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Differential quadrature method for nonlinear vibration of orthotropic plates with finite deformation and transverse shear effect

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Abstract

Based on the Reddy's theory of plates with the effect of higher-order shear deformations, the governing equations for nonlinear vibration of orthotropic plates with finite deformations are presented. The nonlinear free vibration is analyzed by the differential quadrature method. The differential quadrature approach suggested by Wang and Bert is extended to handle the multiple boundary conditions of the plate. A new technique is also further extended to simplify nonlinear computations and the harmonic balance method is used in deriving the equation of motion. The numerical convergence and comparison studies are carried out to validate the present solutions. The results show that the presented differential quadrature method is fairly reliable and valid. Influences of geometric and material parameters, transverse shear deformations and rotation inertia, as well as vibration amplitudes, on the nonlinear free vibration characteristics of orthotropic plates are studied.

1. Introduction

Plates with various shapes and materials are important structural elements. They are widely used in modern engineering and science. There are many papers for the nonlinear behaviors of

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orthotropic and laminated plates based on the Kirchhoff assumption, which ignores the effect of transverse shear deformation [1–3]. It is more important to consider the effect of transverse shear deformation for orthotropic and laminated plates than for isotropic plates. Sathyamoorthy, Reddy and other researchers [4] studied nonlinear vibration of plates based on the first-order plate theory with a shear correction factor to account for the parabolic shear strain distribution. Reddy [5] developed the theory of plates, taking into account the effect of higher-order shear deformations, which eliminates the need for a shear correction factor. Tennetiand et al. [6–9] analyzed the nonlinear vibration of laminated plates according to the Reddy theory and by using finite element method.

Compared with the standard numerical techniques such as the finite element and finite difference methods, the differential quadrature method (DQM), originated by Bellman and Casti [10], is one of the high-efficiency methods in solving complex linear and nonlinear problems of solid and structural mechanics. Bert and Malik [11] provided an excellent overview of the publications on differential quadrature method. Bert et al. [12,13] employed successfully the DQM to solve the static and free vibration problems of isotropic and orthotropic plates. In order to simplify application and to improve efficiency and accuracy of the DQM, Wang and Bert [14,15] presented an approach to deal with boundary conditions. In addition, Chen [16] introduced the special matrix product to express the formulation of the nonlinear partial differential operator in an explicit matrix form. There are a lot of papers related to linear vibration of Mindlin Plates with various boundary conditions by using DQM [17,18]. To the best of the authors' knowledge, there have been no reports on the nonlinear vibration of plates with finite deformations by DQM.

In this paper, the DQM is further extended to deal with the nonlinear free vibration of orthotropic plates based on the Reddy's theory of plates with the effect of higher-order transverse shear deformation. Moreover, the differential quadrature approach presented by Wang and Bert (DQWB) is also further extended to handle the boundary conditions of higher-order moments of plates with finite deformations. A technique similar to paper [16] is extended to simplify nonlinear computations and the harmonic balance method [19] is used to derive the equations of motion. The numerical convergence and comparison studies are carried out to validate the present DQ solutions. Good convergence and agreement are achieved. Furthermore, numerical examples show the effects of geometric and material parameters of orthotropic plate on nonlinear vibration characteristics. At the same time, the effects of transverse shear deformations and rotation inertia on nonlinear vibration behaviors of orthotropic plate are studied, too.

2. Mathematical Model

Consider an orthotropic rectangular plate with the sides of lengths a and b along x and y axes, respectively, and thickness h. Let $X = x/a$, $Y = y/a$, the dimensionless displacements be $U(X, Y) = u(x, y)/h$, $V(X, Y) = v(x, y)/h$, $W(X, Y) = w(x, y)/h$, and $\Phi(X, Y)$ and $\Psi(X, Y)$ be the mid-plane rotations about the y and x axes, respectively.

Based on Reddy's theory of plates taking into account the effect of higher-order shear deformations [4], the governing equations of nonlinear free vibration of orthotropic plates with finite deformations can be written as

$$\begin{aligned}
&A_{11}U_{,XX} + A_{66}\lambda^2 U_{,YY} + (A_{12}+A_{66})\lambda V_{,XY} + W_{,X}(A_{66}\beta^{-1}\lambda^2 W_{,YY} + A_{11}\beta^{-1}W_{,XX})\\
&\quad + (A_{12}+A_{66})\lambda^2\beta^{-1}W_{,XY}W_{,Y} = \beta^{-2}\ddot{U},\\
&A_{22}\lambda^2 V_{,YY} + A_{66}V_{,XX} + (A_{12}+A_{66})\lambda U_{,XY} + W_{,Y}(A_{22}\lambda^3\beta^{-1}W_{,YY} + A_{66}\lambda\beta^{-1}W_{,XX})\\
&\quad + (A_{12}+A_{66})\lambda\beta^{-1}W_{,XY}W_{,X} = \beta^{-2}\ddot{V},\\
&G_1 W_{,XXXX} + G_2\lambda^4 W_{,YYYY} + G_3\lambda^2 W_{,XXYY} + G_4\beta\Phi_{,XXX} + G_5\beta\lambda^3\Psi_{,YYY}\\
&\quad + G_6(\beta\lambda^2\Phi_{,XYY} + \lambda\beta\Psi_{,YXX}) - G_7(\beta^3\Phi_{,X} + \beta^2 W_{,XX}) - G_8(\lambda\beta^3\Psi_{,Y} + \lambda^2\beta^2 W_{,YY})\\
&\quad + (\lambda^2\beta^2 A_{22}W_{,YY} + \beta^2 A_{12}W_{,XX})(\beta^{-1}\lambda V_{,Y} + \tfrac{1}{2}\beta^{-2}\lambda^2 W_{,Y}^2)\\
&\quad + (\lambda^2\beta^2 A_{12}W_{,YY} + \beta^2 A_{11}W_{,XX})(\beta^{-1}U_{,X} + \tfrac{1}{2}\beta^{-2}W_{,X}^2)\\
&\quad + 2A_{66}\beta^2\lambda W_{,XY}(\beta^{-1}\lambda U_{,Y} + \beta^{-1}V_{,X} + \lambda\beta^{-2}W_{,Y}W_{,X})\\
&\quad + \beta W_{,X}[A_{11}U_{,XX} + A_{66}\lambda^2 U_{,YY} + (A_{12}+A_{66})\lambda V_{,XY}\\
&\quad + W_{,X}(A_{66}\beta^{-1}\lambda^2 W_{,YY} + A_{11}\beta^{-1}W_{,XX}) + (A_{12}+A_{66})\lambda^2\beta^{-1}W_{,XY}W_{,Y}]\\
&\quad + \lambda\beta W_{,Y}[A_{22}\lambda^2 V_{,YY} + A_{66}V_{,XX} + (A_{12}+A_{66})\lambda U_{,XY}\\
&\quad + W_{,Y}(A_{22}\lambda^3\beta^{-1}W_{,YY} + A_{66}\lambda\beta^{-1}W_{,XX}) + (A_{12}+A_{66})\lambda\beta^{-1}W_{,XY}W_{,X}]\\
&= \ddot{W} + G_{13}\beta^{-2}\ddot{W}_{,XX} + G_{13}\beta^{-2}\lambda^2\ddot{W}_{,YY} + G_{14}\beta^{-1}\ddot{\Phi}_{,X} + G_{14}\beta^{-1}\lambda\ddot{\Psi}_{,Y}\\
&\quad - G_4 W_{,XXX} - G_6\lambda^2 W_{,XYY} + G_{10}\beta\Phi_{,XX} + G_{11}\beta\lambda^2\Phi_{,YY} + G_9\beta\lambda\Psi_{,YX}\\
&\quad + G_7(\beta^3\Phi + \beta^2 W_{,X}) = G_{15}\beta^{-1}\ddot{\Phi} + G_{16}\beta^{-2}\ddot{W}_{,X}\\
&\quad - G_5\lambda^3 W_{,YYY} - G_6\lambda W_{,XXY} + G_{11}\beta\Psi_{,XX} + G_9\beta\lambda\Phi_{,XY} + G_{12}\beta\lambda^2\Psi_{,YY}\\
&\quad + G_8(\beta^3\Psi + \lambda\beta^2 W_{,Y}) = G_{15}\beta^{-1}\ddot{\Psi} + G_{16}\beta^{-2}\lambda\ddot{W}_{,Y},
\end{aligned} \tag{1}$$

