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# Use of equivalent-damper method for free vibration analysis of a beam carrying multiple two degree-of-freedom spring–damper–mass systems

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**Abstract**

Although the dynamic characteristics of a beam carrying multiple two degree-of-freedom (dof) spring–damper–mass systems is one of the important topics in engineering, the information in this aspect is rare. The object of this paper is to replace the effect of each 2-dof spring–damper–mass system, composed of two springs, two dashpots and one lumped mass, by a set of *equivalent dampers*, so that the natural frequencies of a beam carrying any number of 2-dof spring–damper–mass systems may be solved from a beam supported by the same number of sets of equivalent dampers. Instead of using both the real part ($\overline{\varpi}_{jR}$) and the imaginary part ($\overline{\varpi}_{jI}$) of a complex eigenvalue, this paper uses the implicit-form complex eigenvalue, $\overline{\varpi}_j$, to derive the mathematical expressions, therefore, much more compact formulations were obtained. To confirm the reliability of the presented theory, all the numerical results obtained from the *equivalent-damper method* (EDM) were compared with those obtained from the conventional finite element method (FEM) and good agreement was achieved. Since the order of the overall property matrices for the equations of motion of the entire structural system derived from the EDM is much less than that derived from the FEM, the computer time required by the EDM is much less than that required by the FEM, particularly in the forced vibration analysis of a structural system using the step-by-step integration method, where the CPU time consumed is proportional to the total number of time steps. In addition, the EDM also provides a simple approach for evaluating the damping effect of each spring–damper–mass system. Furthermore, the presented *equivalent dampers* will provide an alternative choice for the effective vibration absorbers, because the damping effects of the *equivalent dampers* are dependent on the physical properties of their

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constituent parts (i.e., the springs, the dashpots and the lumped masses) and will be more flexible (or adjustable) than the damping effects of the classical dampers.


## 1. Introduction

Vibration analyses of structures with rigidly attached or elastically mounted equipments, such as engines, motors, oscillators, etc., are the important topics in structural engineering and, in general, each of the above-mentioned equipments can be simplified as a concentrated element, thus, a lot of researchers devoted themselves to studying the dynamic characteristics of beams carrying various rigidly attached or elastically mounted concentrated elements. For example, Bapat and Bapat [1] have studied the free vibration characteristics of a beam carrying multiple concentrated masses. Gürgöze [2] has obtained the exact solution for the natural frequencies of a beam and robs with point masses. Laura et al. [3,4] have researched the free vibration characteristics of an elastically restrain-free beam and a cantilever beam carrying a tip mass. Rossi et al. [5] have studied the exact solutions for the frequencies and mode shapes of a Timoshenko beam carrying a spring–mass system with three types of boundary conditions. Nicholson and Bergman [6,7] have used the Green functions to determine the free and forced vibration responses of an undamped and damped simply supported beam with an elastically mounted mass. Ozguven and Candir [8] have determined the optimum parameters of a beam with two vibration absorbers by modifying each of the vibration absorbers as a single-degree-of-freedom (dof) spring–-damper–mass system. Dowell [9] has derived the frequency equation of a beam carrying a spring–mass system by means of the Lagrange method. Wu et al. have determined the natural frequencies and mode shapes of a Timoshenko beam carrying multiple 1-dof spring–mass systems using the numerical assembly technique [10] and those of a Bernoulli–Euler cantilever beam carrying multiple 1-dof spring–mass systems using the analytical-and-numerical-combined method [11]. Laura et al. [12], Ercoli and Laura [13], Rossit and Laura [14] and Larrondo et al. [15] have performed the free vibration analysis of a Bernoulli–Euler beam with elastically mounted concentrated masses by means of various analytical approaches. The theoretical results were compared with the experimental ones and satisfactory agreement was achieved. Gürgöze [16] has used the Lagrange method to derive the frequency equation of a clamped-free Bernouli–Euler beam mounted with a tip mass and a spring–mass system. Yoshimura et al. [17,18], Lin and Trethewey [19,20] and Frýba [21] have studied the forced vibration responses of the beam under a moving vehicle, where the vehicle is modelled as a lumped mass elastically supported by a spring. Wu et al. [22,23] have studied the free vibrations of beams with 1-dof spring–damper–mass systems with the analytical-and-numerical-combined method and the conventional finite element method (FEM), while Chang et al. [24], Jen and Magrab [25], Wu and Whittaker [26] and Wu [27] have investigated those of beams carrying single and multiple 2-dof spring–mass systems *without dampers* by means of various approaches. In spite of the fact that the dynamic behaviours of beams carrying multiple 2-dof spring–damper–mass systems are also important problems in engineering, from the foregoing literature review, one sees that the material concerned is not found yet. Therefore, this paper attempts to study the last problem and to present some information in this aspect.

For convenience, a beam without attachment is called the *bare beam* and that carries any number of 2-dof spring–damper–mass systems is called the *loading beam*. In this paper, each 2-dof spring–damper–mass system is firstly replaced by four effective dampers with damping coefficients $c^{(v)}_{\mathrm{eff},ij}$ $(i,j = 1,2)$. The last effective dampers are, in turn, replaced by a set of *equivalent dampers* with damping coefficients, $c^{(v)}_{\mathrm{eq},i}$ and $c^{(v)}_{\mathrm{eq},k}$. Incorporating the expansion theorem, the equation of motion for the loading beam is derived analytically based on the last equivalent dampers together with the natural frequencies and mode shapes of the bare beam. Since the damping coefficients $c^{(v)}_{\mathrm{eff},ij}$ $(i,j = 1,2)$ for the effective dampers are functions of the eigenvalues of the loading beam, the cut and trial procedure is used to find the eigenvalues of the loading beam. Because the loading beam is a damped structural system, its eigenvalues are in complex form. Hence, one needs to guess two values, one for real part and one for imaginary part, to determine the eigenvalue in each iteration procedure. To overcome the difficulties encountered, the relation between the real part and the imaginary part of a complex eigenvalue is derived. By means of the last relationship, one only needs to guess one value to determine the desired eigenvalue in each cut and trial procedure.

## 2. Effective dampers for a 2-dof spring–damper–mass system

Fig. 1(a) shows a uniform beam carrying a 2-dof spring–damper–mass system located at $x = x^{(v)}_i$ and $x = x^{(v)}_k$, in which $m^{(v)}_e$, $J^{(v)}_e$, $k^{(v)}_y$ and $c^{(v)}$ are, respectively, the lumped mass, mass moment of inertia, spring constants and damping coefficients of the spring–damper–mass system, while $a^{(v)}_1$ and $a^{(v)}_2$ are, respectively, the distances between the lumped mass $m^{(v)}_e$ and the two springs $k^{(v)}_y$ (or dampers $c^{(v)}$). Besides, $u_v$ and $\theta_v$ are, respectively, the translational and rotational displacements of the lumped mass $m^{(v)}_e$; $u_i$ and $\theta_i$ are, respectively, the transverse displacement and rotational angle (or slope) of the beam at node ⓘ; and $u_k$ and $\theta_k$ are those at node ⓚ. It is noted that all the superscripts $v$ in the foregoing symbols represent the $v$th 2-dof spring–damper–mass system attached to the beam.

If the external loads on the 2-dof spring–damper–mass system are zero, i.e., $F_v = M_v = 0$, then from Fig. 1(a) one obtains

$$m^{(v)}_e \ddot{u}_v - F^{(v)}_i - F^{(v)}_k = 0, \tag{1a}$$

$$J^{(v)}_e \ddot{\theta}_v + F^{(v)}_i a^{(v)}_1 - F^{(v)}_k a^{(v)}_2 = 0, \tag{1b}$$

where $F^{(v)}_i$ and $F^{(v)}_k$ are, respectively, the forces at nodes ⓘ and ⓚ of the beam, and are given by

$$F^{(v)}_i = k^{(v)}_y (u_i - u_v + a^{(v)}_1 \theta_v) + c^{(v)} (\dot{u}_i - \dot{u}_v + a^{(v)}_1 \dot{\theta}_v), \tag{2}$$

$$F^{(v)}_k = k^{(v)}_y (u_k - u_v - a^{(v)}_2 \theta_v) + c^{(v)} (\dot{u}_k - \dot{u}_v - a^{(v)}_2 \dot{\theta}_v). \tag{3}$$

The last two equations are obtained based on the force equilibrium between the beam and the 2-dof spring–damper–mass system.

