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# The vibrations of a real 3-string: the timbre of the tritare

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**Abstract**

The tritare is a new stringed musical instrument comprised of six networks of strings instead of six single strings. Each of the networks is called 3-string. We analyze the timbre of the tritare by studying the vibrations of the 3-strings. We show that for a real 3-string, i.e. a physical model rather than a theoretical model, the frequency spectrum is composed of only non-harmonic frequencies which leads to a very unique tone color.


## 1. Introduction

Stringed instruments have played a central role in the history of music and in contemporary music. Among stringed instruments, the guitar is one of the most widely played. Many types of guitars have been invented to take advantage of one or more aspects of string vibrations. For example, acoustic guitars can be considered as a system of coupled vibrators where the string vibrations excite those of the set of pieces making up its resonator. By comparison, the string vibrations on an electric guitar transfer almost no energy to its body. In the present paper, we investigate the timbral properties of a new stringed instrument called the tritare.

Vibrations in all traditional stringed instruments occur along string sections whose two extremities are fixed. The new stringed instrument, which we have called tritare, is comprised of planar networks of string sections instead of single string sections. Each network of strings

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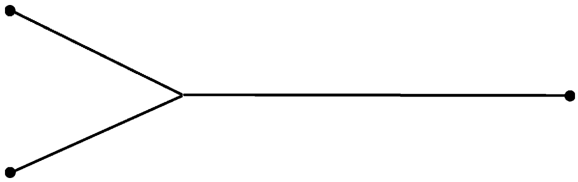

Fig. 1. Schematic of one of the six superposed 3-strings of the tritare, which vibrate perpendicularly to the planes containing them at rest.

consists of three tightly stretched flexible sections of string connected at one common extremity. This common extremity, or junction point, is mobile while all other extremities are fixed (see Fig. 1). The acoustic properties of these networks, called 3-strings, can be studied by analyzing the small amplitude vibrations that are perpendicular to the plane containing the network at rest. The numbers of studies on the vibrations of string networks are still very limited. In-depth mathematical investigations of both the linear [1], and nonlinear [2], vibrations of networks of the above kind have been carried out. The present papers aim is to apply certain results of Ref. [1] to the specific 3-strings used for the tritare. We shall explain why certain important acoustic features of the theoretical model of a 3-string considered in Ref. [1] do not apply to all material 3-strings which can effectively be constructed. In particular, we shall see that the tritare's networks only vibrate according to a spectrum of essentially non-harmonic frequencies. A certain number of particular 3-strings composed of three strings with different linear densities will be examined to exemplify possible timbres of the tritare. The uniqueness of the sounds they produce will be highlighted using Fourier analysis.

The rest of this paper is organized as follows. To have a clear idea of what distinguishes the vibrations of the tritare networks from those of an ordinary string, we recall in Section 2 the main aspects of the vibrations of the standard vibrating string. The vibrations of a 3-string are described in general terms in Section 3. In Section 4, we apply the results of Section 3 to two sets of 3-strings to illustrate the timbral possibilities for the tritare. Finally, Section 5 contains the conclusion and a short discussion on perspectives.

## 2. The vibrations of a standard string

In this section, we recall the small-amplitude transverse vibrations of a homogenous string, which is assumed to be perfectly flexible and elastic. The string is under uniform tension $\tau$ and, in its equilibrium position, lies along the $x$-axis between $x = 0$ and $L$, where it is fixed. Let $\rho$ be the linear density of the string. We assume that all particles of the string move in one plane and the tension $\tau$ is sufficiently large for the effects of gravity to be negligible. All internal and external frictions are also neglected.

Under the above assumptions, it is easily shown that the displacement at time $t$ of the point on the string whose abscissa is $x$, denoted $u(x,t)$, verifies the wave equation [3]

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where $c = \sqrt{\tau/\rho}$. The function $u(x, t)$ must also satisfy the boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0. \tag{2}$$

The vibrations begin at a certain instant of time, usually selected to be $t = 0$. The initial conditions of the string are usually expressed as

$$u(x, 0) = F(x), \quad u_t(x, 0) = G(x), \tag{3}$$

where $F(x)$ and $G(x)$, $0 \leqslant x \leqslant L$, are given functions representing the displacements and the velocities of all points along the string at time $t = 0$.

Using the classical method of separation of variables to solve Eqs. (1)–(3), we set

$$u(x, t) = X(x)T(t),$$

where $X$ is a function of $x$ only and $T$ is a function of $t$ only. Letting $-\lambda^2$ represent the separation constant, it is straightforward to show that the eigenvalues of the above problem are $\lambda = n\pi/L$, $n \in \mathbf{N}^*$. Summing over all possible values of $\lambda$, it follows that the most general expression for $u(x, t)$ is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right),$$

where $a_n$ and $b_n$ are constants determined by the initial conditions (3).

The frequency $f_1 = c/2L$ corresponding to $n = 1$ is the frequency associated with the fundamental mode of vibration. Those corresponding to $n > 1$, i.e. $f_n = nc/2L$, are called harmonics of the fundamental, because each $f_n$ is an integer multiple of $f_1$. The timbre of a note played on a given instrument depends on the relative magnitudes of the vibration modes, $a_n$ and $b_n$, corresponding to the different harmonics. It is this relationship between the vibration modes that distinguishes the timbre of different musical instruments. One way to measure the contribution of the different modes, and therefore to analyze the timbre of a sound, is to calculate the proportion of the sound's total energy present in each mode.

The kinetic $K_n$ and potential $V_n$ energies of the $n$th mode, $n \in \mathbf{N}^*$, are given by

$$K_n = \frac{\rho}{2}\int_0^L [(u_n)_t(x, t)]^2 \,\mathrm{d}x$$

and

$$V_n = \frac{\rho c^2}{2}\int_0^L [(u_n)_x(x, t)]^2 \,\mathrm{d}x.$$

The total energy $E_n$ of the $n$th harmonic is then simply the sum $K_n + V_n$. The total energy $E$ of the string is the sum of each $E_n$ for $n \in \mathbf{N}^*$.

Let us now consider the energies associated with the vibrations of a string whose initial conditions correspond to plucking the string to the height $h$ at $x = (L/2)(1 + 1/m)$, where $m \in \mathbf{R}$, $m > 1$. The energies $K_n$, $V_n$ and $E_n$ of the $n$th mode are then functions of $m$ and $h$. It is easy to

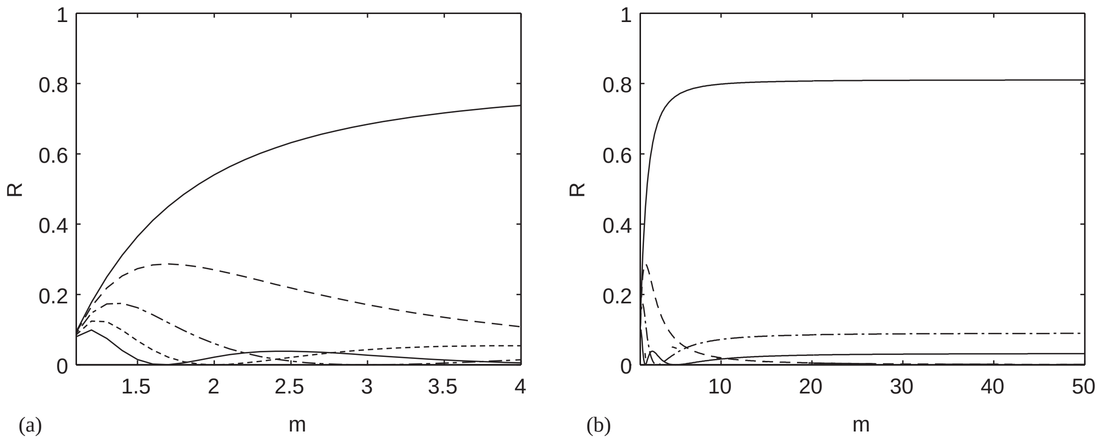


Fig. 2. $R(m,n)$ in terms of $m$. The curve legends, are ——, $n=1$; – – –, 2; – · –, 3; · · ··, 4; ——, 5.