where $\lambda = a/b$, $\beta = a/h$ and $T = t/t_0$, $t_0 = (a^2/h)\sqrt{\rho/E_1}$, t is the time, E_1 is the Young's modulus in the x-direction, ρ is the density. Obviously, Eq. (1) is a set of nonlinear partial-differential equations, as in which include the product of unknown variables.

For convenience, we assume that the edge of the plate is simply supported. So the boundary conditions can be given as

$$\begin{aligned}
&U = V = W = P_1 = M_1 = \Psi = 0, \quad X = 0\\
&U = V = W = P_1 = M_1 = \Psi = 0, \quad X = 1\\
&U = V = W = P_2 = M_2 = \Phi = 0, \quad Y = 0\\
&U = V = W = P_2 = M_2 = \Phi = 0, \quad Y = 1,
\end{aligned} \tag{2}$$

in which the moment M_i and higher-order moment P_i are expressed as

$$\begin{aligned}
M_1 &= D_1\Phi_{,X} + D_2\Psi_{,Y} + D_3 W_{,XX} + D_4 W_{,YY},\\
M_2 &= D_5\Phi_{,X} + D_6\Psi_{,Y} + D_7 W_{,XX} + D_8 W_{,YY},
\end{aligned}$$

$$P_1 = D_9 \Phi_{,X} + D_{10} \Psi_{,Y} + D_{11} W_{,XX} + D_{12} W_{,YY},$$
$$P_2 = D_{13} \Phi_{,X} + D_{14} \Psi_{,Y} + D_{15} W_{,XX} + D_{16} W_{,YY}. \quad (3)$$

From Eqs. (3) and (2), the boundary conditions with simply supported edge can be simplified as

$$\begin{aligned} &U = V = W = \Psi = \Phi_{,X} = W_{,XX} = 0, \quad X = 0, \\ &U = V = W = \Psi = \Phi_{,X} = W_{,XX} = 0, \quad X = 1, \\ &U = V = W = \Phi = \Psi_{,Y} = W_{,YY} = 0, \quad Y = 0, \\ &U = V = W = \Phi = \Psi_{,Y} = W_{,YY} = 0, \quad Y = 1. \end{aligned} \quad (4)$$

All coefficients G_i and D_i in the above equations and boundary conditions can be found in Appendix A.

3. DQM on nonlinear vibration of orthotropic plates

The DQM approximates the partial derivative of a function, with respect to a spatial variable at a given discrete point, as a weighted linear sum of the function values at all discrete points chosen in the solution domain of the spatial variable. Consider a function F (representing U,V,W,Φ and Ψ) of the variables in the domain $(0 \leqslant X \leqslant 1, 0 \leqslant Y \leqslant 1)$ with $N \times N$ grid points along x and y axes, respectively. Then, the first-order partial derivative of the function $F(X, Y)$ at a given discrete point $X = X_i$ along any line $Y = Y_i$ parallel to x-axis may be approximated by

$$(F_{,X})_i = \left(\frac{\partial F}{\partial X}\right)_{x=x_i} = \sum_{k=1}^{N} A_{ik} F_k \equiv A_{ik} F_k, \quad i,k = 1,2,....,N. \quad (5)$$

Here, $F_k = F(X_k, Y_l)$, A_{ik} $(i,k = 1,2\dots,N)$ are the weighting coefficients of the first-order partial derivative and they may be obtained from the paper [11]. At the same time, for convenience of writing, we employ here summation convention, namely, the terms with repeat subscript in an expression (for example, k in Eq. (5)) express the sum about the subscripts from 1 to N.

The weighting coefficients of the higher-order partial derivatives with respect to x can be computed by matrix multiplication once A_{ik}'s are determined. Thus, one has

$$\begin{aligned} &(F_{,XX})_i = A_{ij} A_{jk} F_k = B_{ik} F_k, \\ &(F_{,XXX})_i = A_{ij} B_{jk} F_k = C_{ik} F_k, \\ &(F_{,XXXX})_i = A_{ij} C_{jk} F_k = B_{ij} B_{jk} F_k = D_{ik} F_k, \end{aligned} \quad (6)$$

where B_{ik}, C_{ik} and D_{ik} are the weighting coefficients of the second-, third- and fourth-order partial derivatives, respectively. The formulae in the y-direction are similar.

DQWB approach [14,15] is now further extended to handle the higher-order boundary conditions of plates taking into account the effect of transverse shear deformations. The essence of the DQWB approach is that boundary conditions are applied during formulation of the weighting coefficients for inner grid points.

Using Eqs. (5) and (6) in matrix form and noticing the boundary conditions about deflections $W_1 = W|_{X=0} = 0$, $W_N = W|_{X=1} = 0$ and $W_{1,XX} = W_{N,XX} = 0$ from Eq. (4), one obtains the equivalent equations as follows [14]:

$$\begin{aligned}(W_{,X})_i &= \bar{A}_{ik}W_k, \quad \bar{A}_{i1} = \bar{A}_{iN} = 0,\\ (W_{,XX})_i &= \bar{B}_{ik}W_k, \quad \bar{B}_{i1} = \bar{B}_{iN} = 0,\\ (W_{,XXX})_i &= \bar{C}_{ik}W_k, \quad \bar{C}_{i1} = \bar{C}_{iN} = 0,\\ (W_{,XXXX})_i &= \bar{D}_{ik}W_k, \quad \bar{D}_{i1} = \bar{D}_{iN} = 0.\end{aligned} \tag{7}$$

From the boundary conditions (4), it can be seen that the weighting coefficients $\bar{A}_{ik}$, $\bar{B}_{ik}$, $\bar{C}_{ik}$ and $\bar{D}_{ik}$ for U, V and W are similar.

Eq. (7) can be written as the following matrix forms:

$$\mathbf{W}_{,X} = \bar{A}\mathbf{W}, \quad \mathbf{W}_{,XX} = \bar{B}\mathbf{W}, \quad \mathbf{W}_{,XXX} = \bar{C}\mathbf{W}, \quad \mathbf{W}_{,XXXX} = \bar{D}\mathbf{W}, \tag{8}$$

where, $\mathbf{W} = [W_2, W_3, \dots W_{N-2}, W_{N-1}]^{\mathrm{T}}$ is a desired $(N-2)$ line vector, $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are $(N-2)\times(N-2)$ coefficient matrices. Similarly, we may obtain the formulae for U and V.