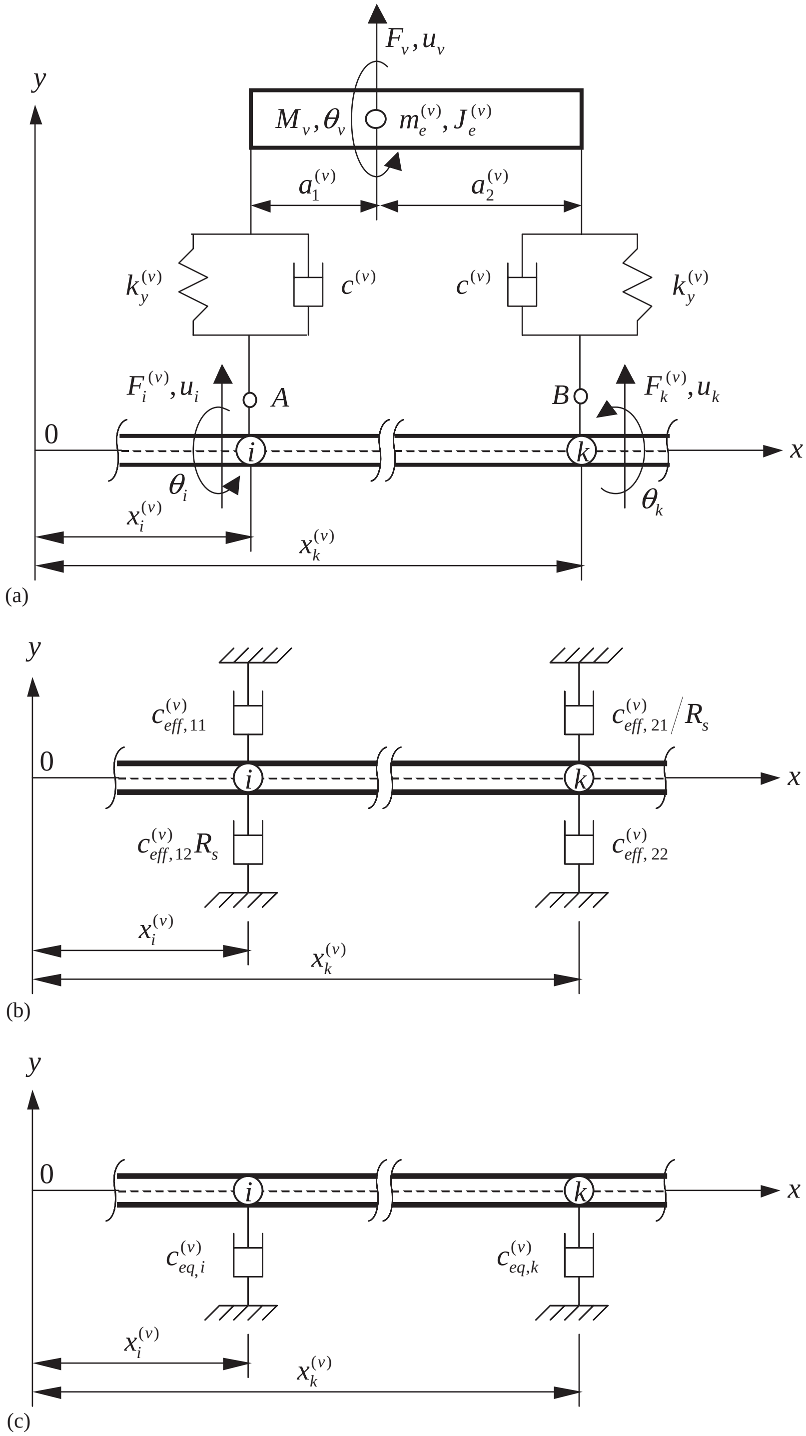


Fig. 1. (a) A 2-dof spring–damper–mass system attaching to a uniform beam can be replaced by (b) four effective dampers with damping coefficients $c^{(v)}_{\mathrm{eff},ij}$ $(i,j = 1,2)$ or (c) two equivalent dampers with damping coefficients $c^{(v)}_{\mathrm{eq},i}$ and $c^{(v)}_{\mathrm{eq},k}$.

Substituting Eqs. (2) and (3) into Eqs. (1a) and 1(b), one obtains

$$\begin{aligned} m_e^{(v)}\ddot{u}_v + c^{(v)}[-\dot{u}_i - \dot{u}_k + 2\dot{u}_v - (a_1^{(v)} - a_2^{(v)})\dot{\theta}_v] \\ + k_y^{(v)}[-u_i - u_k + 2u_v - (a_1^{(v)} - a_2^{(v)})\theta_v] = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} J_e^{(v)}\ddot{\theta}_v + c^{(v)}[a_1^{(v)}\dot{u}_i - a_2^{(v)}\dot{u}_k - (a_1^{(v)} - a_2^{(v)})\dot{u}_v + (a_1^{(v)^2} + a_2^{(v)^2})\dot{\theta}_v] \\ + k_y^{(v)}[a_1^{(v)}u_i - a_2^{(v)}u_k - (a_1^{(v)} - a_2^{(v)})u_v + (a_1^{(v)^2} + a_2^{(v)^2})\theta_v] = 0. \end{aligned} \tag{4b}$$

Writing the last two equations in matrix form yields

$$\begin{aligned} &\begin{bmatrix} m_e^{(v)} & 0 \\ 0 & J_e^{(v)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_v \\ \ddot{\theta}_v \end{Bmatrix} + c^{(v)} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix} + c^{(v)} \begin{bmatrix} 2 & -(a_1^{(v)} - a_2^{(v)}) \\ -(a_1^{(v)} - a_2^{(v)}) & a_1^{(v)^2} + a_2^{(v)^2} \end{bmatrix} \begin{Bmatrix} \dot{u}_v \\ \dot{\theta}_v \end{Bmatrix} \\ &+ k_y^{(v)} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} u_i \\ u_k \end{Bmatrix} + k_y^{(v)} \begin{bmatrix} 2 & -(a_1^{(v)} - a_2^{(v)}) \\ -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix} \begin{Bmatrix} u_v \\ \theta_v \end{Bmatrix} = 0. \end{aligned} \tag{5}$$

If the loading beam performs harmonic free vibration, one has

$$u_i = \bar{u}_i e^{\bar{\omega}t}, \quad u_k = \bar{u}_k e^{\bar{\omega}t}, \quad u_v = \bar{u}_v e^{\bar{\omega}t}, \quad \theta_v = \bar{\theta}_v e^{\bar{\omega}t}, \tag{6}$$

where

$$\bar{\omega} = \bar{\omega}_R + i\bar{\omega}_I. \tag{7}$$

In the last expressions, $\bar{\omega}_R$ and $\bar{\omega}_I$ are, respectively, the real and imaginary parts of the eigenvalue ($\bar{\omega}$) of the loading beam, while $\bar{u}_i, \bar{u}_k, \bar{u}_v$ and $\bar{\theta}_v$ are, respectively, the amplitudes of $u_i, u_k, u_v$ and $\theta_v$, $t$ is time and $i = \sqrt{-1}$.