show that

$$E_n(m,h)=\begin{cases}\dfrac{16\rho c^2h^2m^4(\cos n\pi/2m)^2}{\pi^2Ln^2(m^2-1)^2} & \text{if } n \text{ is odd,}\\[2ex] \dfrac{16\rho c^2h^2m^4(\sin n\pi/2m)^2}{\pi^2Ln^2(m^2-1)^2} & \text{if } n \text{ is even.}\end{cases}$$

Now, since we have conservation of energy, the total energy $E(m,h)$ of all the modes of the string is the same as its initial energy, which is the potential energy associated with its initial displacement. One easily shows that

$$E(m,h)=\frac{2\rho c^2m^2h^2}{L(m^2-1)}.$$

The fraction of total energy in the $n$th harmonic is therefore given by

$$R(m,n)=\frac{E_n(m,h)}{E(m,h)}=\begin{cases}\dfrac{8m^2(\cos n\pi/2m)^2}{\pi^2n^2(m^2-1)} & \text{if } n \text{ is odd,}\\[2ex] \dfrac{8m^2(\sin n\pi/2m)^2}{\pi^2n^2(m^2-1)} & \text{if } n \text{ is even.}\end{cases} \tag{4}$$

The graphs in Fig. 2 gives $R(m,n)$ in terms of $m$ for $n=1,2,3,4,5$. Observe that $m\to\infty$ corresponds to plucking the string at $x=L/2$ and it is well known that the corresponding frequency spectrum does not contain any even-numbered harmonics.

## 3. The vibrations of a real 3-string

The sounds produced by the tritare can be modeled as the small-amplitude transverse oscillations of one or more of its six 3-strings. Each of these 3-strings can be comprised of sections of strings of different linear densities and these densities can also vary from one 3-string to another.

Let $x_i$ be the arclength parameter for the $i$th string at rest whose length at rest is assumed to be $l_i$, $i = 1, 2, 3$. We also assume that the junction point of the sections of string is located at $x_i = 0$, $i = 1, 2, 3$. Therefore $0 \leqslant x_i \leqslant l_i$. Let $u^i(x_i, t)$ denote the functions which give the deviation from the rest position at time $t$ of the point at arclength position $x_i$ on the $i$th string. Also, let $\rho_i$ and $\tau_i$ be the linear density and the tension, respectively, in the $i$th string.

Under the same assumptions as those applied in Section 2, one can show that the following equations describe the vibrations of the 3-string [1]

$$u^i_{tt} = c_i^2 u^i_{x_i x_i}, \quad i = 1, 2, 3, \tag{5}$$

$$u^1(0, t) = u^2(0, t) = u^3(0, t), \tag{6}$$

$$u^i(l_i, t) = 0, \quad i = 1, 2, 3, \tag{7}$$

$$\sum_{i=1}^{3} c_i^2 \rho_i u^i_{x_i}(0, t) = 0, \tag{8}$$

where $c_i = \sqrt{\tau_i/\rho_i}$. We are interested in the solution to problem (5)–(8) subjected to the initial conditions

$$u^i(x_i, 0) = F^i(x_i), \quad u^i_t(x_i, 0) = G^i(x_i), \quad i = 1, 2, 3, \tag{9}$$

where the functions $F^i$ and $G^i$ are respectively continuous and piecewise continuous on $[0, l_i]$.

A normalization of the strings facilitates the solution of the above problem. Let $x = \pi x_i/l_i$ for $i = 1, 2, 3$. It follows that $0 \leqslant x \leqslant \pi$ parameterizes each string. If one defines $v^i(x, t) = u^i(l_i x/\pi, t)$ for $i = 1, 2, 3$, then problem (5)–(9) becomes

$$v^i_{tt} = \frac{\pi^2 c_i^2}{l_i^2} v^i_{xx}, \quad i = 1, 2, 3, \tag{10}$$

$$v^1(0, t) = v^2(0, t) = v^3(0, t), \tag{11}$$

$$v^i(\pi, t) = 0, \quad i = 1, 2, 3, \tag{12}$$

$$\sum_{i=1}^{3} \frac{c_i^2 \rho_i}{l_i} v^i_x(0, t) = 0, \tag{13}$$

$$v^i(x, 0) = F^i(l_i x/\pi), \quad v^i_t(x, 0) = G^i(l_i x/\pi), \quad i = 1, 2, 3. \tag{14}$$

To solve Eqs. (10)–(14), we use the method of separation of variables. We therefore set

$$v^i(x, t) = X^i(x) T(t), \quad i = 1, 2, 3, \tag{15}$$

where the $X^i$ are functions of $x$ only and $T$ is a function of $t$ only. The substitution of Eq. (15) into Eqs. (10)–(14) produces

$$T(t) = K_1 \cos \lambda t + K_2 \sin \lambda t,$$

$$X^i(x) = A^i \cos \frac{\lambda l_i x}{\pi c_i} + B^i \sin \frac{\lambda l_i x}{\pi c_i}, \quad i = 1, 2, 3,$$

where $-\lambda^2$ is the separation constant and $K_1, K_2, A^i, B^i$, $i = 1, 2, 3$, are arbitrary constants. Condition (11) leads to

$$A^1 = A^2 = A^3, \tag{16}$$

while Eq. (13) gives

$$\sum_{i=1}^{3} v_i B^i = 0, \tag{17}$$

where we define $v_i = c_i \rho_i$. Finally, Eqs. (12), (16) and (17) give the following system of equations:

$$\begin{aligned} &A^1 \cos \frac{l_1 \lambda}{c_1} - \frac{1}{v_1} (B^2 v_2 + B^3 v_3) \sin \frac{l_1 \lambda}{c_1} = 0, \\ &A^1 \cos \frac{l_i \lambda}{c_i} + B^i \sin \frac{l_i \lambda}{c_i} = 0, \quad i = 2, 3. \end{aligned} \tag{18}$$

We shall have a non-trivial solution to Eqs. (10)–(14) if and only if $\lambda$ is such that Eq. (18) has a non-trivial solution for $A^1$, $B^2$ and $B^3$. It is easy to see that this occurs if and only if $\lambda$ is such that

$$\sum_{i=1}^{3} \left[ \frac{v_i}{v_1} \cos\left( \frac{l_i \lambda}{c_i} \right) \prod_{\substack{j=1 \\ j \neq i}}^{3} \sin\left( \frac{l_j \lambda}{c_j} \right) \right] = 0. \tag{19}$$

This equation has an infinite number of roots $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \lambda_{n+1} \leqslant \cdots$, with $\lim_{n\to\infty} \lambda_n = \infty$, whose actual values depend on the values of the $v_i$, $l_i$ and $c_i$.

Let $\lambda$ be a solution to Eq. (19). Then either $\sin(l_i \lambda / c_i) \neq 0$ for $i = 1, 2, 3$, or there exists an $i_1 \in \{1, 2, 3\}$ such that $\sin(l_{i_1} \lambda / c_{i_1}) = 0$. These two cases are linked to whether or not the real numbers $c_i / l_i$, $i = 1, 2, 3$ are incommensurate.

