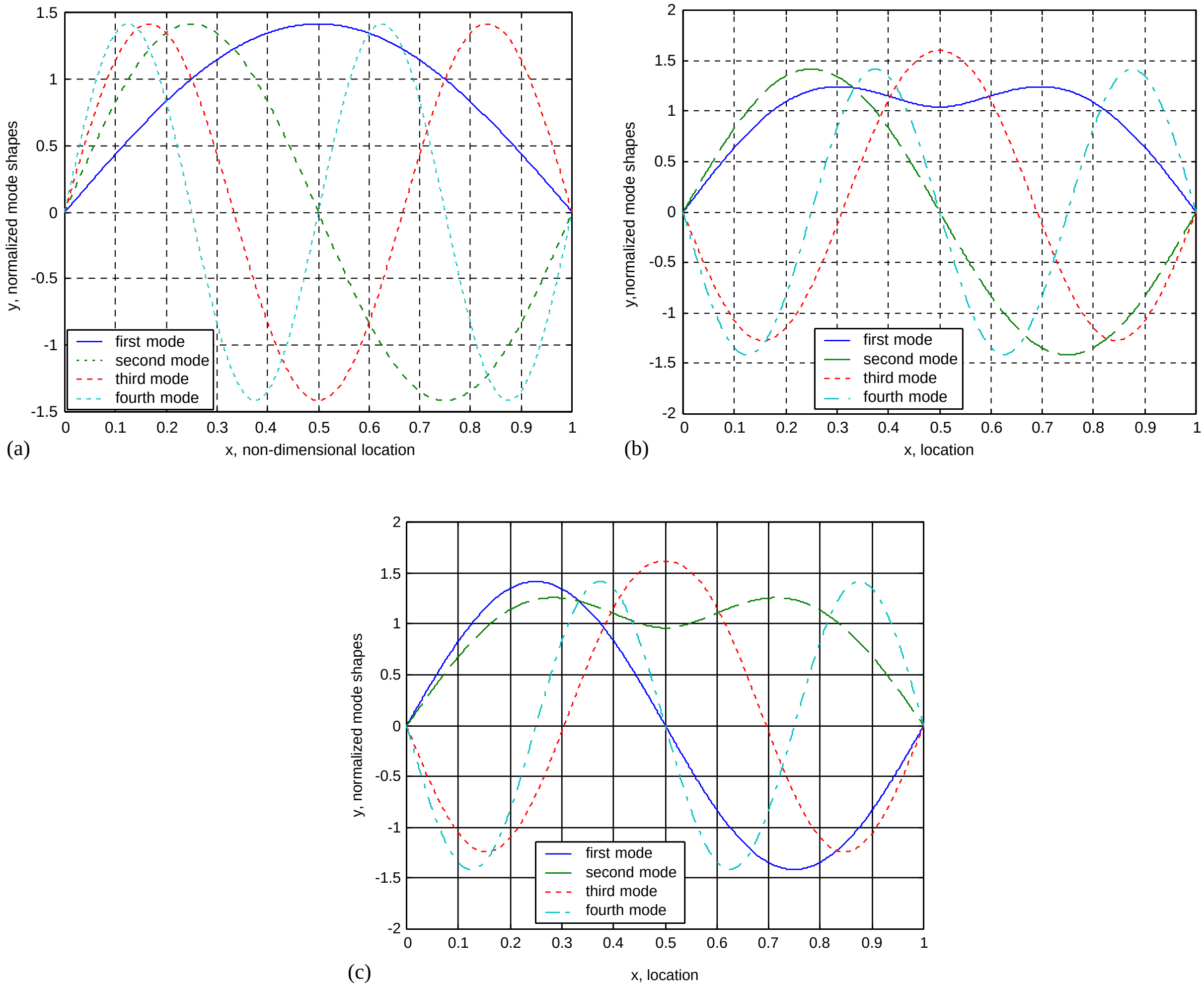

Fig. 4. Variations of lowest four mode shapes before and after the "cross-over" point: (a) stiffness $S = 0\ \mathrm{N/m}$, (b) stiffness $S = 800{,}000\ \mathrm{N/m}$, (c) stiffness $S = 1{,}000{,}000\ \mathrm{N/m}$.