From Eq. (6) one obtains

$$\begin{aligned} &u_i = \dot{u}_i/\bar{\omega}, \quad \ddot{u}_i = \dot{u}_i\bar{\omega}, \quad u_k = \dot{u}_k/\bar{\omega}, \quad \ddot{u}_k = \dot{u}_k\bar{\omega}, \\ &u_v = \dot{u}_v/\bar{\omega}, \quad \ddot{u}_v = \dot{u}_v\bar{\omega}, \quad \theta_v = \dot{\theta}_v/\bar{\omega}, \quad \ddot{\theta}_v = \dot{\theta}_v\bar{\omega}. \end{aligned} \tag{8}$$

Introducing Eq. (8) into Eq. (5) leads to

$$\begin{aligned} &\bar{\omega} \begin{bmatrix} m_e^{(v)} & 0 \\ 0 & J_e^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_v \\ \dot{\theta}_v \end{Bmatrix} \\ &+ c^{(v)} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix} + c^{(v)} \begin{bmatrix} 2 & -(a_1^{(v)} - a_2^{(v)}) \\ -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix} \begin{Bmatrix} \dot{u}_v \\ \dot{\theta}_v \end{Bmatrix} \\ &+ \frac{k_y^{(v)}}{\bar{\omega}} \begin{bmatrix} -1 & -1 \\ a_1^{(v)} & -a_2^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix} + \frac{k_y^{(v)}}{\bar{\omega}} \begin{bmatrix} 2 & -(a_1^{(v)} - a_2^{(v)}) \\ -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix} \begin{Bmatrix} \dot{u}_v \\ \dot{\theta}_v \end{Bmatrix} = 0 \end{aligned} \tag{9}$$

or

$$\begin{Bmatrix} \dot{u}_v \\ \dot{\theta}_v \end{Bmatrix} = \begin{bmatrix} -\bar{\omega} m_e^{(v)} - 2c^{(v)} - \frac{2k_y^{(v)}}{\bar{\omega}} & a_1^{(v)} c^{(v)} - a_2^{(v)} c^{(v)} + \frac{k_y^{(v)}(a_1^{(v)} - a_2^{(v)})}{\bar{\omega}} \\ a_1^{(v)} c^{(v)} - a_2^{(v)} c^{(v)} + \frac{k_y^{(v)}(a_1^{(v)} - a_2^{(v)})}{\bar{\omega}} & -\bar{\omega} J_e^{(v)} - a_1^{(v)^2} c^{(v)} - a_2^{(v)^2} c^{(v)} - \frac{k_y^{(v)}(a_1^{(v)^2} + a_2^{(v)^2})}{\bar{\omega}} \end{bmatrix}^{-1} \times \begin{bmatrix} -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} & -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} \\ a_1^{(v)} c^{(v)} + \frac{a_1^{(v)} k_y^{(v)}}{\bar{\omega}} & -a_2^{(v)} c^{(v)} - \frac{a_2^{(v)} k_y^{(v)}}{\bar{\omega}} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix}. \tag{10}$$

Substituting $u_i, u_k, u_v$ and $\theta_v$ given by Eq. (8) and $\dot{u}_v$ and $\dot{\theta}_v$ given by Eq. (10) into Eqs. (2) and (3), one obtains

$$\begin{Bmatrix} F_i^{(v)} \\ F_k^{(v)} \end{Bmatrix} = \left( \begin{bmatrix} c^{(v)} + \frac{k_y^{(v)}}{\bar{\omega}} & 0 \\ 0 & c^{(v)} + \frac{k_y^{(v)}}{\bar{\omega}} \end{bmatrix} + \begin{bmatrix} -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} & a_1^{(v)} c^{(v)} + \frac{a_1^{(v)} k_y^{(v)}}{\bar{\omega}} \\ -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} & -a_2^{(v)} c^{(v)} - \frac{a_2^{(v)} k_y^{(v)}}{\bar{\omega}} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} & -c^{(v)} - \frac{k_y^{(v)}}{\bar{\omega}} \\ a_1^{(v)} c^{(v)} + \frac{a_1^{(v)} k_y^{(v)}}{\bar{\omega}} & -a_2^{(v)} c^{(v)} - \frac{a_2^{(v)} k_y^{(v)}}{\bar{\omega}} \end{bmatrix} \right) \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix}, \tag{11}$$

where

$$W_{11} = \left[ -\bar{\omega} J_e^{(v)} - a_1^{(v)^2} c^{(v)} - a_2^{(v)^2} c^{(v)} - \frac{k_y^{(v)}(a_1^{(v)^2} + a_2^{(v)^2})}{\bar{\omega}} \right] / \Delta, \tag{12a}$$

$$W_{12} = -\left[ a_1^{(v)} c^{(v)} - a_2^{(v)} c^{(v)} + \frac{k_y^{(v)}(a_1^{(v)} - a_2^{(v)})}{\bar{\omega}} \right] / \Delta = W_{21}, \tag{12b}$$

$$W_{22} = \left[ -\bar{\omega} m_e^{(v)} - 2c^{(v)} - \frac{2k_y^{(v)}}{\bar{\omega}} \right] / \Delta \tag{12c}$$

and

$$\Delta = \begin{vmatrix} -\bar{\omega} m_e^{(v)} - 2c^{(v)} - \frac{2k_y^{(v)}}{\bar{\omega}} & a_1^{(v)} c^{(v)} - a_2^{(v)} c^{(v)} + \frac{k_y^{(v)}(a_1^{(v)} - a_2^{(v)})}{\bar{\omega}} \\ a_1^{(v)} c^{(v)} - a_2^{(v)} c^{(v)} + \frac{k_y^{(v)}(a_1^{(v)} - a_2^{(v)})}{\bar{\omega}} & -\bar{\omega} J_e^{(v)} - a_1^{(v)^2} c^{(v)} - a_2^{(v)^2} c^{(v)} - \frac{k_y^{(v)}(a_1^{(v)^2} + a_2^{(v)^2})}{\bar{\omega}} \end{vmatrix}. \tag{12d}$$

From Eq. (11), one obtains the following relationship between $\{F_i \quad F_k\}$ and $\{\dot{u}_i \quad \dot{u}_k\}$:

$$\begin{Bmatrix} F_i^{(v)} \\ F_k^{(v)} \end{Bmatrix} = \begin{bmatrix} c_{\text{eff},11}^{(v)} & c_{\text{eff},12}^{(v)} \\ c_{\text{eff},21}^{(v)} & c_{\text{eff},22}^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix}, \tag{13}$$

where

$$c_{\text{eff},11} = X + W_{11}X^2 - a_1^{(v)}W_{21}X^2 - a_1^{(v)}W_{12}X^2 + a_1^{(v)^2}W_{22}X^2, \tag{14a}$$

$$c_{\text{eff},12} = W_{11}X^2 - a_1^{(v)}W_{21}X^2 + a_2^{(v)}W_{12}X^2 - a_1^{(v)}a_2^{(v)}W_{22}X^2, \tag{14b}$$

$$c_{\text{eff},21} = W_{11}X^2 + a_2^{(v)}W_{21}X^2 - a_1^{(v)}W_{12}X^2 - a_1^{(v)}a_2^{(v)}W_{22}X^2, \tag{14c}$$

$$c_{\text{eff},22} = X + W_{11}X^2 + a_2^{(v)}W_{21}X^2 + a_2^{(v)}W_{12}X^2 + a_2^{(v)^2}W_{22}X^2, \tag{14d}$$

with

$$X = c^{(v)} + \frac{k_y^{(v)}}{\bar{\omega}}. \tag{15}$$

Eq. (13) indicates that the 2-dof spring–damper–mass system shown in Fig. 1(a) can be replaced by four effective dampers with damping coefficients $c_{\text{eff},ij}^{(v)}$ $(i,j = 1,2)$ given by Eq. (14) and shown in Fig. 1(b).