For the boundary conditions of $\Phi(X, Y)$ and $\Psi(X, Y)$, it is necessary to modify the corresponding coefficient matrices.

From the boundary conditions (4), the corresponding DQ approximate equations of the boundary conditions for $\Phi(X, Y)$ are given as

$$\Phi_1 = \Phi_N = 0, \quad Y = 0, 1, \tag{9a}$$

$$\sum_{k=1}^{N} A_{1k}\Phi_k = 0, \sum_{k=1}^{N} A_{Nk}\Phi_k = 0, \quad X = 0, 1, \tag{9b}$$

$$\Psi_1 = \Psi_N = 0, \quad X = 0, 1, \tag{9c}$$

$$\sum_{k=1}^{N} A_{1k}\Psi_k = 0, \sum_{k=1}^{N} A_{Nk}\Psi_k = 0, \quad Y = 0, 1. \tag{9d}$$

From Eqs. (9b) and (9d), the boundary values of (Φ_1, Φ_N) at $X = 0$, 1 and the boundary values of $(\Psi_1, \ \Psi_N)$ at $Y = 0$, 1 can be, respectively, expressed in terms of the values of Φ_i and Ψ_j at the inner points:

$$\begin{aligned}\Phi_1|_{X=0} &= \frac{A_{1N}\sum_{k=2}^{N-1} A_{Nk}\Phi_k - A_{NN}\sum_{k=2}^{N-1} A_{1k}\Phi_k}{A_{NN}A_{11} - A_{1N}A_{N1}},\\ \Phi_N|_{X=1} &= \frac{A_{N1}\sum_{k=2}^{N-1} A_{1k}\Phi_k - A_{11}\sum_{k=2}^{N-1} A_{Nk}\Phi_k}{A_{NN}A_{11} - A_{1N}A_{N1}},\\ \Psi_1|_{Y=0} &= \frac{A_{1N}\sum_{k=2}^{N-1} A_{Nk}\Psi_k - A_{NN}\sum_{k=2}^{N-1} A_{1k}\Psi_k}{A_{NN}A_{11} - A_{1N}A_{N1}},\\ \Psi_N|_{Y=1} &= \frac{A_{N1}\sum_{k=2}^{N-1} A_{1k}\Psi_k - A_{11}\sum_{k=2}^{N-1} A_{Nk}\Psi_k}{A_{NN}A_{11} - A_{1N}A_{N1}}.\end{aligned} \tag{10}$$

Similar to the way above, substituting Eqs. (9a), (9c) and (10) into Eqs. (5) and (6), we have

$$\begin{aligned}
&\mathbf{\Phi},_Y = \bar{A}\mathbf{\Phi}, \quad \mathbf{\Phi},_{YY} = \bar{B}\mathbf{\Phi}, \\
&\mathbf{\Phi},_X = \bar{\bar{A}}\mathbf{\Phi}, \quad \mathbf{\Phi},_{XX} = \bar{\bar{B}}\mathbf{\Phi}, \quad \mathbf{\Phi},_{XXX} = \bar{\bar{C}}\mathbf{\Phi}, \\
&\mathbf{\Psi},_X = \bar{A}\mathbf{\Psi}, \quad \mathbf{\Psi},_{XX} = \bar{B}\mathbf{\Psi}, \\
&\mathbf{\Psi},_Y = \bar{\bar{A}}\mathbf{\Psi}, \quad \mathbf{\Psi},_{YY} = \bar{\bar{B}}\mathbf{\Psi}, \quad \mathbf{\Psi},_{YYY} = \bar{\bar{C}}\mathbf{\Psi}.
\end{aligned} \tag{11}$$

Here, the forms of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are similar to those of $\mathbf{W}$, $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{\bar{C}}$ are $(N-2)\times(N-2)$ modified coefficient matrices.

Further dividing the two-dimensional domain into $N_x \times N_y$ grid points along x and y axes, respectively, the DQ formulation, in matrix form, for the partial derivative of a function $F(X,Y)$ (representing U, V and W) in two-dimensional domain may be given as follows [16]:

$$\begin{aligned}
&\mathbf{F}_{,X} = \bar{A}_x\mathbf{F}, \quad \mathbf{F}_{,XX} = \bar{B}_x\mathbf{F}, \quad \mathbf{F}_{,XXX} = \bar{C}_x\mathbf{F}, \quad \mathbf{F}_{,XXXX} = \bar{D}_x\mathbf{F}, \\
&\mathbf{F}_{,Y} = \mathbf{F}\bar{A}_y^{\mathrm{T}}, \quad \mathbf{F}_{,YY} = \mathbf{F}\bar{B}_y^{\mathrm{T}}, \quad \mathbf{F}_{,YYY} = \mathbf{F}\bar{C}_y^{\mathrm{T}}, \quad \mathbf{F}_{,YYYY} = \mathbf{F}\bar{D}_y^{\mathrm{T}}, \\
&\mathbf{F}_{,XY} = \bar{A}_x\mathbf{F}\bar{A}_y^{\mathrm{T}}, \qquad \mathbf{F}_{,XYY} = \bar{A}_x\mathbf{F}\bar{B}_y^{\mathrm{T}}, \\
&\mathbf{F}_{,XXY} = \bar{B}_x\mathbf{F}\bar{A}_y^{\mathrm{T}}, \qquad \mathbf{F}_{,XXYY} = \bar{B}_x\mathbf{F}\bar{B}_y^{\mathrm{T}}.
\end{aligned} \tag{12}$$

The DQ formulations in matrix form for the partial derivatives of the functions $\Phi(x, y)$ and $\Psi(x, y)$ in two-dimensional domain are similar; hence we have

$$\begin{aligned}
&\mathbf{\Phi}_{,Y} = \mathbf{\Phi}\bar{A}_y^{\mathrm{T}}, \quad \mathbf{\Phi}_{,YY} = \mathbf{\Phi}\bar{B}_y^{\mathrm{T}} \quad \mathbf{\Phi}_{,XYY} = \bar{\bar{A}}_x\mathbf{\Phi}\bar{B}_y^{\mathrm{T}}, \\
&\mathbf{\Phi}_{,X} = \bar{\bar{A}}_x\mathbf{\Phi}, \quad \mathbf{\Phi}_{,XX} = \bar{\bar{B}}_x\mathbf{\Phi}, \quad \mathbf{\Phi}_{,XXX} = \bar{\bar{C}}_x\mathbf{\Phi}, \mathbf{\Phi}_{,XY} = \bar{\bar{A}}_x\mathbf{\Phi}\bar{A}_y^{\mathrm{T}}, \\
&\mathbf{\Psi}_{,X} = \bar{A}_x\mathbf{\Psi}, \quad \mathbf{\Psi}_{,XX} = \bar{B}_x\mathbf{\Psi}, \quad \mathbf{\Psi}_{,XXY} = \bar{B}_x\mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}}, \\
&\mathbf{\Psi}_{,Y} = \mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}}, \quad \mathbf{\Psi}_{,YY} = \mathbf{\Psi}\bar{\bar{B}}_y^{\mathrm{T}}, \quad \mathbf{\Psi}_{,YYY} = \mathbf{\Psi}\bar{\bar{C}}_y^{\mathrm{T}}, \mathbf{\Psi}_{,XY} = \bar{A}_x\mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}}.
\end{aligned} \tag{13}$$

Other than $\mathbf{W}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ (please see Eqs. (8) and (11)), the unknown variables $\mathbf{F}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are rectangular unknown matrices. $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ and $\bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}$ with subscripts x and y stand for the DQ weighting coefficient matrices for the first-, second-, third- and fourth-order partial derivatives along x and y directions, respectively. The superscript T means the transpose of the matrices.

