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Short Communication

An analysis of a nonlinear elastic force van der Pol oscillator equation

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The one-dimensional, nonlinear elastic force van der Pol oscillator equation is

$$\ddot{x} + |x|x = \varepsilon(1 - x^2)\,\dot{x}, \tag{1}$$

where ε is a positive parameter. The aim of this paper is to see if Eq. (1) has a limit-cycle and calculate an approximation to this periodic solution. The assumption that Eq. (1) has a limit-cycle is premised on the fact that it has the same general structure as the van der Pol equation [1,2]. The existence of limit-cycles is a feature of great practical importance in science and engineering because many devices can be modelled by such nonlinear systems. Since most, if not all nonlinear differential equations cannot be solved analytically, it is then of great importance to investigate the possible existence of a limit-cycle. After putting Eq. (1) in system form, the fixed points will be determined and the local stability properties of the fixed point determined. The Lienard–Levinson–Smith Theorem will then be used to show that a unique, stable limit-cycle exists. The method of harmonic balance can now be applied to calculate an analytic approximation to the periodic solution. The result obtained is then compared with numerical integration of Eq. (1).

Eq. (1) can be written in the form of a coupled, first-order system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -|x|x + \varepsilon(1 - x^2)y. \tag{2}$$

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Inspection of these equations show that there is a single fixed point at $(\bar{x}, \bar{y}) = (0, 0)$. Since the trajectories in phase space are solutions of

$$\frac{dy}{dx} = \frac{-|x|x + \varepsilon(1 - x^2)y}{y}, \tag{3}$$

then the curve along which the trajectories in phase space have zero slope is

$$y_0(x) = \left(\frac{1}{\varepsilon}\right)\frac{|x|x}{1 - x^2}. \tag{4}$$

Similarly, the curve along which $dy/dx = \infty$ is the x-axis. From these two equations it can be concluded that the path of the trajectory for a point far from the fixed point moves towards the origin [3]. The argument for this is given below.

The stability of the fixed point at $(\bar{x}, \bar{y}) = (0, 0)$ is studied by assuming that $|x|$ and $|y|$ are small and then writing in Eq. (1) as

$$\ddot{x} + |x|x \simeq \varepsilon\dot{x}, \quad \varepsilon > 0. \tag{5}$$

The equations of the trajectories, under this condition, are given by

$$\frac{dx}{dt} \simeq y, \quad \frac{dy}{dt} \simeq -|x|x + \varepsilon y. \tag{6}$$

Now consider the function $V(x, y)$ given by

$$V(x, y) = \frac{y^2}{2} + \tfrac{1}{3}|x|x^2 \geqslant 0. \tag{7}$$

Therefore,

$$\frac{dV}{dt} = y\frac{dy}{dt} + |x|x\frac{dx}{dt} = (-|x|x + \varepsilon y)y + |x|xy,$$

and making the substitution for $\dot{x}$ and $\dot{y}$ from Eq. (6) gives

$$\frac{dV}{dt} = \varepsilon y^2 \geqslant 0. \tag{8}$$

Thus, trajectories near the fixed point, $(\overline{x}, \overline{y}) = (0, 0)$, move away from it, and it can be concluded that the fixed point is unstable.

Consequently, a typical trajectory of a point starting far from the fixed point spirals towards the fixed point and the one in the neighbourhood of the fixed point spirals outwards. Since such trajectories cannot cross each other, it follows that there exists at least one closed curve in the phase space. Hence Eq. (1) has at least one stable limit-cycle.

This result also follows from the Lineard–Levinson–Smith theorem [1,4]. The equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{9}$$

has a unique periodic solution if f and g are continuous, and (1) if there exists some number $a > 0$ such that $f(x) < 0$ for $-a < x < a$, and $f(x) > 0$ otherwise; (2) $xg(x) > 0$ for $|x| > 0$; (3) $f(x) = f(-x)$

and $g(x) = -g(-x)$, with

$$\int_0^\infty g(x)\,\mathrm{d}x = \infty; \tag{10}$$

(4)

$$\int_0^\infty f(x)\,\mathrm{d}x = \infty, \tag{11}$$

and (5)

$$G(x) = \int_0^\infty g(u)\,\mathrm{d}u, \tag{12}$$

where

$$G(-a) = G(a). \tag{13}$$

Applying this theorem to Eq. (1) then

$$g(x) = |x|x, \tag{14}$$

and

$$f(x) = \varepsilon(x^2 - 1). \tag{15}$$

Since all these conditions of the theorem hold, it follows that Eq. (1) has a unique, stable limit-cycle.