## 3. Equivalent dampers for a 2-dof spring–damper–mass system

The expansion of Eq. (13) gives

$$F_i^{(v)} = c_{\text{eff},11}^{(v)}\dot{u}_i + c_{\text{eff},12}^{(v)}\dot{u}_k = [c_{\text{eff},11}^{(v)} + c_{\text{eff},12}^{(v)}(\dot{u}_k/\dot{u}_i)]\dot{u}_i, \tag{16a}$$

$$F_k^{(v)} = c_{\text{eff},21}^{(v)}\dot{u}_i + c_{\text{eff},22}^{(v)}\dot{u}_k = [c_{\text{eff},21}^{(v)}(\dot{u}_i/\dot{u}_k) + c_{\text{eff},22}^{(v)}]\dot{u}_k. \tag{16b}$$

Substituting $\dot{u}_i$ and $\dot{u}_k$ given by Eq. (8) into Eq. (16) and writing the resulting expressions in matrix form yields

$$\begin{Bmatrix} F_i^{(v)} \\ F_k^{(v)} \end{Bmatrix} = \begin{bmatrix} c_{\text{eff},11}^{(v)} + c_{\text{eff},12}^{(v)}R_s & 0 \\ 0 & (c_{\text{eff},21}^{(v)}/R_s) + c_{\text{eff},22}^{(v)} \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{u}_k \end{Bmatrix}, \tag{17a}$$

where

$$R_s = u_k/u_i = \bar{Y}_s(x_k^{(v)})/\bar{Y}_s(x_i^{(v)}). \tag{17b}$$

In the last expression, $\bar{Y}_s(x_i^{(v)})$ and $\bar{Y}_s(x_k^{(v)})$, respectively, represent the modal displacements of the $s$th mode shape at the attaching points of the $v$th 2-dof spring–damper–mass system, $x = x_i^{(v)}$ and $x = x_k^{(v)}$.

Eq. (17a) reveals that the $v$th 2-dof spring–damper–mass system attached to the beam may be replaced by a set of *equivalent dampers* with damping coefficients, $c_{\text{eq},i}^{(v)}$ and $c_{\text{eq},k}^{(v)}$, given by (see

Fig. 1(c))

$$c_{\mathrm{eq},i}^{(v)} = c_{\mathrm{eff},11}^{(v)} + c_{\mathrm{eff},12}^{(v)} R_s, \tag{18a}$$

$$c_{\mathrm{eq},k}^{(v)} = (c_{\mathrm{eff},21}^{(v)}/R_s) + c_{\mathrm{eff},22}^{(v)}. \tag{18b}$$

From the above formulations, one sees that the magnitudes of the damping coefficients, $c_{\mathrm{eq},i}^{(v)}$ and $c_{\mathrm{eq},k}^{(v)}$, of the $v$th *equivalent dampers* are dependent on the magnitudes of the two spring constants $k_y^{(v)}$, the two damping coefficients $c^{(v)}$ and the one lumped mass $m_e^{(v)}$ of the $v$th 2-dof spring–damper–mass system attached to the beam. Therefore, the damping effect of an equivalent damper will be more flexible (or adjustable) than that of the classical dashpot. Furthermore, the last equivalent-damper method (EDM) also provides a simple approach for evaluating the overall damping effect of a spring–damper–mass system.

## 4. Eigenvalue equation for a uniform beam with multiple equivalent dampers

The equation of motion for a uniform beam carrying $p$ 2-dof spring–damper–mass systems takes the form [11,26]

$$EI\,\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\,\frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{v=1}^{p} F_i^{(v)}\delta(x - x_i^{(v)}) + \sum_{v=1}^{p} F_k^{(v)}\delta(x - x_k^{(v)}), \tag{19}$$

where $E$ is the Young's modulus, $I$ is the moment of inertia for cross-sectional area of the beam, $\bar{m}$ is the mass per unit length of the beam, $y(x,t)$ is the transverse deflection of the beam at position $x$ and time $t$, $F_i^{(v)}$ and $F_k^{(v)}$ are, respectively, the interaction force at the contacting points, $x = x_i^{(v)}$ and $x = x_k^{(v)}$, between the $v$th 2-dof spring–damper–mass system and the beam, and $\delta(\cdot)$ is the Dirac delta function.

Based on the expansion theorem [28] or the mode superposition method [29], the transverse deflection of the beam is given by

$$y(x,t) = \sum_{s=1}^{n'} \bar{Y}_s(x) q_s(t), \tag{20}$$

where $\bar{Y}_s(x)$ represents the $s$th mode shape of the bare beam, $q_s(t)$ is the generalized coordinate, and $n'$ is the total modes considered.

If the entire system performs free harmonic vibration, the generalized coordinate $q_s(t)$ takes the form

$$q_s(t) = \bar{q}_s e^{\bar{\omega} t}, \tag{21}$$

where $\bar{q}_s$ is the amplitude of $q_s(t)$ and $\bar{\omega}$ is the eigenvalue of the loading beam taking the form given by Eq. (7).

From Eqs. (20) and (21), one has

$$\dot{y}(x,t) = \bar{\omega} \sum_{s=1}^{n'} \bar{Y}_s(x) q_s(t). \tag{22}$$

According to the theory presented in the last subsection, if the $v$th 2-dof spring–damper–mass system attached to the bare beam is replaced by two equivalent dampers with damping coefficients given by $c_{\text{eq},i}^{(v)}$ and $c_{\text{eq},k}^{(v)}$ as shown in Fig. 1(c) and Eq. (18), then the interaction forces at the contacting points, $x = x_i^{(v)}$ and $x = x_k^{(v)}$, between the $v$th 2-dof spring–damper–mass system and the bare beam, $F_i^{(v)}$ and $F_i^{(v)}$, are, respectively, given by

$$F_i^{(v)} = -c_{\text{eq},i}^{(v)}\bar{\omega}\sum_{s=1}^{n'}\bar{Y}_s(x_i^{(v)})q_s(t), \tag{23a}$$

$$F_k^{(v)} = -c_{\text{eq},k}^{(v)}\bar{\omega}\sum_{s=1}^{n'}\bar{Y}_s(x_k^{(v)})q_s(t). \tag{23b}$$

Substituting Eqs. (20) and (23) into Eq. (19), multiplying the resulting expression by $\bar{Y}_r(x)$ and integrating the entire equation over the beam length $l$, one obtains

$$\begin{aligned}&\int_0^l\sum_{s=1}^{n'}\bar{Y}_r(x)EI\bar{Y}''''_s(x)q_s(t)\,\mathrm{d}x + \int_0^l\sum_{s=1}^{n'}\bar{Y}_r(x)\bar{m}\bar{Y}_s(x)\ddot{q}_s(t)\,\mathrm{d}x\\ &= -\int_0^l\sum_{v=1}^{p}\bar{Y}_r(x_i^{(v)})c_{\text{eq},i}^{(v)}\bar{\omega}\sum_{s=1}^{n'}\bar{Y}_s(x_i^{(v)})q_s(t)\delta(x-x_i^{(v)})\,\mathrm{d}x\\ &\quad -\int_0^l\sum_{v=1}^{p}\bar{Y}_r(x_k^{(v)})c_{\text{eq},k}^{(v)}\bar{\omega}\sum_{s=1}^{n'}\bar{Y}_s(x_k^{(v)})q_s(t)\delta(x-x_k^{(v)})\,\mathrm{d}x.\end{aligned} \tag{24}$$

If the mode shapes $\bar{Y}_s(x)$ ($s = 1-n'$) are normalized with respect to $\bar{m}$, then application of the orthogonal properties of the normal mode shapes to Eq. (24) leads to

$$\begin{aligned}\ddot{q}_r(t) + \Omega_r^2 q_r(t) = &-\sum_{v=1}^{p}\sum_{s=1}^{n'}c_{\text{eq},i}^{(v)}\bar{\omega}\bar{Y}_r(x_i^{(v)})\bar{Y}_s(x_i^{(v)})q_s(t)\\ &-\sum_{v=1}^{p}\sum_{s=1}^{n'}c_{\text{eq},k}^{(v)}\bar{\omega}\bar{Y}_r(x_k^{(v)})\bar{Y}_s(x_k^{(v)})q_s(t), \quad r = 1,\ldots,n',\end{aligned} \tag{25}$$

where $\Omega_r$ represents the $r$th natural frequency of the bare beam.