Applying DQ matrix formulas (12) and (13), the nonlinear vibration equation (1) can be discretized at each discrete point on all inner grids of the two-dimensional domain as

$$\begin{aligned}
&A_{11}\bar{B}_x\mathbf{U} + A_{66}\lambda^2\mathbf{U}\bar{B}_y^{\mathrm{T}} + (A_{12}+A_{66})\lambda\bar{A}_x\mathbf{V}\bar{A}_y^{\mathrm{T}} + (\beta^{-1}\bar{A}_x\mathbf{W})\circ(A_{66}\lambda^2\mathbf{W}\bar{B}_y^{\mathrm{T}} + A_{11}\bar{B}_x\mathbf{W}) \\
&\quad + (A_{12}+A_{66})(\lambda\bar{A}_x\mathbf{W}\bar{A}_y^{\mathrm{T}})\circ(\lambda\beta^{-1}\mathbf{W}\bar{A}_y^{\mathrm{T}}) = \beta^{-2}\ddot{\mathbf{U}}, \\
&A_{22}\lambda^2\mathbf{V}\bar{B}_y^{\mathrm{T}} + A_{66}\bar{B}_x\mathbf{V} + (A_{12}+A_{66})\lambda\bar{A}_x\mathbf{U}\bar{A}_y^{\mathrm{T}} + (\lambda\beta^{-1}\mathbf{W}\bar{A}_y^{\mathrm{T}})\circ(A_{22}\lambda^2\mathbf{W}\bar{B}_y^{\mathrm{T}} + A_{66}\bar{B}_x\mathbf{W}) \\
&\quad + (A_{12}+A_{66})(\lambda\bar{A}_x\mathbf{W}\bar{A}_y^{\mathrm{T}})\circ(\beta^{-1}\bar{A}_x\mathbf{W}) = \beta^{-2}\ddot{\mathbf{V}},
\end{aligned}$$

$$
\begin{aligned}
& G_1\bar{D}_x\mathbf{W} + G_2\lambda^4\mathbf{W}\bar{D}_y^{\mathrm{T}} + G_3\lambda^2\bar{B}_x\mathbf{W}\bar{B}_y^{\mathrm{T}} + G_4\beta\bar{\bar{C}}_x\mathbf{\Phi} + G_5\beta\lambda^3\mathbf{\Psi}\bar{\bar{C}}_Y^{\mathrm{T}} \\
&\quad + G_6(\beta\lambda^2\bar{\bar{A}}_x\mathbf{\Phi}\bar{B}_y^{\mathrm{T}} + \beta\lambda\bar{B}_x\mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}}) - G_7(\beta^3\bar{\bar{A}}_x\mathbf{\Phi} + \beta^2\bar{B}_x\mathbf{W}) - G_8(\lambda\beta^3\mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}} + \beta^2\lambda^2\mathbf{W}\bar{B}_y^{\mathrm{T}}) \\
&\quad + [\lambda^2\beta^2A_{22}(\mathbf{W}\bar{B}_y^{\mathrm{T}}) + \beta^2A_{12}(\bar{B}_x\mathbf{W})] \circ [\beta^{-1}\lambda(\mathbf{V}\bar{A}_y^{\mathrm{T}}) + \tfrac{1}{2}\beta^{-2}\lambda^2(\mathbf{W}\bar{A}_y^{\mathrm{T}}) \circ (\mathbf{W}\bar{A}_y^{\mathrm{T}})] \\
&\quad + [\lambda^2\beta^2A_{12}(\mathbf{W}\bar{B}_y^{\mathrm{T}}) + \beta^2A_{11}(\bar{B}_x\mathbf{W})] \circ [\beta^{-1}(\bar{A}_x\mathbf{U}) + \tfrac{1}{2}\beta^{-2}(\bar{A}_x\mathbf{W}) \circ (\bar{A}_x\mathbf{W})] \\
&\quad + 2A_{66}\beta^2\lambda(\bar{A}_x\mathbf{W}\bar{B}_y^{\mathrm{T}}) \circ [\beta^{-1}\lambda(\mathbf{U}\bar{A}_y^{\mathrm{T}}) + \beta^{-1}(\bar{A}_x\mathbf{V}) + \lambda\beta^{-2}(\bar{A}_x\mathbf{W}) \circ (\mathbf{W}\bar{A}_y^{\mathrm{T}})] \\
&\quad + \beta(\bar{A}_x\mathbf{W}) \circ [A_{11}\bar{B}_x\mathbf{U} + A_{66}\lambda^2\mathbf{U}\bar{B}_y^{\mathrm{T}} + (A_{12} + A_{66})\lambda\bar{A}_x\mathbf{V}\bar{A}_y^{\mathrm{T}} \\
&\quad + (\beta^{-1}\bar{A}_x\mathbf{W}) \circ (A_{66}\lambda^2\mathbf{W}\bar{B}_y^{\mathrm{T}} + A_{11}\bar{B}_x\mathbf{W}) + (A_{12} + A_{66})(\lambda\bar{A}_x\mathbf{W}\bar{A}_y^{\mathrm{T}}) \circ (\lambda\beta^{-1}\mathbf{W}\bar{A}_y^{\mathrm{T}})] \\
&\quad + \lambda\beta(\mathbf{W}\bar{A}_y^{\mathrm{T}}) \circ [A_{22}\lambda^2\mathbf{V}\bar{B}_y^{\mathrm{T}} + A_{66}\bar{B}_x\mathbf{V} + (A_{12} + A_{66})\lambda\bar{A}_x\mathbf{U}\bar{A}_y^{\mathrm{T}} \\
&\quad + (\lambda\beta^{-1}\mathbf{W}\bar{A}_y^{\mathrm{T}}) \circ (A_{22}\lambda^2\mathbf{W}\bar{B}_y^{\mathrm{T}} + A_{66}\bar{B}_x\mathbf{W}) + (A_{12} + A_{66})(\lambda\bar{A}_x\mathbf{W}\bar{A}_y^{\mathrm{T}}) \circ (\beta^{-1}\bar{A}_x\mathbf{W})] \\
&= \ddot{\mathbf{W}} + G_{13}\beta^{-2}\bar{B}_x\ddot{\mathbf{W}} + G_{13}\beta^{-2}\lambda^2\ddot{\mathbf{W}}\bar{B}_y^{\mathrm{T}} + G_{14}\beta^{-1}\bar{\bar{A}}_x\ddot{\mathbf{\Phi}} + G_{14}\beta^{-1}\lambda\ddot{\mathbf{\Psi}}\bar{\bar{A}}_y^{\mathrm{T}} \\
&\quad - G_4\bar{C}_x\mathbf{W} - G_6\lambda^2\bar{A}_x\mathbf{W}\bar{B}_y^{\mathrm{T}} + G_{10}\beta\bar{\bar{B}}_x\mathbf{\Phi} + G_{11}\beta\lambda\mathbf{\Phi}\bar{B}_y^{\mathrm{T}} \\
&\quad + G_9\beta\lambda\bar{A}_x\mathbf{\Psi}\bar{\bar{A}}_y^{\mathrm{T}} + G_7(\beta^3\mathbf{\Phi} + \beta^2\bar{A}_x\mathbf{W}) = G_{15}\beta^{-1}\ddot{\mathbf{\Phi}} + G_{16}\beta^{-2}\bar{A}_x\ddot{\mathbf{W}} \\
&\quad - G_5\lambda^3\mathbf{W}\bar{C}_y^{\mathrm{T}} - G_6\lambda\bar{B}_x\mathbf{W}\bar{A}_y^{\mathrm{T}} + G_{11}\beta\bar{B}_x\mathbf{\Psi} + G_9\beta\lambda\bar{\bar{A}}_x\mathbf{\Phi}\bar{A}_y^{\mathrm{T}} \\
&\quad + G_{12}\beta\lambda^2\mathbf{\Psi}\bar{\bar{B}}_y^{\mathrm{T}} + G_8(\beta^3\mathbf{\Psi} + \lambda\beta^2\mathbf{W}\bar{A}_y^{\mathrm{T}}) = G_{15}\beta^{-1}\ddot{\mathbf{\Psi}} + G_{16}\beta^{-2}\lambda\ddot{\mathbf{W}}\bar{A}_y^{\mathrm{T}}.
\end{aligned}
\tag{14}
$$