We first consider the case where some of the $c_i/l_i$ are commensurate, i.e. when $c_i l_j / c_j l_i \in \mathbf{Q}$ for some $i, j = 1, 2, 3$, $i \neq j$. Recall that we are examining the situation corresponding to a real 3-string, i.e. a material 3-string constructed by a human being or a machine. Such a 3-string will have three sections of strings whose lengths, tensions and linear densities are described by numbers in $\mathbf{R}$. To build a 3-string where some of the $c_i/l_i$, $i = 1, 2, 3$, are commensurate thus corresponds to randomly choosing a rational number in a given interval of real numbers. But it is well known that the set of rational numbers in any given interval of real numbers is a set of measure zero. The building of a 3-string thus almost surely leads to incommensurate $c_i/l_i$, $i = 1, 2, 3$, [4]. Consequently, even if the commensurate case is a mathematical possibility, it is not interesting in practice.

We can therefore restrict our attention to the case where the numbers $c_i/l_i$, $i = 1, 2, 3$, are incommensurate, i.e. $c_i l_j / c_j l_i \notin \mathbf{Q}$ for all $i, j = 1, 2, 3$, $i \neq j$. Let $\lambda$ be a solution of Eq. (19) such that there exists an $i_1 \in \{1,2,3\}$ with $\sin(l_{i_1}\lambda/c_{i_1}) = 0$. Then $\lambda = n_1\pi c_{i_1}/l_{i_1}$ for some positive integer $n_1$. In this case Eq. (19) reduces to

$$\frac{v_{i_1}}{v_1}\cos n_1\pi \prod_{\substack{j=1\\ j\neq i_1}}^{3} \sin\left(\frac{c_{i_1}l_j}{c_j l_{i_1}}\, n_1\pi\right) = 0,$$

from which it follows that there exists an $i_2 \in \{1,2,3\}$, $i_2 \neq i_1$, and a positive integer $n_2 = c_{i_1}l_{i_2}n_1/c_{i_2}l_{i_1}$. The numbers $c_{i_1}/l_{i_1}$ and $c_{i_2}/l_{i_2}$ would thus be commensurate, which is a contradiction. Consequently, Eq. (19) has an infinite number of solutions $\lambda_k = \alpha_k$, $k \in \mathbf{N}^*$, and each $\alpha_k$ is such that $\sin(l_i\alpha_k/c_i) \neq 0$ for $i = 1,2,3$. The corresponding eigenvalue is simple and the eigenfunction is given by

$$P_k(x) = \left[\cos\frac{l_1\alpha_k x}{\pi c_1} + \left(\frac{v_2}{v_1}\cot\frac{l_2\alpha_k}{c_2} + \frac{v_3}{v_1}\cot\frac{l_3\alpha_k}{c_3}\right)\sin\frac{l_1\alpha_k x}{\pi c_1},\right.$$
$$\left.\cos\frac{l_2\alpha_k x}{\pi c_2} - \left(\cot\frac{l_2\alpha_k}{c_2}\right)\sin\frac{l_2\alpha_k x}{\pi c_2}, \cos\frac{l_3\alpha_k x}{\pi c_3} - \left(\cot\frac{l_3\alpha_3}{c_3}\right)\sin\frac{l_3\alpha_k x}{\pi c_3}\right]^{\mathrm{T}}.$$

The solution of Eqs. (10)–(14) can then be written as [1]

$$[v^1(x,t), v^2(x,t), v^3(x,t)]^{\mathrm{T}} = \sum_{k=1}^{\infty}(a_k\cos\alpha_k t + \hat{a}_k\sin\alpha_k t)P_k(x),$$

where

$$a_k = \frac{\langle\langle F, P_k\rangle\rangle}{\langle\langle P_k, P_k\rangle\rangle}, \quad \hat{a}_k = \frac{\langle\langle G, P_k\rangle\rangle}{\alpha_k\langle\langle P_k, P_k\rangle\rangle}$$

for

$$F(x) = [F^1(l_1x/\pi), F^2(l_2x/\pi), F^3(l_3x/\pi)]^{\mathrm{T}},$$

$$G(x) = [G^1(l_1x/\pi), F^2(l_2x/\pi), G^3(l_3x/\pi)]^{\mathrm{T}},$$

and where the scalar product is defined by

$$\langle\langle \Phi(x), \Psi(x)\rangle\rangle = \int_0^{\pi}\left(\sum_{i=1}^{3} l_i\rho_i\phi_i(x)\psi_i(x)\right)\mathrm{d}x$$

for $\Phi(x) = [\phi_1(x), \phi_2(x), \phi_3(x)]^{\mathrm{T}}$ and $\Psi(x) = [\psi_1(x), \psi_2(x), \psi_3(x)]^{\mathrm{T}}$. The solution of problem (5)–(9) is finally obtained through the substitution

$$u^i(x_i,t) = v^i(\pi x_i/l_i, t), \quad i = 1,2,3.$$

To determine the energy of the $n$th mode in a vibrating 3-string, one has to sum the kinetic $K_n^i$ and the potential $V_n^i$ energies of this mode in the three sections of string $i = 1,2,3$. These energies are respectively given by

$$K_n^i = \frac{\rho}{2}\int_0^l [(u_n^i)_t(x_i,t)]^2\,\mathrm{d}x_i$$

and

$$V_n^i = \frac{\rho\, c^2}{2}\int_0^l [(u_n^i)_x(x_i,t)]^2\,\mathrm{d}x_i.$$

The total energy $\mathscr{E}_n(m,h)$ of the $n$th mode is thus

$$\mathscr{E}_n(m,h) = \sum_{i=1}^{3}(K_n^i + V_n^i).$$

In the next section, we shall use the ratio $\mathscr{R}(n,m) = \mathscr{E}_n(m,h)/\mathscr{E}(m,h)$, where $\mathscr{E}(m,h)$ is the total energy of all modes of the 3-string. In the case where the 3-string is initially plucked, the energy $\mathscr{E}(m,h)$ is always equal to the potential energy associated with the initial displacement of the 3-string.

## 4. The tritare's timbre

The six 3-strings of the tritare have the same geometric configuration. In each 3-string, the three sections of string form two angles of $5\pi/6$ at the junction point and one angle of $\pi/3$. The length of the string section between the two angles of $5\pi/6$, which is denoted $l_1$, is approximately twice as long as the two other sections $l_2$ and $l_3$, which are approximately equal to a determined constant $l$ (see Fig. 1). The tensions $\tau_1$, $\tau_2$, $\tau_3$ in the 3-strings are thus related by approximately $\tau_1 = \tau\sqrt{3}$, $\tau_2 = \tau_3 = \tau$, where $\tau$ is a determined constant. The three string sections of each 3-string can have different linear densities. Frets are installed under the first and longest string section. As mentioned previously, in practice, the lengths, tensions and linear densities of the three sections of the 3-string, as well as the fret placements, are such that the numbers $c_i/l_i$ with $l_1 \leqslant 2$ are almost surely incommensurate.

To analyze the timbre of a 3-string, we shall now consider the first five modes of vibration resulting from the plucking of its first or third section to a height $h$ at $x_j = l_j/m$, where $m \in \mathbf{R}$, $m > 1$, and $j = 1$ or $j = 3$. The following cases are examples of 3-strings which could be installed on a tritare. We shall show the graphs representing the portion of the total energy in each of the first five modes. The frequencies of the modes will be given in the captions of the figures. These frequencies are expressed in terms of the constants $l$ and $c = \sqrt{\tau/\rho}$, where $\tau$ is the tension in the second and third string sections of the 3-string.