Introducing Eq. (21) into Eq. (25) gives

$$\begin{aligned}&\Omega_r^2\bar{q}_r + \sum_{v=1}^{p}\sum_{s=1}^{n'}c_{\text{eq},i}^{(v)}\bar{\omega}\bar{Y}_r(x_i^{(v)})\bar{Y}_s(x_i^{(v)})\bar{q}_s\\ &\quad + \sum_{v=1}^{p}\sum_{s=1}^{n'}c_{\text{eq},k}^{(v)}\bar{\omega}\bar{Y}_r(x_k^{(v)})\bar{Y}_s(x_k^{(v)})\bar{q}_s = -\bar{\omega}^2\bar{q}_r, \quad r = 1,\ldots,n',\end{aligned} \tag{26}$$

Writing Eq. (26) in matrix form yields the following eigenvalue equation:

$$([\mathbf{A}] + \bar{\omega}^2[\mathbf{B}])\{\bar{q}\} = 0, \tag{27}$$

where

$$[\mathbf{A}]_{n'\times n'} = [\mathbf{\Omega}^2]_{n'\times n'} + [\mathbf{A'}]_{n'\times n'}, \tag{28a}$$

$$[\mathbf{B}]_{n'\times n'} = [\mathbf{I}]_{n'\times n'} = \lceil 11\ldots 1 \rfloor_{n'\times n'}, \tag{28b}$$

$$[\mathbf{A'}]_{n'\times n'} = \sum_{v=1}^{p} c_{\mathrm{eq},i}^{(v)} \bar{\omega} [\bar{\mathbf{Y}}(x_i^{(v)})]_{n'\times n'} + \sum_{v=1}^{p} c_{\mathrm{eq},k}^{(v)} \bar{\omega} [\bar{\mathbf{Y}}(x_k^{(v)})]_{n'\times n'}, \tag{28c}$$

$$[\bar{\mathbf{Y}}(x)]_{n'\times n'} = \begin{bmatrix} \bar{Y}_1(x)\bar{Y}_1(x) & \bar{Y}_1(x)\bar{Y}_2(x) & \cdots & \bar{Y}_1(x)\bar{Y}_{n'}(x) \\ \bar{Y}_2(x)\bar{Y}_1(x) & \bar{Y}_2(x)\bar{Y}_2(x) & \cdots & \bar{Y}_2(x)\bar{Y}_{n'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Y}_{n'}(x)\bar{Y}_1(x) & \bar{Y}_{n'}(x)\bar{Y}_2(x) & \cdots & \bar{Y}_{n'}(x)\bar{Y}_{n'}(x) \end{bmatrix}, \tag{28d}$$

$$\{\bar{q}\}_{n'\times 1} = \{\, \bar{q}_1 \quad \bar{q}_2 \quad \ldots \quad \bar{q}_{n'} \,\}_{n'\times 1}, \tag{28e}$$

$$[\mathbf{\Omega}^2]_{n'\times n'} = \lceil \Omega_1^2 \quad \Omega_2^2 \quad \ldots \quad \Omega_{n'}^2 \rfloor_{n'\times n'}. \tag{28f}$$

In Eqs. (27) and (28), the symbols, {}, [ ] and ⌈ ⌋, represent the column matrix, square matrix and diagonal matrix, respectively.

## 5. Element property matrices of a 2-dof spring–damper–mass system

To confirm the reliability of the foregoing theories presented for the title problem by means of the conventional FEM, one requires the element stiffness, damping and mass matrices of a 2-dof spring–damper–mass system, and they are derived in this section.

From the free-body diagram of the 2-dof spring–damper–mass system (cf. Fig. 1(a)), one has

$$\sum F_y = F_i^{(v)} + F_k^{(v)} + F_v - m_e^{(v)} \ddot{u}_v = 0, \tag{29}$$

$$\sum M_z = -F_i^{(v)} a_1^{(v)} + F_k^{(v)} a_2^{(v)} + M_v - J_e^{(v)} \ddot{\theta}_v = 0. \tag{30}$$

Introducing Eqs. (2) and (3) into Eqs. (29) and (30), one obtains

$$\begin{aligned} F_v = & -F_i^{(v)} - F_k^{(v)} + m_e^{(v)} \ddot{u}_v = k_y^{(v)} [-u_i - u_k + 2u_v - (a_1^{(v)} - a_2^{(v)})\theta_v] \\ & + c^{(v)} [-\dot{u}_i - \dot{u}_k + 2\dot{u}_v - (a_1^{(v)} - a_2^{(v)})\dot{\theta}_v] + m_e^{(v)} \ddot{u}_v, \end{aligned} \tag{31}$$

$$\begin{aligned} M_v &= F_i^{(v)} a_1^{(v)} - F_k^{(v)} a_2^{(v)} + J_e^{(v)} \ddot{\theta}_v \\ &= k_y^{(v)} [a_1^{(v)} u_i - a_2^{(v)} u_k - (a_1^{(v)} - a_2^{(v)}) u_v + (a_1^{(v)^2} + a_2^{(v)^2})\theta_v] \\ &\quad + c^{(v)} [a_1^{(v)} \dot{u}_i - a_2^{(v)} \dot{u}_k - (a_1^{(v)} - a_2^{(v)}) \dot{u}_v + (a_1^{(v)^2} + a_2^{(v)^2})\dot{\theta}_v] + J_e^{(v)} \ddot{\theta}_v. \end{aligned} \tag{32}$$

Writing Eqs. (2), (3), (31) and (32) in matrix form yields

$$\{\mathbf{F}^{(v)}\} = [\mathbf{k}^{(v)}]\{\mathbf{u}\} + [\mathbf{c}^{(v)}]\{\dot{\mathbf{u}}\} + [\mathbf{m}^{(v)}]\{\ddot{\mathbf{u}}\}, \tag{33}$$

where

$$\{\mathbf{F}^{(v)}\} = [\,F_i^{(v)} \quad F_k^{(v)} \quad F_v \quad M_v\,]^{\mathrm{T}}, \tag{34}$$

$$\{\mathbf{u}\} = [\,u_i \quad u_k \quad u_v \quad \theta_v\,]^{\mathrm{T}}, \tag{35}$$

$$\{\dot{\mathbf{u}}\} = [\,\dot{u}_i \quad \dot{u}_k \quad \dot{u}_v \quad \dot{\theta}_v\,]^{\mathrm{T}}, \tag{36a}$$

$$\{\ddot{\mathbf{u}}\} = [\,\ddot{u}_i \quad \ddot{u}_k \quad \ddot{u}_v \quad \ddot{\theta}_v\,]^{\mathrm{T}}, \tag{36b}$$

$$[\mathbf{k}^{(v)}] = k_y^{(v)} \begin{bmatrix} 1 & 0 & -1 & a_1^{(v)} \\ 0 & 1 & -1 & -a_2^{(v)} \\ -1 & -1 & 2 & -(a_1^{(v)} - a_2^{(v)}) \\ a_1^{(v)} & -a_2^{(v)} & -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix}, \tag{37}$$

$$[\mathbf{c}^{(v)}] = c^{(v)} \begin{bmatrix} 1 & 0 & -1 & a_1^{(v)} \\ 0 & 1 & -1 & -a_2^{(v)} \\ -1 & -1 & 2 & -(a_1^{(v)} - a_2^{(v)}) \\ a_1^{(v)} & -a_2^{(v)} & -(a_1^{(v)} - a_2^{(v)}) & (a_1^{(v)^2} + a_2^{(v)^2}) \end{bmatrix}, \tag{38}$$

$$[\mathbf{m}^{(v)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_e^{(v)} & 0 \\ 0 & 0 & 0 & J_e^{(v)} \end{bmatrix}. \tag{39}$$

In Eqs. (37)–(39), $[\mathbf{k}^{(v)}]$, $[\mathbf{c}^{(v)}]$ and $[\mathbf{m}^{(v)}]$, respectively, represent the element stiffness matrix, damping matrix and mass matrix for the $v$th 2-dof spring–damper–mass system attached to the beam, as shown in Fig. 1(a).