Table 1
Lowest four natural frequencies for cases of large constraint stiffness ($S = 10,000,000\,\mathrm{N/m}$ and $S = 100,000,000\,\mathrm{N/m}$) and the case for stiff intermediate support from Ref. [15]

Lowest four natural frequencies (Hz)	Constraint stiffness $S = 10,000,000\,\mathrm{N/m}$	Constraint stiffness $S = 100,000,000\,\mathrm{N/m}$	Stiff support [15] ($S = \infty$)
f_1	18.738	18.738	18.614
f_2	28.004	29.143	29.122
f_3	74.802	74.802	74.566
f_4	80.317	93.306	94.362

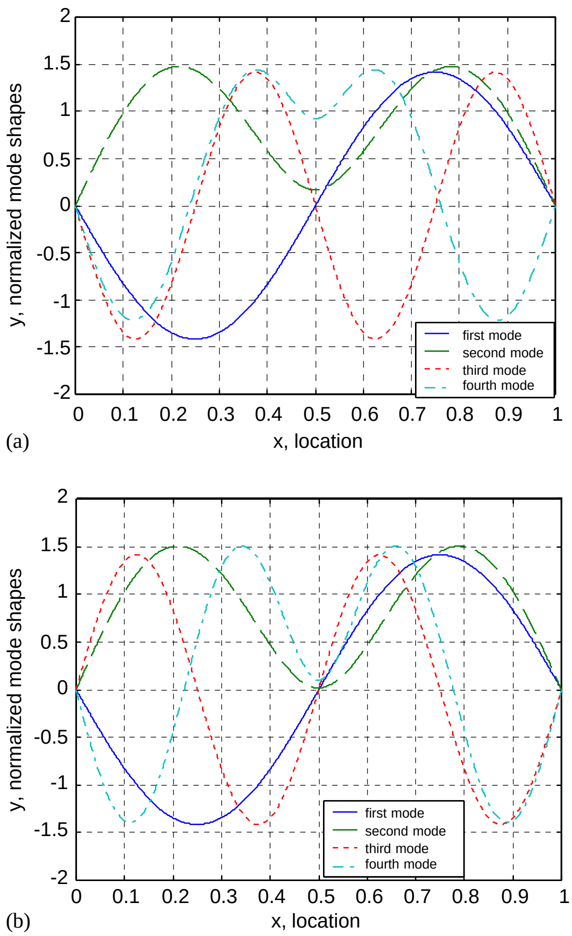

Fig. 5. Variations of lowest four mode shapes for large support stiffness: (a) stiffness $S = 10,000,000\,\mathrm{N/m}$, (b) stiffness $S = 100,000,000\,\mathrm{N/m}$.

third frequencies increase as the stiffness of the support increases. Fig. 4a–c shows the lowest 4 mode shapes for different constraint stiffness. Curves in Fig. 4a ($S = 0\,\mathrm{N/m}$) are typical mode shapes for a simply supported beam. Curves in Fig. 4b ($S = 800,000\,\mathrm{N/m}$) and Fig. 4c ($S = 1,000,000\,\mathrm{N/m}$) are shown in positions "A" and "B", respectively, in Fig. 3. Although the shapes in Fig. 4b and Fig. 4c look similar, however, the first mode is symmetric and the second mode is anti-symmetric in Fig. 4b, contradictorily, the first mode is anti-symmetric and the second mode is symmetric in Fig. 4c. There is a "cross over" at point "C" in Fig. 3, and before this point, the first mode is symmetric and after this point, the first mode is anti-symmetric. When the constraint stiffness is large enough, the system can be regarded as a system with an intermediate stiff support (deflection of the intermediate support $\delta = 0$). Table 1 shows the comparison results of the stiff support from [15] and the results from this article for cases of large constraint stiffness