Here, the symbol "$\circ$" expresses Hadamard product of matrices defined as

$$A \circ B = [\, a_{ij} \quad b_{ij} \,] \in C^{N\times M},$$

where $A = [a_{ij}]$, $B = [b_{ij}] \in C^{N\times M}$, $C^{N\times M}$ denotes the set of $N \times M$ real matrices.

We have to point out that the boundary conditions (4) have been applied when the coefficients in Eq. (14) are calculated. Thus, the boundary conditions (4) must not be reconsidered when Eq. (14) is solved.

Using Hadamard and Kronecker products of matrices [16,20], and ignoring in-plane inertia, the coupled nonlinear formulations can be converted into an explicit matrix form as follows:

$$L_1\bar{\mathbf{U}} + L_2\bar{\mathbf{V}} + (L_7\bar{\mathbf{W}}) \circ (L_1\bar{\mathbf{W}}) + (L_8\bar{\mathbf{W}}) \circ (L_2\bar{\mathbf{W}}) = 0, \tag{15a}$$

$$L_2\bar{\mathbf{U}} + L_3\bar{\mathbf{V}} + (L_8\bar{\mathbf{W}}) \circ (L_3\bar{\mathbf{W}}) + (L_7\bar{\mathbf{W}}) \circ (L_2\bar{\mathbf{W}}) = 0, \tag{15b}$$

$$
\begin{aligned}
& L_4\bar{\mathbf{W}} + H_7\ddot{\mathbf{\Phi}} + H_8\ddot{\mathbf{\Psi}} + (L_5\bar{\mathbf{W}}) \circ [L_7\bar{\mathbf{U}} + \tfrac{1}{2}(L_7\bar{\mathbf{W}}) \circ (L_7\bar{\mathbf{W}})] \\
&\quad + (L_6\bar{\mathbf{W}}) \circ [L_8\bar{\mathbf{V}} + \tfrac{1}{2}(L_8\bar{\mathbf{W}}) \circ (L_8\bar{\mathbf{W}})] \\
&\quad + \frac{2A_{66}\beta^2}{A_{12} + A_{66}}(L_2\bar{\mathbf{W}}) \circ [L_8\bar{\mathbf{U}} + L_7\bar{\mathbf{V}} + (L_7\bar{\mathbf{W}}) \circ (L_8\bar{\mathbf{W}})]
\end{aligned}
$$

$$= \ddot{\bar{\mathbf{W}}} + G_{13}\beta^{-2}\bar{B}_x\ddot{\bar{\mathbf{W}}} + G_{13}\beta^{-2}\lambda^2\ddot{\mathbf{W}}\bar{B}_y^T + G_{14}\beta^{-1}\bar{\bar{A}}_x\ddot{\bar{\mathbf{\Phi}}} + G_{14}\beta^{-1}\lambda\ddot{\mathbf{\Psi}}\bar{\bar{A}}_y^{\mathrm{T}}, \tag{15c}$$

$$H_1\bar{\mathbf{W}} + H_2\bar{\mathbf{\Phi}} + H_3\bar{\mathbf{\Psi}} = G_{15}\beta^{-1}\ddot{\bar{\mathbf{\Phi}}} + G_{16}\beta^{-2}\bar{A}_x\ddot{\bar{\mathbf{W}}}, \tag{15d}$$

$$H_4\bar{\mathbf{W}} + H_5\bar{\mathbf{\Phi}} + H_6\bar{\mathbf{\Psi}} = G_{15}\beta^{-1}\ddot{\mathbf{\Psi}} + G_{16}\beta^{-2}\lambda\ddot{\mathbf{W}}\bar{A}_y^{\mathrm{T}}, \tag{15e}$$

in which the expressions of L_i and H_i are listed in Appendix B, and $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$, $\bar{\mathbf{W}}$, $\bar{\mathbf{\Phi}}$ and $\bar{\mathbf{\Psi}}$ are vectors generated by stacking the rows of the corresponding rectangular matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ into one column vector.

In order to avoid the ill-conditioning matrix and easily decouple, the coupled nonlinear equations (15) are changed to the equivalent forms by algebraic operation. The process given in the present paper is different from the approach in Ref. [16].

From Eqs. (15a) and (15b), the unknown vectors $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ in terms of $\bar{\mathbf{W}}$ can be expressed as

$$\begin{aligned} \bar{\mathbf{U}} &= L_9^{-1}L_{23}^{-1}H_{21}(\bar{\mathbf{W}}) - L_9^{-1}L_{32}^{-1}H_{12}(\bar{\mathbf{W}}), \\ \bar{\mathbf{V}} &= L_{10}^{-1}L_{12}^{-1}H_{21}(\bar{\mathbf{W}}) - L_{10}^{-1}L_{21}^{-1}H_{12}(\bar{\mathbf{W}}), \end{aligned} \tag{16}$$

where

$$\begin{gathered} L_{21} = L_1 + L_2, \quad L_{12} = L_1 - L_2, \quad L_{32} = L_2 + L_3, \quad L_{23} = L_2 - L_3, \\ L_9 = L_{32}^{-1}L_{21} - L_{23}^{-1}L_{12}, \quad L_{10} = L_{21}^{-1}L_{32} - L_{12}^{-1}L_{23}, \\ H_{12}(\bar{\mathbf{W}}) = (L_{21}\bar{\mathbf{W}}) \circ (L_7\bar{\mathbf{W}}) + (L_{32}\bar{\mathbf{W}}) \circ (L_8\bar{\mathbf{W}}), \\ H_{21}(\bar{\mathbf{W}}) = (L_{12}\bar{\mathbf{W}}) \circ (L_7\bar{\mathbf{W}}) + (L_{23}\bar{\mathbf{W}}) \circ (L_8\bar{\mathbf{W}}). \end{gathered}$$

Substituting Eq. (16) into Eq. (15c), the coupling equations (15a)–(15c) can be decoupled. By applying the harmonic balance method [19] and neglecting the higher harmonic component, Eqs. (15c)–(15e) may be solved iteratively.