## 6. Solution of the problem

### 6.1. By using FEM

The property matrices of the entire bare beam are firstly determined using the standard assembling technique of the FEM. Next, the contribution of each 2-dof spring–damper–mass system to the overall property matrices of the loading beam is taken into consideration by adding the associated element property matrices of each system. Since each spring–damper–mass system is considered as a 2-dof finite element, the total degree of freedom of the loading beam increases by 2 when one more 2-dof spring–damper–mass system is attached to the beam. Finally, a set of equations of motion can be obtained after assembling all the element property matrices of all the 2-dof spring–damper–mass systems and imposing the prescribed boundary conditions of the beam:

$$[\bar{\mathbf{M}}]\{\ddot{\mathbf{u}}(t)\} + [\bar{\mathbf{C}}]\{\dot{\mathbf{u}}(t)\} + [\bar{\mathbf{K}}]\{\mathbf{u}(t)\} = 0, \tag{40}$$

where $[\bar{\mathbf{M}}]$, $[\bar{\mathbf{C}}]$ and $[\bar{\mathbf{K}}]$, respectively, represent the overall mass, damping and stiffness matrices, and $\{\ddot{\mathbf{u}}(t)\}$, $\{\dot{\mathbf{u}}(t)\}$ and $\{\mathbf{u}(t)\}$, respectively, represent the overall acceleration, velocity and displacement vectors for the entire loading beam. In this paper, the eigenvalue problem derived from Eq. (40) was solved with the subroutine EISPACK [30,31].

### 6.2. By using EDM

For a bare beam carrying any number of 2-dof spring–damper–mass systems, if the effect of each 2-dof spring–damper–mass system is replaced by a set of *equivalent dampers* as shown in Fig. 1(c), then the eigenvalue equation for the associated loading beam is given by Eq. (27). Non-trivial solution of Eq. (27) requires that

$$\left|[\mathbf{A}] + \bar{\omega}^2[\mathbf{B}]\right| = 0. \tag{41}$$

From Eqs. (13)–(17), it is seen that the damping coefficients, $c_{\mathrm{eq},i}^{(v)}$ and $c_{\mathrm{eq},k}^{(v)}$, for the two dampers equivalent to the $v$th 2-dof spring–damper–mass system are functions of eigenvalue of the loading beam. Hence, the cut and trial procedures are used to find the eigenvalue $\bar{\omega}$ in Eq. (41).

Since the eigenvalue $\bar{\omega}$ is a complex number (see Eq. (7)), one has to guess two values, the real part $\bar{\omega}_R$ and the imaginary part $\bar{\omega}_I$, of the eigenvalue $\bar{\omega}$ in each cut and trial procedure. To reduce the difficulties encountered, the following relationship between $\bar{\omega}_R$ and $\bar{\omega}_I$ is employed [29]:

$$\bar{\omega}_R = -\frac{\xi_r}{\sqrt{1-\xi_r^2}}\,\bar{\omega}_I, \quad r = 1, 2, \ldots. \tag{42}$$

In the last equation, $\xi_r$ is the damping ratio associated with the $r$th mode shape of the loading beam and is given by

$$\xi_r = C_r^*/2m_r^*\Omega_r, \tag{43}$$

where $\Omega_r$ is the $r$th undamped natural frequency of the loading beam, while $C_r^*$ and $m_r^*$ are, respectively, the generalized damping coefficient and generalized mass with respect to the $r$th normal mode shape $\bar{Y}_r(x)$, and are given by

$$\begin{aligned} C_r^* &= \sum_{v=1}^{p}\int_0^l \bar{Y}_r(x)c_{\mathrm{eq},i}^{(v)}\bar{Y}_r(x)\delta(x-x_i^{(v)})\,\mathrm{d}x + \sum_{v=1}^{p}\int_0^l \bar{Y}_r(x)c_{\mathrm{eq},k}^{(v)}\bar{Y}_r(x)\delta(x-x_k^{(v)})\mathrm{d}\,x \\ &= \sum_{v=1}^{p} c_{\mathrm{eq},i}^{(v)}\bar{Y}_r^2(x_i^{(v)}) + \sum_{v=1}^{p} c_{\mathrm{eq},k}^{(v)}\bar{Y}_r^2(x_k^{(v)}), \end{aligned} \tag{44}$$

$$m_r^* = \int_0^l \bar{Y}_r(x)\bar{m}\bar{Y}_r(x)\,\mathrm{d}x = 1. \tag{45}$$

By using Eqs. (42)–(45), one only needs to guess the real part $\bar{\omega}_R$ or the imaginary part $\bar{\omega}_I$ for the eigenvalue $\bar{\omega}$ in each cut and trial procedure. If the assumed eigenvalue $\bar{\omega}$ satisfies Eq. (41), the last value of $\bar{\omega}$ is one of the eigenvalues of the loading beam; otherwise, iteration with a new eigenvalue $\bar{\omega}$ is required.

## 7. Numerical results

The dimensions and material constants for the undamped uniform cantilever beam studied in this section are: total beam length $l = 1.0\,m$; diameter $d = 0.05$ m; Young's modulus $E = 2.069 \times 10^{11}\,\mathrm{N/m^2}$ and mass density $\rho = 7.8367 \times 10^3\,\mathrm{kg/m^3}$. In addition, the total number of modes used for the expansion theorem [28] or the mode superposition method [29] as shown in Eq. (20) is $n' = 8$.

### 7.1. Reliability of the theory and the computer programs

As shown in Fig. 2, a 2-dof spring–damper–mass system is attached to the beam at $x_i^{(1)} = 0.8$ m and $x_k^{(1)} = 1.0$ m. The spring constants, damping coefficients, lumped mass and moment of inertia of the spring–damper–mass system are given by $k_y^{(1)} = 6.34761 \times 10^6\,\mathrm{N/m}$, $c^{(1)} = 0.0\,\mathrm{Ns/m}$, $m_e^{(1)} = 1.53875$ kg and $J_e^{(1)} = 1.53875\,\mathrm{kg\,m^2}$, while the distance between the lumped mass $m_e^{(1)}$ and the two springs (or dampers) are respectively $a_1^{(1)} = 0.06667$ m and $a_2^{(1)} = 0.13333$ m. In order to confirm the reliability of the theory presented and the computer programs developed, the last information is exactly the same as that of Ref. [26]. It is noted that the damping coefficient $c^{(1)}$ for the 2-dof spring–damper–mass system studied here is set to be zero because the elastically mounted system studied in Ref. [26] is a 2-*dof spring–mass* system (without damper).

Table 1 shows the first five eigenvalues $\bar{\omega}_j\,(j = 1-5)$ of the beam with a 2-dof spring–damper–mass system, as shown in Fig. 2. From the table, one sees that the imaginary parts of the eigenvalues, $\bar{\omega}_{jI}$, obtained from the EDM presented in this paper are very close to the corresponding ones obtained from Ref. [26] and the conventional FEM. In addition, all the real parts of the eigenvalues, $\bar{\omega}_{jR}$, are negligible, as one may see from the third and fourth rows of

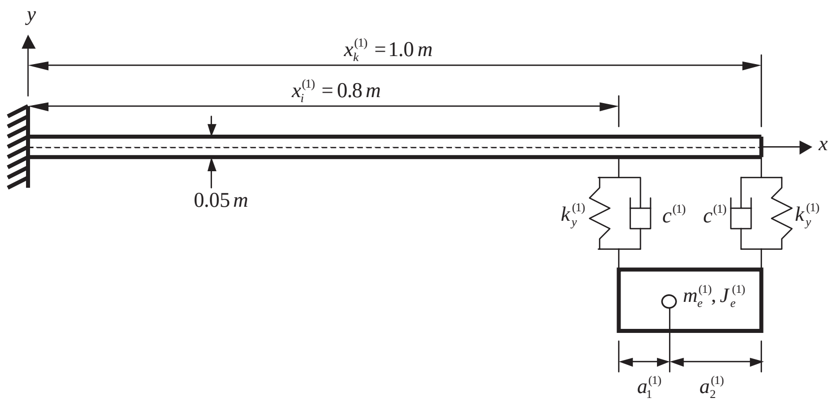


Fig. 2. A uniform cantilever beam carrying a 2-dof spring–damper–mass system.