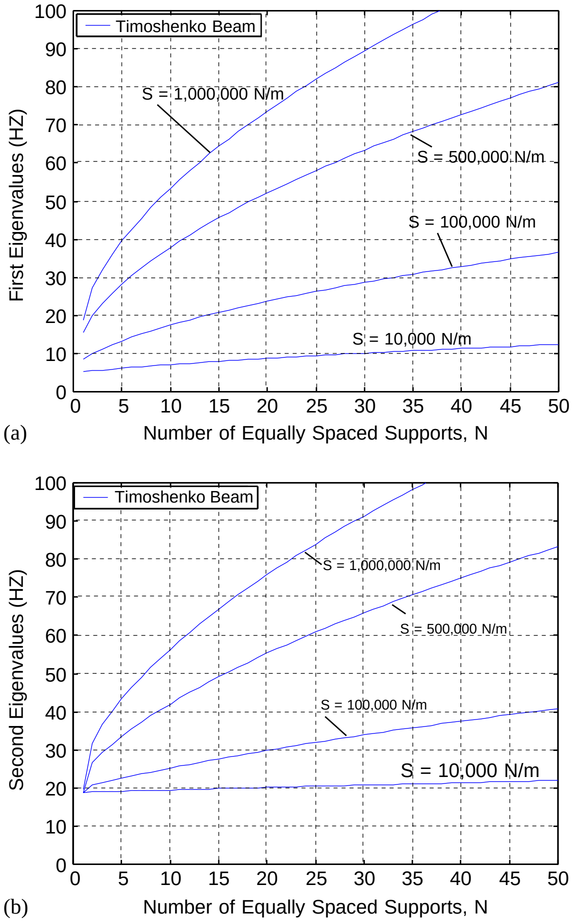

Fig. 6. Variations of first and second eigenvalues of a multi-span simply supported beam with equal spans as the number of supports N varies ($S = 10,000, 100,000, 500,000$ and $1{,}000{,}000\,\mathrm{N/m}$). (a) First eigenvalues, (b) second eigenvalues.

for the same example shown above. Fig. 5a and 5b shows the mode shapes for the cases of large constraint stiffness, $S = 10,000,000\,\text{N/m}$ and $S = 100,000,000\,\text{N/m}$, respectively.

For multi-span example cases, the system is the same described above, but increasing the number of constraints. Fig. 6a and 6b shows the variations of the first and second natural frequencies by increasing the number of flexible supports N for different support stiffness ($S = 10000, 100000, 500000\text{N/m}$ and $1000000\,\text{N/m}$). Each support is assumed to have the same stiffness and with equal spans. As the number of supports is increased, the spans become shorter and shorter. When a span is short enough, then the shear deformation effects cannot be ignored, in which case the results of the Euler–Bernoulli beam model are no longer valid. The Timoshenko beam model, in which the shear deformation effect has been considered, and, thus, its applications are much wider than those of the traditional Euler–Bernoulli beam model.

5. Conclusions

A hybrid analytical/numerical solution method is developed that permits the efficient evaluation of eigensolutions for a vibration beam with an arbitrary finite number of flexible constraints. The method utilizes a numerical implementation of a transfer matrix solution to an analytical form of the equation of motion. There are only four undetermined coefficients in the method proposed in this article which can be solved by the application of boundary conditions. The dimension of the matrix is independent of the number of constraints in this method. The main feature of this method is to decrease the dimension of the matrix involved in the finite element methods and some other analytical methods.

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Appendix A. Transfer matrix derivation

The compatibility conditions across the ith support ($i = 1, 2, \ldots, k$) are represented in Eqs. (8a)–(8d).

$$w_{(i)}(x_i^-) = w_{(i+1)}(x_i^+), \tag{8a}$$

$$w'_{(i)}(x_i^-) = w'_{(i+1)}(x_i^+), \tag{8b}$$

$$\varphi'_{(i)}(x_i^-) = \varphi'_{(i+1)}(x_i^+), \tag{8c}$$

$$w'_{(i)}(x_i^-) - \varphi_{(i)}(x_i^-) = w'_{(i+1)}(x_i^+) - \varphi_{(i+1)}(x_i^+) - s_i w(x_i). \tag{8d}$$

By using the general solutions, Eqs. (9a) and (9b), the above equations can be expressed as

$$A_{i+1} + C_{i+1} = A_i \cosh \lambda_1 l_i + B_i \sinh \lambda_1 l_i + C_i \cos \lambda_2 l_i + D_i \sin \lambda_2 l_i, \tag{A.1}$$

$$B_{i+1}\lambda_1 + D_{i+1}\lambda_2 = A_i\lambda_1 \sinh \lambda_1 l_i + B_i\lambda_1 \cosh \lambda_1 l_i - C_i\lambda_2 \sin \lambda_2 l_i + D_i\lambda_2 \cos \lambda_2 l_i, \tag{A.2}$$

$$A_{i+1}q_1\lambda_1 + C_{i+1}q_2\lambda_2 = B_i q_1\lambda_1 \sinh \lambda_1 l_i + A_i q_1\lambda_1 \cosh \lambda_1 l_i + D_i q_2\lambda_2 \sin \lambda_2 l_i + C_i q_2\lambda_2 \sin \lambda_2 l_i, \tag{A.3}$$

$$\begin{aligned} B_{i+1}q_1 - D_{i+1}q_2 = {} & (B_i q_1 \cosh \lambda_1 l_i + A_i q_1 \sinh \lambda_1 l_i - D_i q_2 \cos \lambda_2 l_i + C_i q_2 \sin \lambda_2 l_i) \\ & - s_i(A_i \cosh \lambda_1 l_i + B_i \sinh \lambda_1 l_i + C_i \cos \lambda_2 l_i + D_i \sin \lambda_2 l_i). \end{aligned} \tag{A.4}$$