We start with four cases which correspond to plucking the tritare in an open position (no frets used) for different string densities where $l_1 = 2l$ and $l_2 = l_3 = l$. The linear density values used are $\rho_1 = \rho_2 = \rho_3 = \rho$ (Fig. 3); $\rho_1 = \rho_3 = \rho$, $\rho_2 = 2\rho$ (Fig. 4); $\rho_1 = 4\rho$, $\rho_2 = \rho_3 = \rho$ (Fig. 5); $\rho_1 = 4\rho$, $\rho_2 = 2\rho$, $\rho_3 = \rho$ (Fig. 6). The versions of Eq. (19) used to determine the eigenvalues for these corresponding problems are then, respectively,

$$\cot\left(\frac{2\lambda}{3^{1/4}}\right) + \frac{2}{3^{1/4}}\cot\lambda = 0,$$

$$\cot\left(\frac{2\lambda}{3^{1/4}}\right) + \frac{\sqrt{2}}{3^{1/4}}\cot(\sqrt{2}\lambda) + \frac{1}{3^{1/4}}\cot\lambda = 0,$$

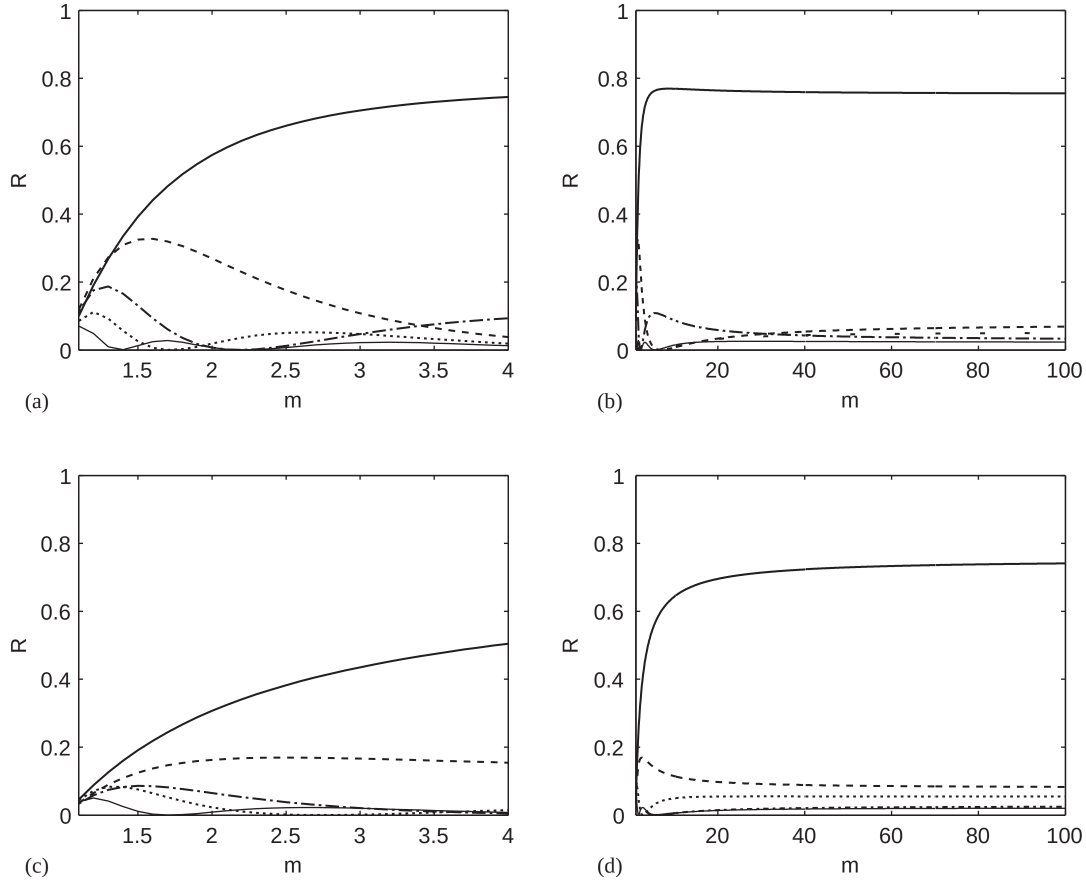


Fig. 3. $\mathscr{R}(m,n)$ in terms of $m$ for $l_1 = 2l$, $l_2 = l_3 = l$, $\rho_1 = \rho_2 = \rho_3 = \rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; – – –, 2; – · –, 3; · · ··, 4; ——, 5. The frequencies of these modes are $f_1 = 1.2981\,c/2\pi l$, $f_2 = 1.8606 f_1$, $f_3 = 2.9396 f_1$, $f_4 = 3.8077 f_1$, $f_5 = 5.8073 f_1$.

$$\cot\left(\frac{4\lambda}{3^{1/4}}\right) + \frac{1}{3^{1/4}}\cot\lambda = 0,$$

$$\cot\left(\frac{4\lambda}{3^{1/4}}\right) + \frac{1}{\sqrt{2}\cdot 3^{1/4}}\cot(\sqrt{2}\lambda) + \frac{1}{2\cdot 3^{1/4}}\cot\lambda = 0.$$

The sought eigenvalues $\lambda_k = \alpha_k$ are in each case given by the positive roots of these equations multiplied by $c$ and divided by $l$ [1].

For a second set of 3-strings, we set $\rho_1 = \rho_3 = \rho, \rho_2 = 3\rho$ and $l_2 = l_3 = l$. These cases correspond to plucking the tritare while fretting section 1, and therefore modifying $l_1$, for the same string densities. The values of $l_1$ considered here are $p\,l/q$ with $p/q = 2, \frac{4}{3}, 1, \frac{2}{3}$. For all these four