Table 1
The first five eigenvalues of a uniform cantilever beam carrying a 2-dof spring–damper–mass system, $\bar{\omega}_j = \bar{\omega}_{jR} \pm \mathrm{i}\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$, as shown in Fig. 2 with $c^{(1)} = 0$

| Methods | Eigenvalues, $\bar{\omega}_j = \bar{\omega}_{jR} \pm \mathrm{i}\bar{\omega}_{jI}$ (rad/s) | | | | | CPU time (s) |
|---|---|---|---|---|---|---|
| | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ | |
| [a]FEM | 3.2644E−08±i141.9957 | −6.4717E−08±i321.8164 | −2.0643E−07±i1524.2575 | 1.9360E−07±i3297.6358 | −1.3349E−06±i4276.0247 | 2.80 |
| [a]EDM | −2.0654E−09±i143.4354 | −5.0923E−07±i324.3061 | −3.0539E−08±i1526.9813 | −1.9980E−07±i3330.0138 | −8.5621E−08±i4281.0267 | 1.32 |
| Ref. [23] | 141.5405 | 321.3685 | 1524.3220 | 3297.6410 | 4276.0260 | — |

[a]FEM refers to the conventional finite element method and EDM refers to the equivalent-damper method presented in this paper.

Table 2
Influence of the spring stiffness ($k_y^{(1)}$) of the spring–damper–mass system on the first five eigenvalues of the uniform cantilever loading beam, $\bar{\omega}_j = \bar{\omega}_{jR} \pm \mathrm{i}\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$, as shown in Fig. 2, with $m_e^{(1)} = 3.2$ kg, $J_e^{(1)} = 3.2\ \mathrm{kg\,m}^2$, $c^{(1)} = 38.5$ Ns/m

| $k_y^{(1)}$ (N/m) | Methods | Eigenvalues, $\bar{\omega}_j = \bar{\omega}_{jR} \pm \mathrm{i}\bar{\omega}_{jI}$ (rad/s) | | | | |
|---|---|---|---|---|---|---|
| | | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ |
| $10^4$ | FEM | −1.1393E−01±i24.6571 | −1.3370E−00±i159.4345 | −1.7985E01±i355.6698 | −5.3471E−00±i1424.6072 | −5.8052E−00±i3966.4966 |
| | EDM | −1.2018E−01±i24.9856 | −1.3372E−00±i159.3698 | −1.7991E01±i355. 8123 | −5.3426E−00±i1424.6719 | −5.7910E−00±i3966.4433 |
| $10^6$ | FEM | −3.4221E−03±i112.3678 | −7.3659E−02±i298.4601 | −1.2826E00±i1537.5339 | −1.5731E01±i3283.6489 | −1.0700E01±i4436.1887 |
| | EDM | −3.4232E−03±i112.3834 | −7.3771E−02±i298.5487 | −1.2842E00±i1537.7218 | −1.5794E01±i3283.3959 | −1.0844E01±i4438.9072 |
| $10^8$ | FEM | −2.8176E−07±i120.3850 | −3.5574E−05±i417.0230 | −1.7478E−03±i2219.5915 | −1.6285E−02±i5704.8618 | −7.3563E−02±i10875.9471 |
| | EDM | −3.9998E−07±i120.4020 | −3.5807E−05±i417.3339 | −1.7595E−03±i2221.9229 | −1.6361E−02±i5712.4976 | −7.3587E−02±i10895.2715 |
| — | FEM (bare beam) | 225.8249 | 1415.2234 | 3962.7231 | 7765.7136 | 12838.6537 |

Table 1, due to the fact that the damping coefficient of the 2-dof system is zero. Hence, it is believed that the EDM is viable for the title problem.

From the final column of Table 1, one sees that the computer time required by the presented EDM is much less than that required by the conventional FEM. This is under one's expectation, because the order of the overall property matrices of the entire structural system derived using the EDM is much less than that derived using the FEM. The last advantage of the EDM will be much more predominant in the forced vibration analysis of a structural system using the step-by-step integration method, because the CPU time consumed is proportional to the total number of time steps.

### 7.2. *Influence of spring constants*

The parameters of the loading beam studied in this subsection are the same as those studied in the last subsection (see Fig. 2), except the following changes: lumped mass, $m_e^{(1)} = 3.2$ kg; mass moment of inertia, $J_e^{(1)} = 3.2\ \mathrm{kg\,m}^2$, damping coefficient, $c^{(1)} = 38.5$ Ns/m; the spring constant, $k_y^{(1)} = 10^4$, $10^6$ or $10^8$ N/m.

Table 2 shows the first five eigenvalues of the loading beam. It is seen that the natural frequencies of the loading beam (i.e., the imaginary parts of the eigenvalues), $\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$, increase with increasing the spring constants of the spring–damper–mass system. In addition, it is also found that the natural frequencies of the loading beam are much lower than the corresponding ones of the bare beam. The last phenomenon indicates that the installation of the spring–damper–mass system to the free end of the beam may reduce the natural frequencies of the bare beam significantly.

### 7.3. Influence of damping coefficients

All the parameters for the present example are exactly the same as those of the last example, except that the spring constant of the spring–damper–mass system is a constant and given by $k_y^{(1)} = 10^6$ N/m and the damping coefficient is taken to be $c^{(1)} = 3.85$, 38.5 or 385 Ns/m.

Table 3 shows the first five eigenvalues of the loading beam. It is found that a spring–damper–mass system with different damping coefficients does not significantly influence the natural frequencies of the loading beam, $\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$. However, the absolute values of decay ratios (i.e., the real parts of the eigenvalues), $\bar{\omega}_{jR}$ $(j = 1, \ldots, 5)$, significantly increase on increasing the magnitude of the damping coefficient. This indicates that the use of higher damping coefficients for the spring–damper–mass system may significantly increase the decay rate of the loading beam.

### 7.4. Influence of mass

For the example studied here, the parameters of the beam are exactly the same as those of the last example, except that the spring constants and damping coefficients of the spring–-damper–mass system are constant and given by $k_y^{(1)} = 10^6$ N/m and $c^{(1)} = 38.5$ Ns/m, while the lumped mass is taken to be $m_e^{(1)} = 1.6$, 3.2 or 6.4 kg. From Table 4, one sees that increasing the magnitude of the lumped mass of a spring–damper–mass system has the effect of reducing the natural frequencies of the loading beam.

Table 3
Influence of the damping coefficient ($c^{(1)}$) of the spring–damper–mass system on the first five eigenvalues of the uniform cantilever loading beam, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$, as shown in Fig. 2 with $m_e^{(1)} = 3.2$ kg, $J_e^{(1)} = 3.2$ kg m$^2$, $k_y^{(1)} = 10^6$ N/m

| $c^{(1)}$ (Ns/m) | Methods | Eigenvalues, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ (rad/s) | | | | |
|---|---|---|---|---|---|---|
| | | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ |
| 3.85 | FEM | −3.4219E−04±i112.3678 | −7.3657E−03±i298.4600 | −1.2826E−01±i1537.5317 | −1.5730E00±i3283.6198 | −1.0701E00±i4436.2674 |
| | EDM | −3.4277E−04±i112.3834 | −7.3771E−03±i298.5487 | −1.2917E−01±i1537.7223 | −1.5784E00±i3288.4359 | −1.0830E00±i4438.9178 |
| 38.5 | FEM | −3.4221E−03±i112.3678 | −7.3659E−02±i298.4601 | −1.2826E00±i1537.5339 | −1.5731E01±i3283.6489 | −1.0700E01±i4436.1887 |
| | EDM | −3.4232E−03±i112.3834 | −7.3771E−02±i298.5487 | −1.2842E00±i1537.7218 | −1.5794E01±i3283.3959 | −1.0844E01±i4438.9072 |
| 385 | FEM | −3.4220E−02±i112.3679 | −7.3659E−01±i298.4627 | −1.2824E01±i1537.6892 | −1.5815E02±i3286.6556 | −1.0635E02±i4427.5132 |
| | EDM | −3.4232E−02±i112.3835 | −7.3777E−01±i298.5575 | −1.2839E01±i1537.5194 | −1.5946E02±i3306.4375 | −1.0739E02±i4414.6265 |
| — | FEM (bare beam) | 225.8249 | 1415.2234 | 3962.7231 | 7765.7136 | 12838.6537 |