The grid spacing pattern in this paper is given as follows [11]:

$$X_i = \tfrac{1}{2}[1 - \cos\frac{(i-1)\pi}{N_x - 1}], \quad i = 1, 2, \ldots N_x,$$

$$Y_i = \tfrac{1}{2}[1 - \cos\frac{(i-1)\pi}{N_y - 1}], \quad i = 1, 2, \ldots N_y, \tag{17}$$

in which, X_i and Y_i are the spacing grids in x and y directions, respectively. Wang and Bert [14] pointed out that DQM can yield good results for static and free vibration analyses of rectangular plates with various aspect ratios, a/b, when the same number of grids points along x and y axes is employed, namely, $N_x = N_y = N$. Thus, in the following computation, we apply the same number of grid points in x and y directions.

4. Results and conclusions

The presented procedure may be, at the same time, employed to solve both linear and nonlinear problems and make comparison between them. In numerical computations, the nonlinear vibration behavior is described by the ratio of nonlinear vibration frequency to the corresponding linear vibration frequency, namely, (ω/ω_0).

4.1. Convergence and comparison studies

In other papers, the convergences of the DQM for linear free vibration and the static geometrically nonlinear analysis of isotropic and orthotropic rectangular plates have been discussed [12,13], respectively. It can be seen that the DQ solution has very good convergence. In this section, the numerical convergence and comparison studies of the DQ solution for geometrically nonlinear free vibration are carried out first by considering the ratio of nonlinear vibration frequency to linear vibration frequency, namely, (ω/ω_0).

For nonlinear free vibration of an isotropic square plate with finite deformations and shear deformation effects, the frequency ratios (ω/ω_0) for various amplitude ratios are obtained by using the present approach with different grids of sampling points. Results are shown in Fig. 1 and Table 1, together with the other solutions for thin plates [21]. From Fig. 1 and Table 1, it is seen that the numerical solutions obtained from DQWB converge rapidly with the grid refinement. The DQ solutions obtained from the grid sizes of 7×7 and 9×9 own the same accuracy. From Table 1, it can be seen that we only employ the grid size of 5×5 to obtain satisfactory results. Thus, in the following computation, we apply 5×5 unequally spaced grids.

Moreover, for thin plates, it is seen that all results are consistent, in spite of that, we employ the Reddy's theory of plates with the effect of higher-order shear deformations and finite deformations or the Kirchhoff theory of thin plates with finite deformations. But for plates in which the thickness-to-width ratio is no longer small, there is the difference between results obtained from different theories of plates. This will be discussed later.

Based on the analysis above, the present method has good reliability and accuracy. Next, we will give typical numerical examples to show the effects of geometric and material parameters, transverse shear deformation and rotation inertia, as well as amplitude of nonlinear vibration on the frequency ratio ω/ω_0. To consider the effect of orthotropy, the material parameters in computation are listed in Table 2.

4.2. Parameter study

Fig. 2 shows the effect of plate length-to-width ratio a/b on the amplitude–frequency ratio curves of nonlinear vibration of orthotropic plates. In computation, we take the plate thickness-to-length ratio $h/a = 0.1$ and the material parameters are listed in Table 2.

It can be seen that the frequency ratio ω/ω_0 increases as a/b changes from 1 to 0.1 or from 1 to 2 for same dimensionless amplitude. It is demonstrated that the difference between nonlinear vibration frequency ω and linear vibration frequency ω_0 of the rectangular plate is larger than that of the square plate under a certain thickness-to-length ratio (h/a) when dimensionless amplitude is under a certain range. At the same time, it is observed that the difference between ω and ω_0 increases with an increase in the dimensionless amplitude.

Fig. 3 shows the effect of thickness-to-length ratio (h/a) of the plate on amplitude–frequency curves of nonlinear vibration of an orthotropic square plate with orthotropy

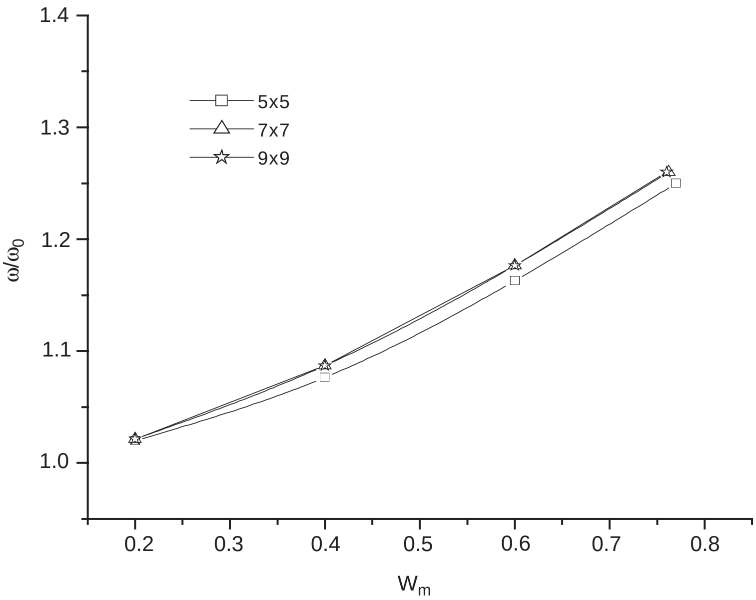

Fig. 1. DQ solution of central deflection of the square plate ($a/h = 100$, $v = 0.3$) for different grids.

Table 1
ω/ω_0 comparison for isotropic square plate ($a/h = 240$, $v = 0.3$)

$W_{\max}$	0.2	0.4	0.6	0.8	1
Present DQ 5×5	1.0186	1.0736	1.1548	1.2641	1.3929
Present DQ 7×7	1.0194	1.0760	1.1641	1.2790	1.4142
Elliptic function [19]	1.0195	1.0757	1.1625	1.2734	1.4024
Peturbation [19]	1.0196	1.0761	1.1642	1.2774	1.4097
FEM [19]	1.0185	1.0716	1.1533	1.2565	1.3752

Table 2
Material parameters in computation in Figs. 2–5

Materials	E_1	E_2	g_{12}	g_{13}	g_{23}	υ_{12}	υ_{21}	E_1/g_{13}
Isotropy	14.7					0.25		2.5
Orthotropy A	145.5	76.4	42.6	25.91	43.23	0.44	0.23	5.6
Orthotropy B	128	8	4.5	4.5	1.6	0.28	0.28	28.4
Orthotropy C	144.79	9.65	4.14	4.14	3.312	0.3	0.3	34.5
Orthotropy D	174.6	6.98	3.49	3.49	1.4	0.25	0.25	50

material B in Table 2. It is seen that an increase in the thickness-to-length ratio results in an increase in the difference between the nonlinear vibration frequency and linear vibration frequency.