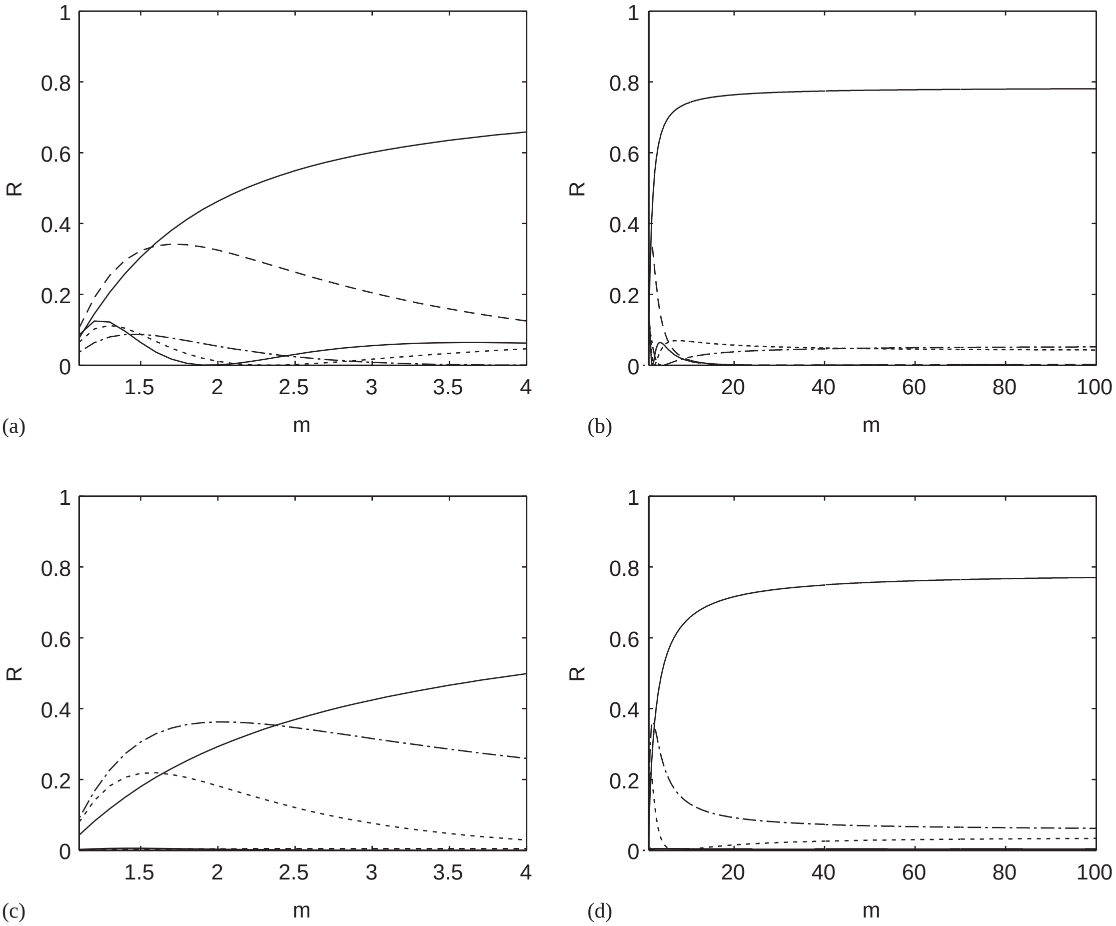


Fig. 4. $\mathcal{R}(m, n)$ in terms of $m$ for $l_1 = 2l$, $l_2 = l_3 = l$, $\rho_1 = \rho_3 = \rho$, $\rho_2 = 2\rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; – – –, 2; – · –, 3; · · ··, 4; ——, 5. The frequencies of these modes are $f_1 = 1.1753\,c/2\pi l$, $f_2 = 1.8182 f_1$, $f_3 = 2.3326 f_1$, $f_4 = 3.0795 f_1$, $f_5 = 3.6480 f_1$.

cases, the equation used to determine the eigenvalues is

$$\cot\left(\frac{p\lambda}{q \cdot 3^{1/4}}\right) + 3^{1/4}\cot(\sqrt{3}\lambda) + \frac{1}{3^{1/4}}\cot\lambda = 0$$

for $p/q = 2, \frac{4}{3}, 1, \frac{2}{3}$. As for the above set of four cases, the sought eigenvalues are in each case given by the positive roots of this equation multiplied by $c$ and divided by $l$. All the above eigenvalues are easily calculated numerically using standard root-finding algorithms.

Let us now comment on the results shown in Figs. 3–10. In all these figures, the fraction of the total energy $\mathcal{R}(m, n)$ contained in the $n$th mode is plotted against $m$ which specifies the location where the 3-string is plucked. A first observation in all these figures when $m \to \infty$ is that all the curves level off and converge to constant values. This result reflects the fact that $m \to \infty$ corresponds to plucking the 3-strings at the junction point.

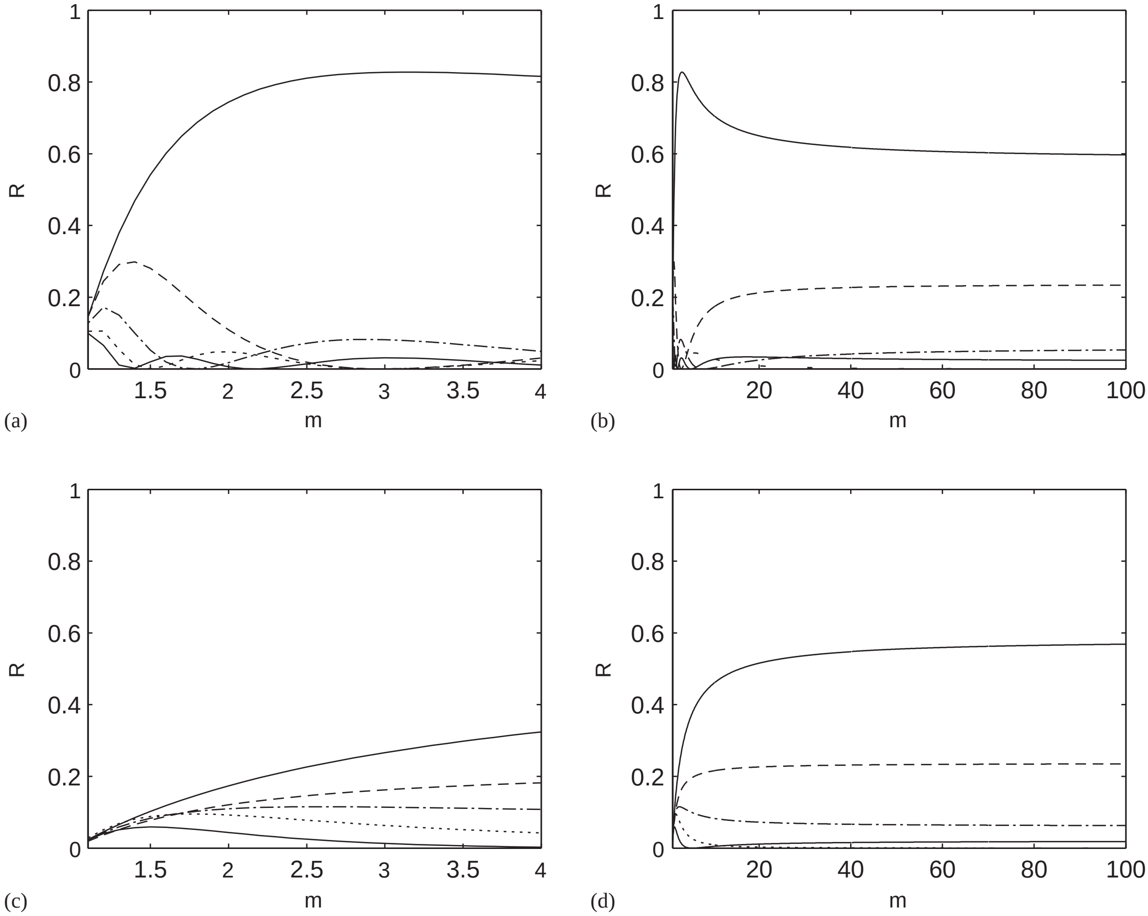


Fig. 5. $\mathcal{R}(m,n)$ in terms of $m$ for $l_1 = 2l$, $l_2 = l_3 = l$, $\rho_1 = 4\rho$, $\rho_2 = \rho_3 = \rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; – – –, 2; – · –, 3; · · ··, 4; ——, 5. The frequencies of these modes are $f_1 = 0.7443\,c/2\pi l, f_2 = 2.0889 f_1, f_3 = 3.1800 f_1, f_4 = 4.1827 f_1, f_5 = 5.1824 f_1$.

It is useful to compare Figs. 3–10 to Fig. 2, which describes the timbre of an ordinary string. Although the fundamental mode $n = 1$ still dominates all other modes in Figs. 3–10 when $m \to \infty$, the portion of energy it contains is always less than in the case of an ordinary string. Also, the portion of the energy contained in the other modes ($n > 1$) is usually very different than for an ordinary string. However, the most dramatic difference remains the fact that the frequencies are non-harmonic because of the nature of the eigenvalue equations.