Table 4
Influence of the lumped mass ($m_e^{(1)}$) of the spring–damper–mass system on the first five eigenvalues of the uniform cantilever loading beam, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ ($j = 1, \ldots, 5$), as shown in Fig. 2 with $J_e^{(1)} = 3.2$ kg m$^2$, $k_y^{(1)} = 10^6$ N/m, $c^{(1)} =$ 38.5 Ns/m

| $m_e^{(1)}$ (kg) | Methods | Eigenvalues, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ (rad/s) | | | | |
|---|---|---|---|---|---|---|
| | | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ |
| 1.6 | FEM | −4.1114E−03±i115.3180 | −7.8279E−02±i320.6025 | −1.2956E00±i1607.9097 | −1.45291E01±i3846.3746 | −2.3416E01±i4644.0892 |
| | EDM | −4.1175E−03±i115.3353 | −7.8411E−02±i320.6977 | −1.2977E00±i1608.0947 | −1.4469E01±i3849.0685 | −2.3782E01±i4650.3591 |
| 3.2 | FEM | −3.4221E−03±i112.3678 | −7.3659E−02±i298.4601 | −1.2826E00±i1537.5339 | −1.5731E01±i3283.6489 | −1.0700E01±i4436.1887 |
| | EDM | −3.4232E−03±i112.3834 | −7.3771E−02±i298.5487 | −1.2842E00±i1537.7218 | −1.5794E01±i3283.3959 | −1.0844E01±i4438.9072 |
| 6.4 | FEM | −2.3749E−03±i106.7775 | −6.7181E−02±i269.5434 | −1.2759E00±i1451.8413 | −1.2383E01±i2895.0846 | −8.2706E00±i4391.1904 |
| | EDM | −2.3707E−03±i106.7902 | −6.7272E−02±i269.6256 | −1.2763E00±i1452.0896 | −1.2457E01±i2899.6881 | −8.3601E00±i4393.1014 |
| — | FEM (bare beam) | 225.8249 | 1415.2234 | 3962.7231 | 7765.7136 | 12838.6537 |

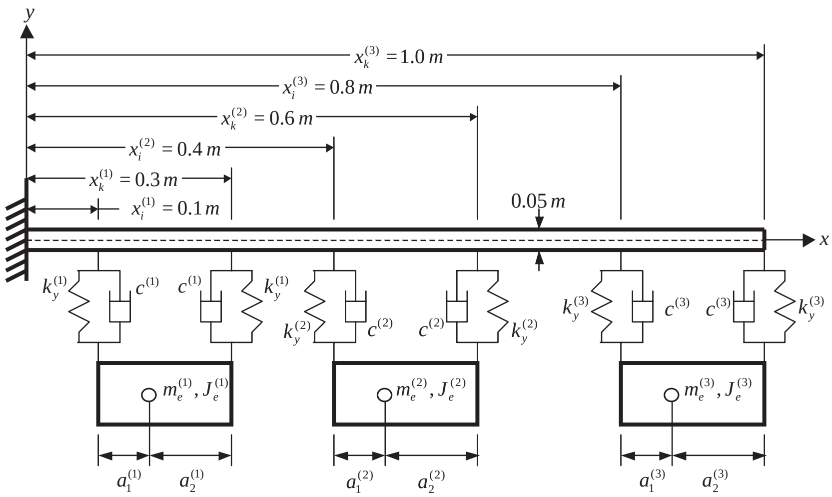


Fig. 3. A uniform cantilever beam carrying three 2-dof spring–damper–mass systems.

### 7.5. *Eigenvalues for a cantilever beam with three spring–damper–mass systems*

Fig. 3 shows the uniform cantilever beam carrying three 2-dof spring–damper–mass systems studied here. The dimensions and material properties of the bare beam are exactly the same as those of the foregoing examples, while the physical properties of the three 2-dof spring–damper–mass systems are listed in Table 5.

The first five eigenvalues of the loading beam, $\bar{\omega}_j = \bar{\omega}_{jI} \pm i\bar{\omega}_{jR}$ ($j = 1, \ldots, 5$), are listed in Table 6. From the table, one sees that the natural frequencies of the loading beam are very close to the corresponding ones of the bare beam. This phenomenon is very different from that for a bare beam carrying a 2-dof spring–damper–mass system at the free end, studied in the previous subsections. In view of the fact that the total spring stiffness, total damping coefficient and total lumped mass for all the 2-dof spring–damper–mass systems of the present example are three times

Table 5
The locations and physical properties of the three 2-dof spring–damper–mass systems shown in Fig. 3

| Numbering of systems (v) | Locations | | Physical properties of the spring–damper–mass systems | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $x_i^{(v)}$ (m) | $x_k^{(v)}$ (m) | $a_1^{(v)}$ (m) | $a_2^{(v)}$ (m) | $k_y^{(v)}$ (N/m) | $c^{(v)}$ (Ns/m) | $m_e^{(v)}$ (kg) | $J_e^{(v)}$ (kg m$^2$) |
| 1 | 0.1 | 0.3 | | | | | | |
| 2 | 0.4 | 0.6 | 0.06667 | 0.13333 | $10^6$ | 38.5 | 1.6 | 3.2 |
| 3 | 0.8 | 1.0 | | | | | | |

Table 6
The first five eigenvalues of the uniform cantilever beam carrying three 2-dof spring–damper–mass systems, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ $(j = 1, \ldots, 5)$, shown in Fig. 3

| Methods | Eigenvalues, $\bar{\omega}_j = \bar{\omega}_{jR} \pm i\bar{\omega}_{jI}$ (rad/s) | | | | |
|---|---|---|---|---|---|
| | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ |
| FEM | −3.1916E−01 ± i231.0626 | −2.3776E01 ± i1697.9132 | −1.2063E01 ± i4038.8097 | −9.9509E00 ± i7798.6297 | −1.3437E01 ± i12865.7140 |
| EDM | −3.0916E−01 ± i231.0626 | −2.3804E01 ± i1698.3038 | −1.2064E01 ± i4038.8646 | −9.9511E00 ± i7798.6533 | −1.3432E01 ± i12865.7313 |
| FEM 1(bare beam) | 225.8249 | 1415.2234 | 3962.7231 | 7765.7136 | 12838.6537 |

those for the 2-dof spring–damper–mass system of the last examples, respectively, one may conclude that the distribution of the 2-dof spring–damper–mass systems along the beam length significantly influences the natural frequencies of the loading beam.

## 8. Conclusions

In this paper, each 2 degree-of-freedom (dof) spring–damper–mass system of a loading beam is replaced by one set of *equivalent dampers*, so that the free vibration analysis of a beam carrying multiple 2-dof spring–damper–mass systems may be performed on the bare beam supported by the same number of sets of equivalent dampers. Since damping effect of the *equivalent dampers* is more flexible (or adjustable) than that of the conventional dampers, the *equivalent dampers* will provide an alternative choice for the effective vibration absorbers. Furthermore, the *equivalent-damper* method presented in this paper also provides a simple approach for evaluating the overall damping effect of a spring–damper–mass system.

Among the three parameters of each 2-dof spring–damper–mass system, the influence on the (damped) natural frequencies of the loading beam due to the spring stiffness or the lumped mass is most predominant and that due to the damping coefficient is negligible. However, the influence on the decay rate of the free vibration responses of the loading beam due to the damping coefficient is most significant and that due to the spring stiffness or the lumped mass is negligible. In addition, the attaching location (or position) of the spring–damper–mass systems along the beam is also an important factor affecting the natural frequencies of the loading beam.

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