Solving for Eqs. (A.1)–(A.4) leads to the following recursion formulae for the constants $A_{i+1}, B_{i+1}, C_{i+1}$ and D_{i+1}:

$$\begin{Bmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ \vdots & & & \\ & & & \\ \cdots & & & t_{44} \end{bmatrix}^{(i)} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix} = \mathbf{T}_{4\times4}^{(i)} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix}, \quad i = 1, 2, \ldots, k.$$

Here, $\mathbf{T}_{4\times4}^{(i)}$ is a transfer matrix composed of the elements

$$\begin{aligned}
t_{11} &= \cosh \lambda_1 l_i, \\
t_{12} &= \sinh \lambda_1 l_i, \\
t_{13} &= 0, \\
t_{14} &= 0, \\
t_{21} &= \sinh \lambda_1 l_i - \frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1} \cosh \lambda_1 l_i, \\
t_{22} &= \cosh \lambda_1 l_i - \frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1} \sinh \lambda_1 l_i, \\
t_{23} &= -\frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1} \cos \lambda_1 l_i, \\
t_{24} &= -\frac{\lambda_2 s_i}{\lambda_1 q_2 + \lambda_2 q_1} \sin \lambda_1 l_i, \\
t_{31} &= 0, \\
t_{32} &= 0, \\
t_{33} &= \cos \lambda_1 l_i, \\
t_{34} &= \sin \lambda_1 l_i, \\
t_{41} &= \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2} \cosh \lambda_1 l_i,
\end{aligned}$$

$$t_{42} = \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2} \sinh \lambda_1 l_i,$$
$$t_{43} = -\sin \lambda_2 l_i + \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2} \cos \lambda_2 l_i,$$
$$t_{44} = \cos \lambda_2 l_i + \frac{\lambda_1 s_i}{\lambda_2 q_1 + \lambda_1 q_2} \sin \lambda_2 l_i.$$

References

[1] L. Fryba, *Vibration of Solids and Structures under Moving Loads*, Noordhoff International Publishing, Groningen, The Netherlands, 1972.
[2] L. Fryba, *Dynamics of Railway Bridges*, Thomas Telford Services, Prague, 1996.
[3] S.P. Timoshenko, On the forced vibration of bridges, *Philosophical Magazine* 6 (1922) 1018.
[4] Y. Cai, S.S. Chen, D.M. Rote, H.T. Coffey, Vehicle/guide interaction for high speed vehicles on flexible guideway, *Journal of Sound and Vibration* 175 (1994) 625–646.
[5] H.P. Lee, Transverse vibration of a Timoshenko beam acted on by an accelerating mass, *Applied Acoustics* 47 (1996) 319–330.
[6] H.P. Lee, Dynamic response of a beam with intermediate point constraints subject to a moving load, *Journal of Sound and Vibration* 171 (1994) 361–368.
[7] R.T. Wang, Vibration of multi-span Timoshenko beams to a moving force, *Journal of Sound and Vibration* 207 (1997) 731–742.
[8] Y.B. Yang, S.S. Liao, B.H. Lin, Impact formulas for vehicle moving over simple and continuous beams, *Journal of Structural Engineering* 121 (1995) 1644–1650.
[9] P.K. Chatterjee, T.K. Datta, C.S. Surana, Vibration of continuous bridges under moving vehicle, *Journal of Sound and Vibration* 169 (1994) 619–632.
[10] M. Ichikawa, Y. Miyakawa, A. Matsuda, Vibration analysis of the continuous beam subjected to a moving mass, *Journal of Sound and Vibration* 230 (2000) 493–506.
[11] I.J. Shames, C.L. Dym, *Energy and Finite Element Methods in Structural Mechanics*, McGraw-Hill, New York, 1985.
[12] H.P. Lin, S.C. Chang, J.D. Wu, Beam vibrations with an arbitrary number of cracks, *Journal of Sound and Vibration* 258 (2002) 987–999.
[13] H.P. Lin, C.K. Chen, Analysis of cracked beams by transfer matrix method, *The 25th National Conference on Theoretical and Applied Mechanics*, Taichung, Taiwan, 2001, pp. 3123–3132.
[14] T.C. Tsai, Y.Z. Wang, Vibration analysis and diagnosis of a cracked shaft, *Journal of Sound and Vibration* 192 (1996) 607–620.
[15] R.D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Krieger Melbourne, FL, 1979.