Figs. 3(a),(b) and 4(a),(b) show that the mode $n = 2$ contains more energy than the fundamental mode for a small interval of $m$, when the first section of the 3-string is plucked. In Fig. 4(c),(d), the modes $n = 3$ and 4 both contain more energy than the fundamental mode for any plucking in approximately the second half ($x_3 > 0.5\,l_3$) of the third section of the 3-string. We also see from Fig. 3 that the first five modes $n = 1, 2, 4, 3, 5$ (in this order) significantly contribute to the timbre

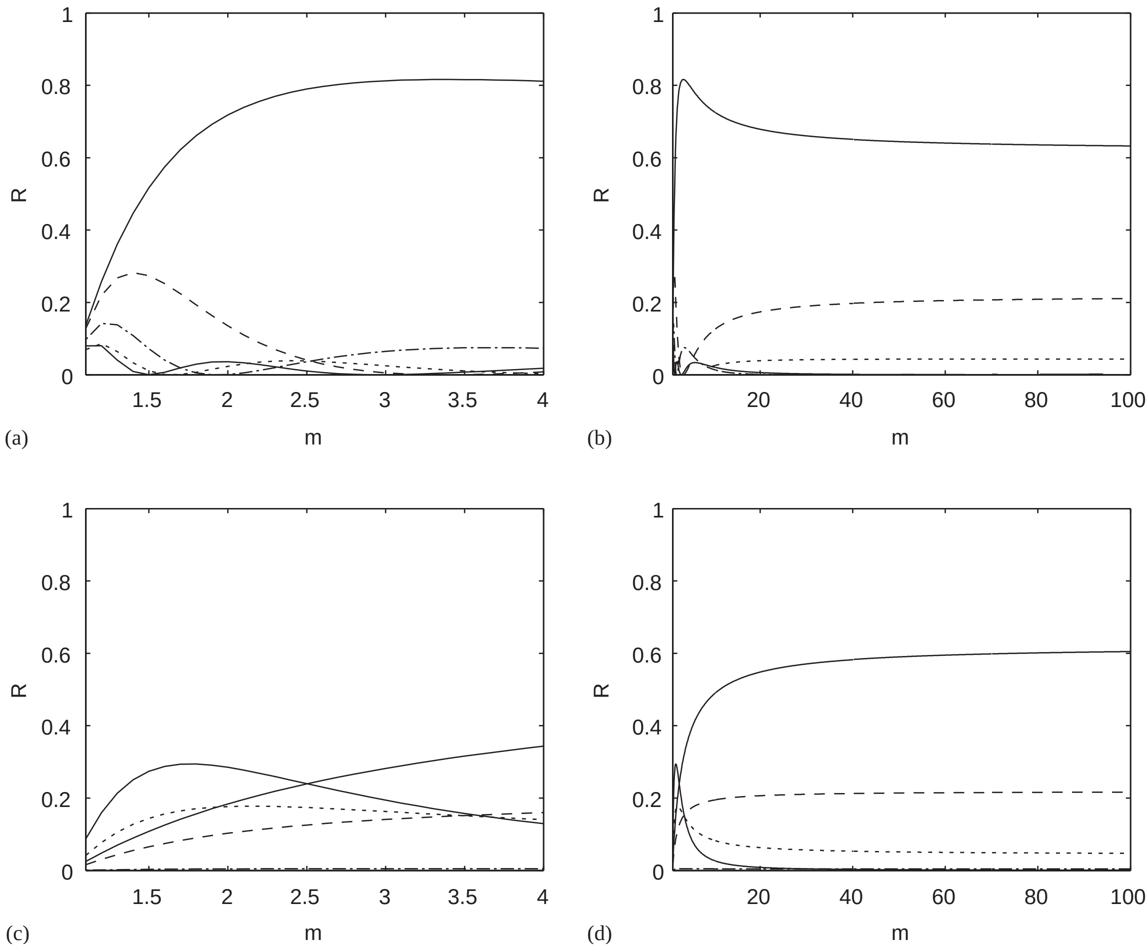


Fig. 6. $\mathscr{R}(m,n)$ in terms of $m$ for $l_1 = 2l$, $l_2 = l_3 = l$, $\rho_1 = 4\rho$, $\rho_2 = 2\rho$, $\rho_3 = \rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; $- - -$, 2; $- \cdot -$, 3; $\cdots$, 4; ——, 5. The frequencies of these modes are $f_1 = 0.7286\, c/2\pi l$, $f_2 = 2.0142 f_1$, $f_3 = 2.9321 f_1$, $f_4 = 3.6110 f_1$, $f_5 = 4.2824 f_1$.

when $m \to \infty$. The situation is different for the ordinary string (Fig. 2), where only the modes $n = 1, 3, 5$, are present when $m \to \infty$. Only the modes $n = 1, 3, 4$ significantly contribute in the case of Fig. 4 when $m \to \infty$.

In Fig. 5 when $m \to \infty$, about 58% and 23% of the total energy is contained in the modes $n = 1$ and 2, respectively, compared to about 81% and 0% for an ordinary string. Fig. 6 corresponds to a case where the mode $n = 3$ contains almost no energy. Also in Fig. 6(c), it is apparent that the mode $n = 5$ dominates all of the other modes in the interval $1 < m \leqslant 2.5$. The above results suggest that by carefully selecting the relative linear densities of the 3-strings and by plucking them at the right locations, one can obtain a desired timbre for the instrument.

For Figs. 7–10, where the string densities are fixed and the length of the first section varies (to illustrate the actual playing of the instrument), we have intervals of $m$ where the modes $n = 2$ (Fig. 7), $n = 3$ (Fig. 8), $n = 4$ (Fig. 9), contain the most energy, when the first section of the 3-string is plucked. The situation is similar but more complex when the third section of the 3-string is plucked. In fact, we note that in some cases, the mode $n = 1$ has less energy than several of the higher modes for certain intervals of $m$. When $m \to \infty$, the most energetic of the modes are $n = 1, 3$ in Fig. 7, $n = 1, 2, 5$ in Fig. 8, $n = 1, 2$ in Fig. 9 and $n = 1, 2, 4$ in Fig. 10. When the third section is plucked in Fig. 7, the modes $n = 2, 5$ are nearly absent for almost all values of $m$. The same observation applies to the mode $n = 4$ in Fig. 9 and to the mode $n = 5$ in Fig. 10, again when the third section is plucked. These results confirm that the tritare's timbre changes as it is played (first section fretted) even if the 3-strings are always plucked in the same location.