Further, the effect of transverse shear deformations on nonlinear vibration behaviors of the orthotropic plate is investigated by comparison of the frequency ω with frequency ω_1, in which ω is the nonlinear frequency, including the effects of transverse shear deformation and rotation inertia, and ω_1 is the corresponding linear frequency obtained from the classical theory of plates, excluding these effects.

The ratios ω/ω_1 have been computed for various non-dimensional amplitudes. The results are graphically presented in Figs. 4 and 5 for various thickness-to-length ratios and material parameters, respectively. It can be seen that, with the increase of h/a or E_1/G_{13}, the effects of transverse shear deformation and rotation inertia on the frequency ratio ω/ω_1 are apparently increased when the dimensionless amplitude is under a certain range, especially at small amplitudes.

5. Concluding remarks

The nonlinear free vibration problem of orthotropic plates with finite deformations and the effect of higher-order transverse shear deformations is studied by using DQM. Good convergence is presented even when only a small number of grid points are used. A wide variety of cases are performed to examine the nonlinear free vibration characteristics of orthotropic plates. The difference between the nonlinear vibration frequency and linear vibration frequency increases with increases in the dimensionless amplitude and the thickness-to-length ratio, and the difference for a rectangular plate is larger than that of a square plate when the dimensionless amplitude is under a

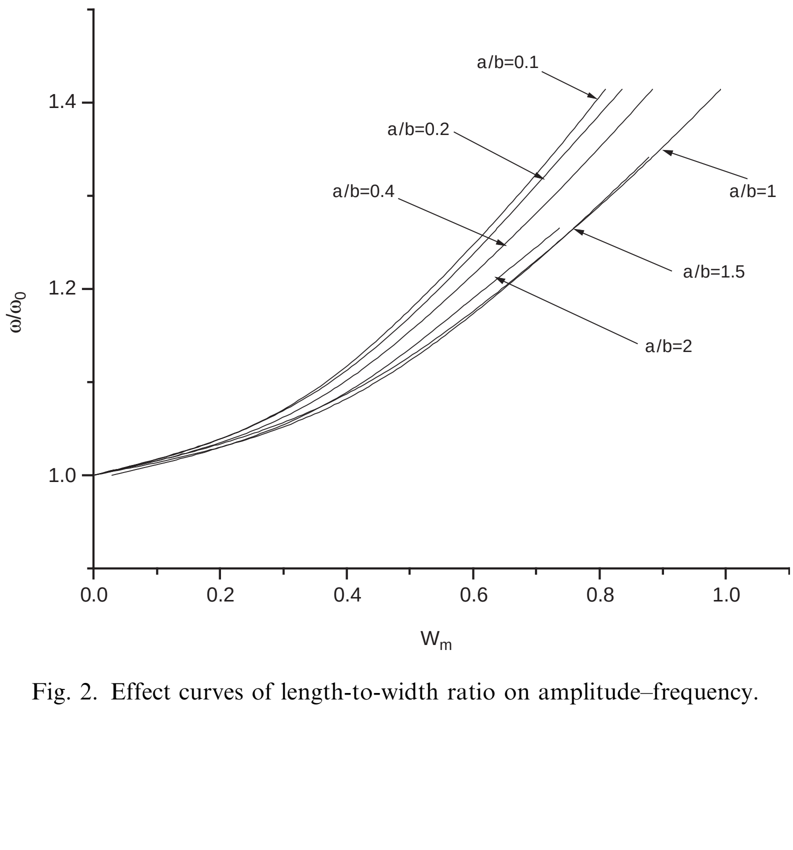

Fig. 2. Effect curves of length-to-width ratio on amplitude–frequency.

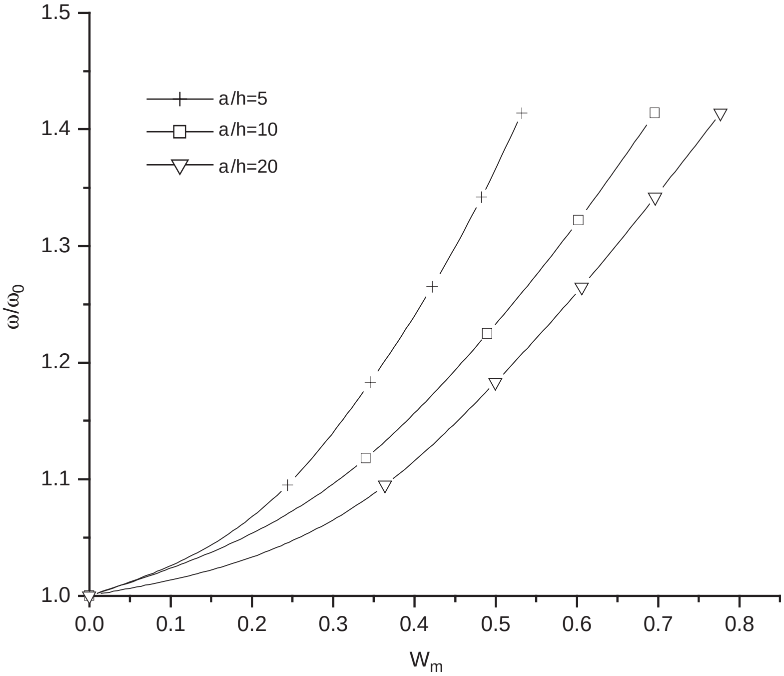

Fig. 3. Effect curves of thickness on amplitude–frequency.

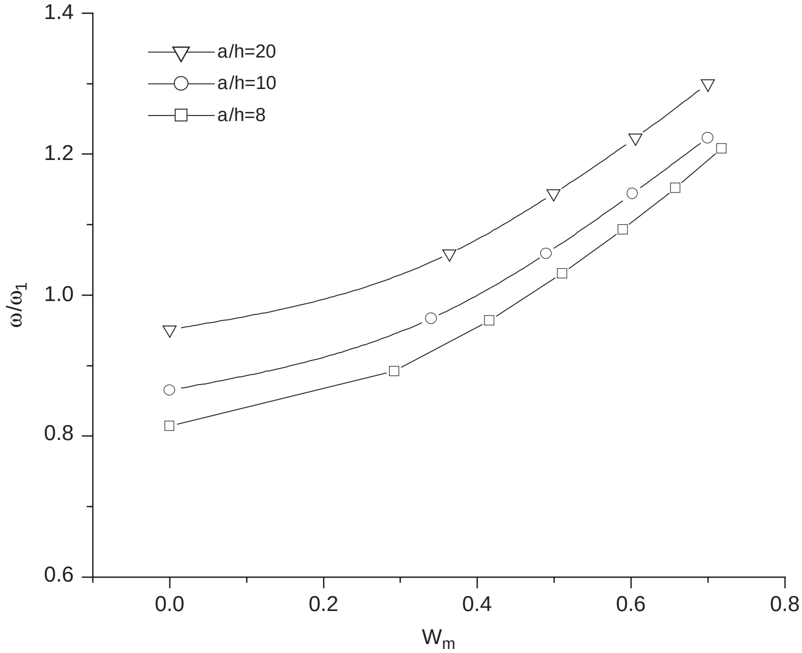

Fig. 4. Effect curves of shear deformation on amplitude–frequency.