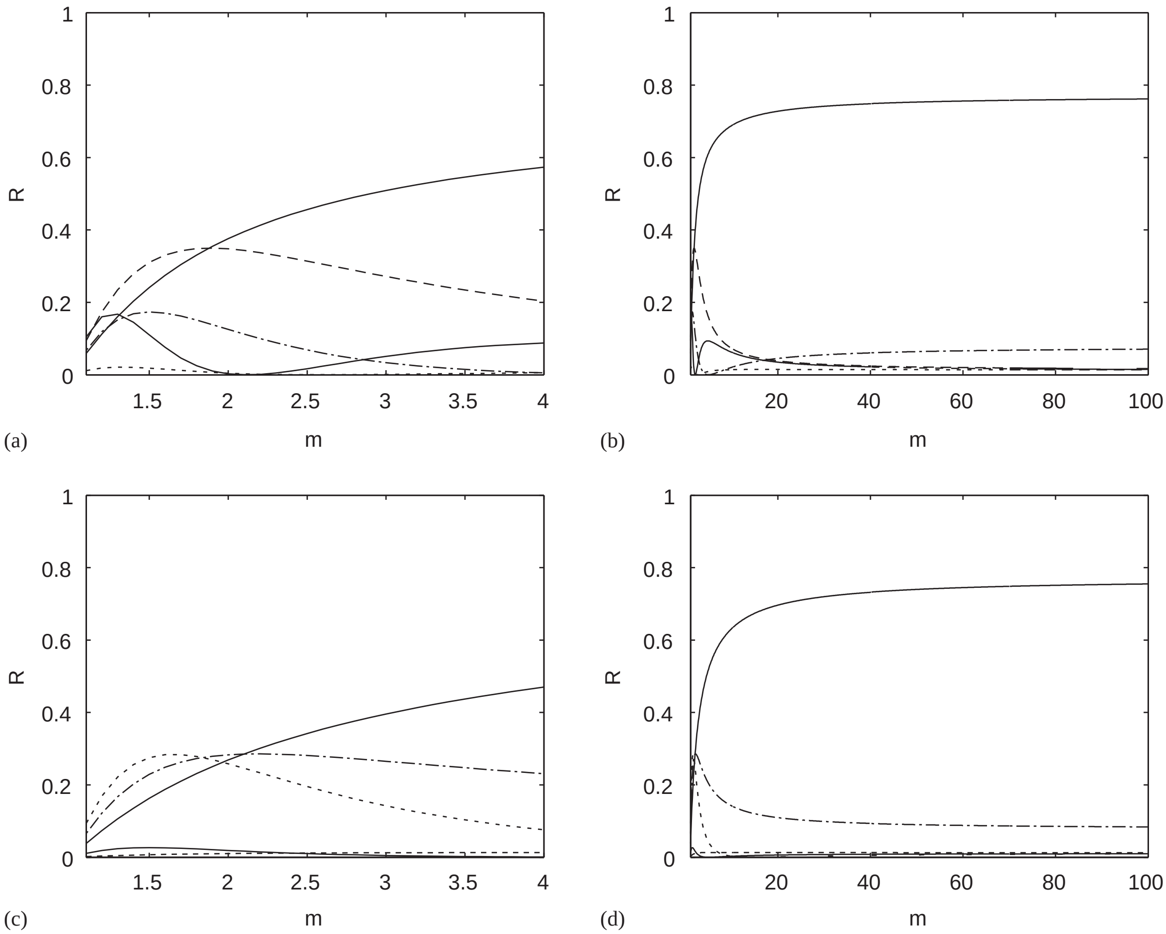


Fig. 7. $\mathscr{R}(m,n)$ in terms of $m$ for $l_1 = 2l$, $l_2 = l_3 = l$, $\rho_1 = \rho_3 = \rho$, $\rho_2 = 3\rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; $- - -$, 2; $- \cdot -$, 3; $\cdots$, 4; ——, 5. The frequencies of these modes are $f_1 = 1.0657\, c/2\pi l$, $f_2 = 1.8258 f_1$, $f_3 = 2.4393 f_1$, $f_4 = 3.1649 f_1$, $f_5 = 3.6871 f_1$.

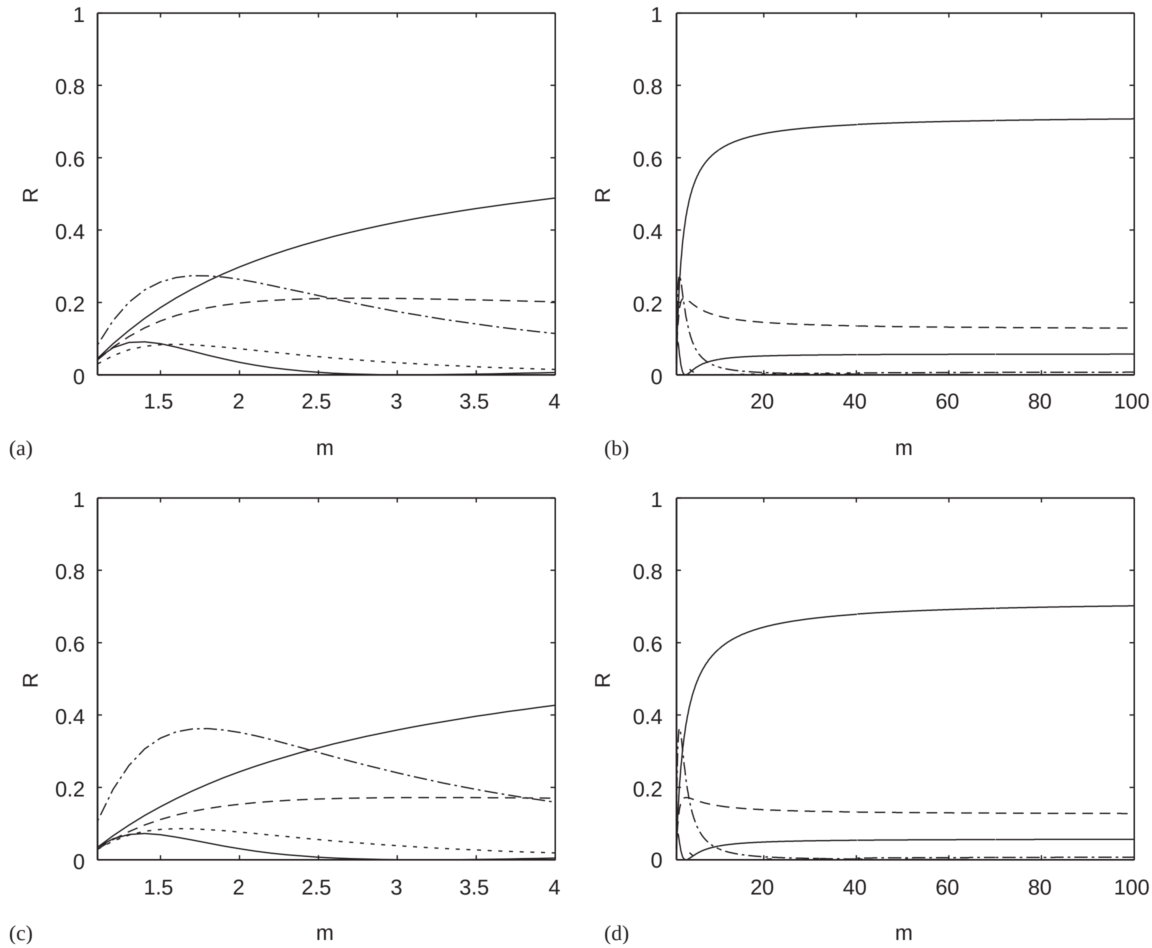


Fig. 8. $\mathcal{R}(m,n)$ in terms of $m$ for $l_1 = 4l/3$, $l_2 = l_3 = l$, $\rho_1 = \rho_3 = \rho$, $\rho_2 = 3\rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; – – –, 2; – · –, 3; · · ··, 4; ——, 5. The frequencies of these modes are $f_1 = 1.1868\, c/2\pi l$, $f_2 = 1.8880 f_1$, $f_3 = 2.6442 f_1$, $f_4 = 2.9252 f_1$, $f_5 = 3.8734 f_1$.

## 5. Conclusion

We have shown that, contrary to ordinary strings, the modes of vibration which occur in a real vibrating 3-string correspond to frequencies which are not integer multiples of a given fundamental frequency. However, in most situations the energy of the fundamental mode is high enough, compared to the energies of the other modes, so that the vibrations of the 3-string sound like that of a definite pitch instrument but without a harmonic or overtone series. This unique frequency spectrum results in a very unique timbre for the tritare. This fact is easily seen when we compare the graphs of Fig. 2 to those of Figs. 3–10.

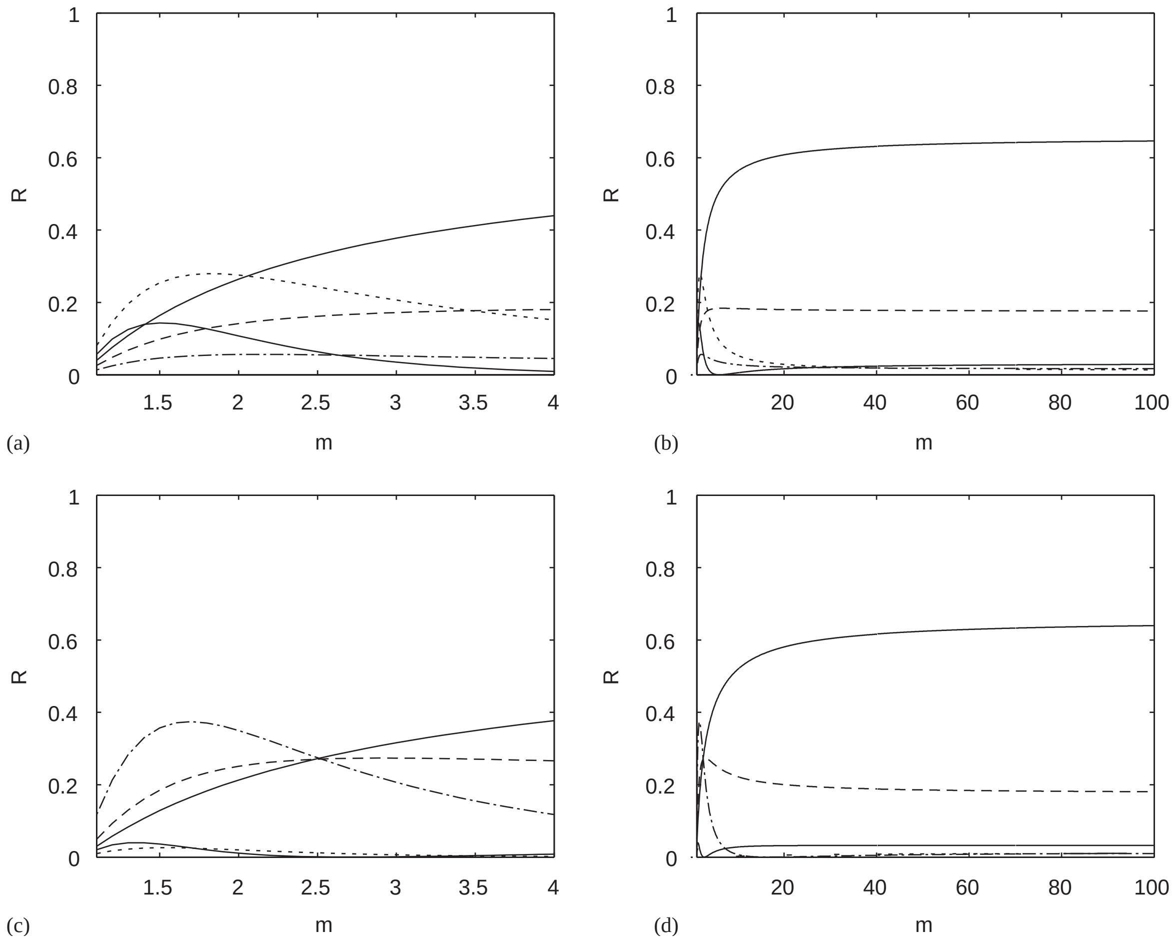


Fig. 9. $\mathscr{R}(m,n)$ in terms of $m$ for $l_1 = l_2 = l_3 = l$, $\rho_1 = \rho_3 = \rho$, $\rho_2 = 3\rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; $- - -$, 2; $- \cdot -$, 3; $\cdot \cdot \cdot \cdot$, 4; ——, 5. The frequencies of these modes are $f_1 = 1.2647\, c/2\pi l$, $f_2 = 1.8602 f_1$, $f_3 = 2.6387 f_1$, $f_4 = 3.0346 f_1$, $f_5 = 3.9210 f_1$.

How many different tritares is it possible to build? If we assume the existence of strings of 20 different linear densities, one can obtain $20^3 = 8000$ distinct 3-strings. A tritare with six different 3-string can then be build in $8000!/7994! = 2.616 \times 10^{23}$ distinct ways. Every person on Earth could thus, in principle, build thousands of billion different unique tritares.

At the time this paper was written, one electrical prototype of the tritare has been built by a professional luthier, based on seven previous prototypes. This new instrument clearly displays a very unique and rich timbre which, when played in certain ways (varying $m$), seems more related to percussion instruments. The tritare could provide innovative musicians and composers with very interesting and new possibilities.

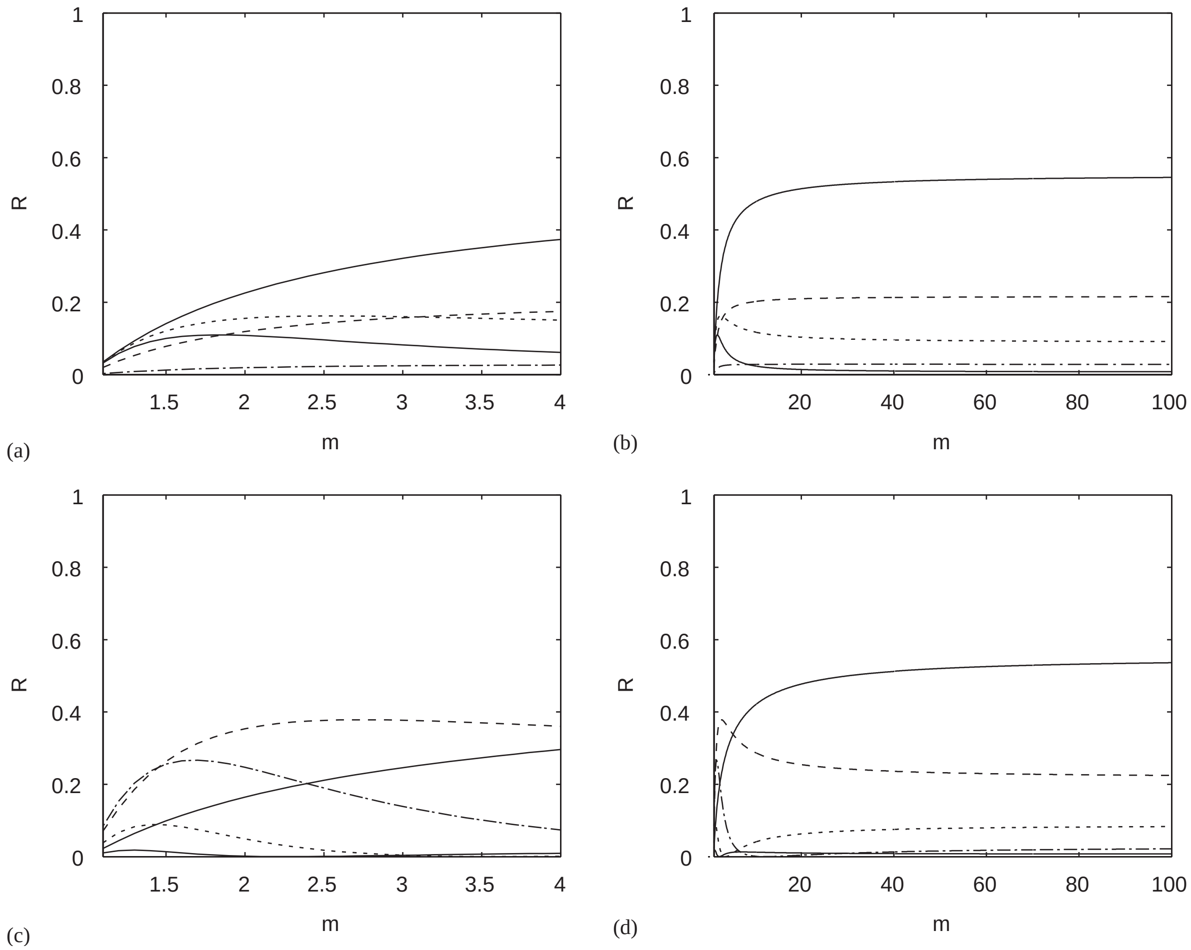


Fig. 10. $\mathscr{R}(m,n)$ in terms of $m$ for $l_1 = l/3$, $l_2 = l_3 = l$, $\rho_1 = \rho_3 = \rho$, $\rho_2 = 3\rho$. The graphs (a),(b) and (c),(d) are for a plucking of the first and third sections, respectively. The curve legends, are ——, $n = 1$; — — —, 2; — · —, 3; · · ·, 4; ——, 5. The frequencies of these modes are $f_1 = 1.3694\,c/2\pi l$, $f_2 = 1.8016 f_1$, $f_3 = 2.4772 f_1$, $f_4 = 3.1748 f_1$, $f_5 = 4.1007 f_1$.

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