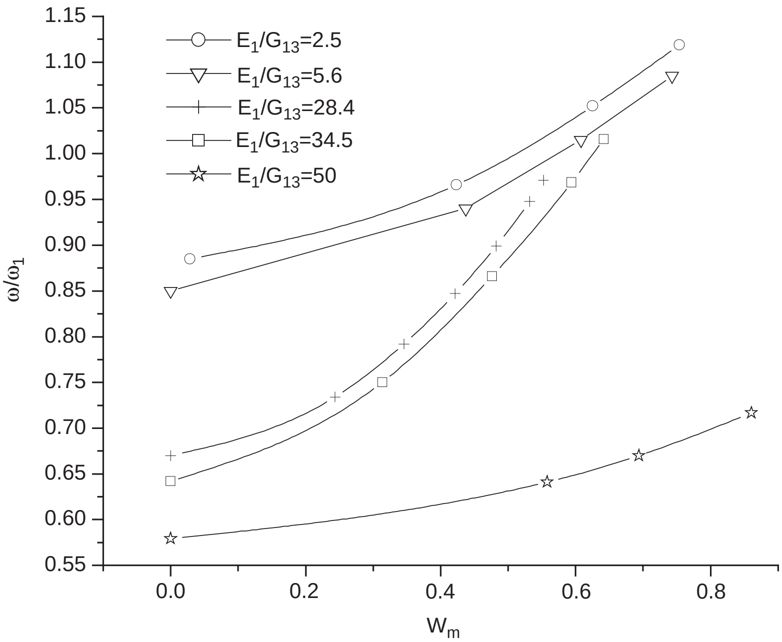

Fig. 5. Effect curves of material parameter on amplitude–frequency.

certain range. The effects of transverse shear deformation and rotation inertia on the frequency ratio are apparently increases when the thickness-to-length ratio or material orthotropy increases. It can be seen that the present DQM is accurate and efficient for solving complex nonlinear problems.

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Appendix A

G_i and D_i in Eqs. (1) and (3) are given by

$$A_{11} = \mu, \quad A_{12} = ev_1\mu, \quad A_{22} = e\mu, \quad A_{66} = g_{12},$$
$$G_1 = -\mu/252, \quad G_2 = -e\mu/252, \quad G_3 = -(2g_{12}/E_1 + v_1 e\mu)/126, \quad G_4 = 4\mu/315, \quad G_5 = 4e\mu/315,$$
$$G_6 = 4(2g_{12}/E_1 + v_1 e\mu)/315, \quad G_7 = -8g_{13}/15E_1, \quad G_8 = -8g_{23}/15E_1, \quad G_{10} = 17\mu/315, \quad G_{15} = 17/315,$$
$$G_9 = 17(g_{12}/E_1 + v_1 e\mu)/315, \quad G_{11} = 17g_{12}/315E_1, \quad G_{12} = 17e\mu/315, \quad G_{13} = -1/252, \quad G_{14} = -G_{16} = 4/315,$$
$$D_1 = h^4\mu E_1/15a, \quad D_2 = h^4 v_1\mu E_2/15b, \quad D_3 = -h^4\mu E_1/60a^2, \quad D_4 = -h^4 v_1 \mu E_2/60b^2,$$
$$D_5 = v_1 e D_1, \quad D_6 = D_2/v_1, \quad D_7 = v_1 e D_3, \quad D_8 = D_4/v_1,$$
$$D_9 = h^6\mu E_1/105a, \quad D_{10} = h^6 v_1\mu E_2/105b, \quad D_{11} = -h^6\mu E_1/336a^2, \quad D_{12} = -h^4 v_1\mu E_2/336b^2,$$
$$D_{13} = v_1 e D_9, \quad D_{14} = D_{10}/v_1, \quad D_{15} = v_1 e D_{11}, \quad D_{16} = D_{12}/v_1.$$
$$\text{where } \mu = 1/(1 - v_1 v_2), \quad e = E_2/E_1.$$

Appendix B

L_i and H_i in Eqs. (19) and (3) are given by

$$L_1 = A_{11}(I_y \otimes \bar{B}_x) + A_{66}\lambda^2(\bar{B}_y \otimes I_x), \quad L_2 = (A_{12} + A_{66})\lambda(\bar{A}_y \otimes \bar{A}_x), \quad L_7 = \beta^{-1}(I_y \otimes \bar{A}_x),$$
$$L_8 = \lambda\beta^{-1}(\bar{A}_y \otimes I_x), \quad L_3 = A_{22}\lambda^2(\bar{B}_y \otimes I_x) + A_{66}(I_y \otimes \bar{B}_x),$$
$$L_5 = \lambda^2\beta^2 A_{12}(\bar{B}_y \otimes I_x) + \beta^2 A_{11}(I_y \otimes \bar{B}_x), \quad L_6 = \lambda^2\beta^2 A_{22}(\bar{B}_y \otimes I_x) + \beta^2 A_{12}(I_y \otimes \bar{B}_x),$$
$$L_4 = G_1(I_y \otimes \bar{D}_x) + G_2\lambda^4(\bar{D}_y \otimes I_x) + G_3\lambda^2(\bar{B}_y \otimes \bar{B}_x) - G_7\beta^2(I_y \otimes \bar{B}_x) - G_8\lambda^2\beta^2(\bar{B}_y \otimes I_x),$$
$$H_7 = G_4\beta(I_y \otimes \bar{\bar{C}}_x) + G_6\beta\lambda^2(\bar{B}_y \otimes \bar{\bar{A}}_x) - G_7\beta^3(I_y \otimes \bar{\bar{A}}_x),$$
$$H_8 = G_5\beta\lambda^3(\bar{\bar{C}}_y \otimes I_x) + G_6\lambda\beta(\bar{\bar{A}}_y \otimes \bar{B}_x) - G_8\lambda\beta^3(\bar{\bar{A}}_y \otimes I_x),$$

$$H_1 = -G_4(I_y \otimes \bar{C}_x) - G_6\lambda^2(\bar{B}_y \otimes \bar{A}_x) + G_7\beta^2(I_y \otimes \bar{A}_x),$$
$$H_2 = G_{10}\beta(I_y \otimes \bar{\bar{B}}_x) + G_{11}\beta\lambda(\bar{B}_y \otimes I_x) + G_7\beta^3(I_y \otimes I_x),$$
$$H_3 = G_9\beta\lambda(\bar{\bar{A}}_y \otimes \bar{A}_x), \quad H_5 = G_9\beta\lambda(\bar{A}_y \otimes \bar{\bar{A}}_x),$$
$$H_4 = -G_5\lambda^3(\bar{C}_y \otimes I_x) - G_6\lambda(\bar{A}_y \otimes \bar{B}_x) + G_8\lambda\beta^2(\bar{A}_y \otimes I_x),$$
$$H_6 = G_{12}\beta\lambda^2(\bar{\bar{B}}_y \otimes I_x) + G_{11}\beta(I_y \otimes \bar{B}_x) + G_8\beta^3(I_y \otimes I_x),$$

where the symbol $\otimes$ denotes the Kronecker product of matrices, I_x and I_y are the unit matrices.

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