To calculate an approximation to the periodic (limit-cycle) solution using the method of harmonic balance [3,5] it is assumed that Eq. (1) has a solution of the form

$$x(t) \approx A\,\cos(\omega t), \tag{16}$$

where a priori, A and ω are unknown. The Fourier expansion of $|\cos u|$ is [3]

$$\cos(u)| = \frac{4}{\pi}\left[\frac{1}{2} + \frac{\cos(2u)}{3} - \frac{\cos(4u)}{15} + \cdots\right]. \tag{17}$$

Therefore,

$$\begin{aligned} |x|x &\approx |A\,\cos(\omega t)|(A\,\cos(\omega t)) \\ &= \frac{4A^2}{\pi}\left[\frac{1}{2} + \frac{\cos(2\omega t)}{3} - \frac{\cos(4\omega t)}{15} + \cdots\right]\cos(\omega t) \\ &= \left(\frac{8A^2}{3\pi}\right)\cos(\omega t) + (\text{HOH terms}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \left(1 - x^2\right)\dot{x} &= \left(1 - A^2\cos^2(\omega t)\right)(-A\omega\,\sin(\omega t)) \\ &= -A\omega\left(1 - \tfrac{1}{4}A^2\right)\sin(\omega t) + (\text{HOH terms}), \end{aligned} \tag{19}$$

where HOH stands for higher order-harmonic. Substitution of these results into Eq. (1) gives

$$\left(-A\omega^2+\frac{8A^2}{3\pi}\right)\cos(\omega t)=-\varepsilon A\omega\left(1-\tfrac{1}{4}A^2\right)\sin(\omega t)+(\text{HOH terms}). \tag{20}$$

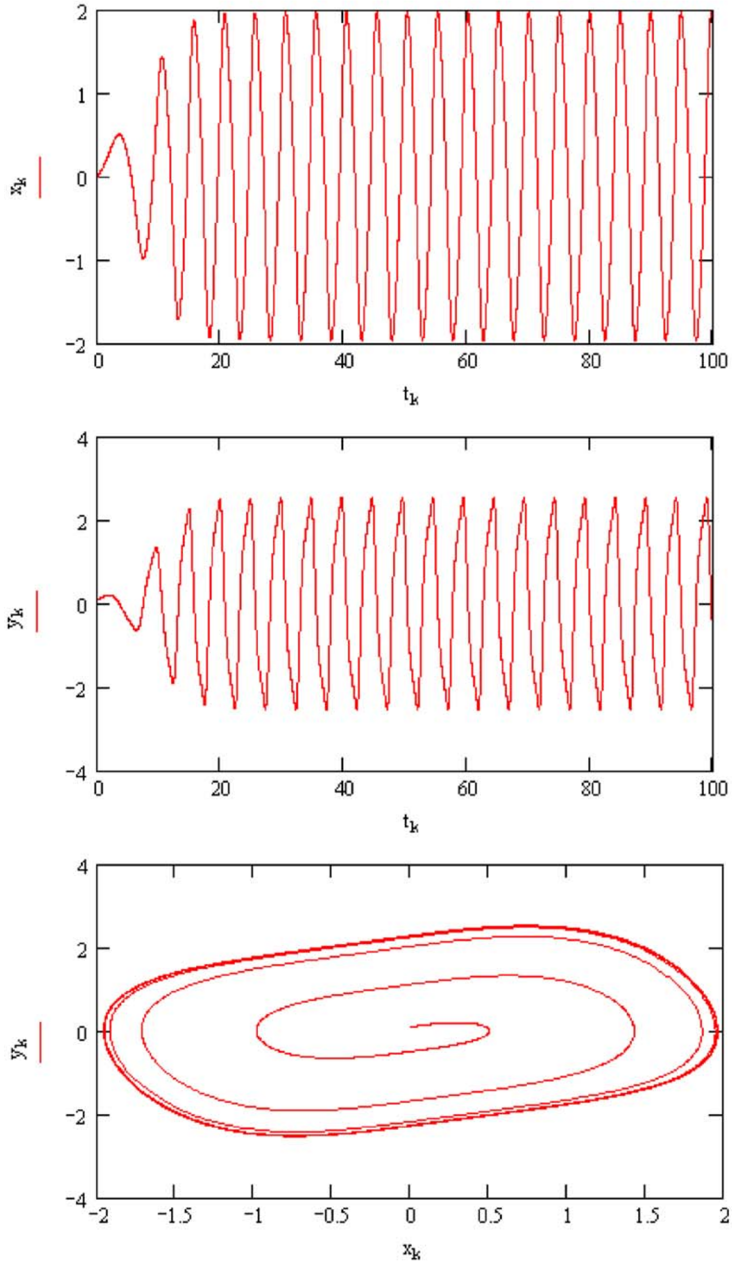

Fig. 1. Plots of x_k and y_k versus k, and x_k versus y_k for $\varepsilon = 0.5$, $\Delta t = 0.01$, and $(x_0, y_0) = (0.0, 0.1)$.

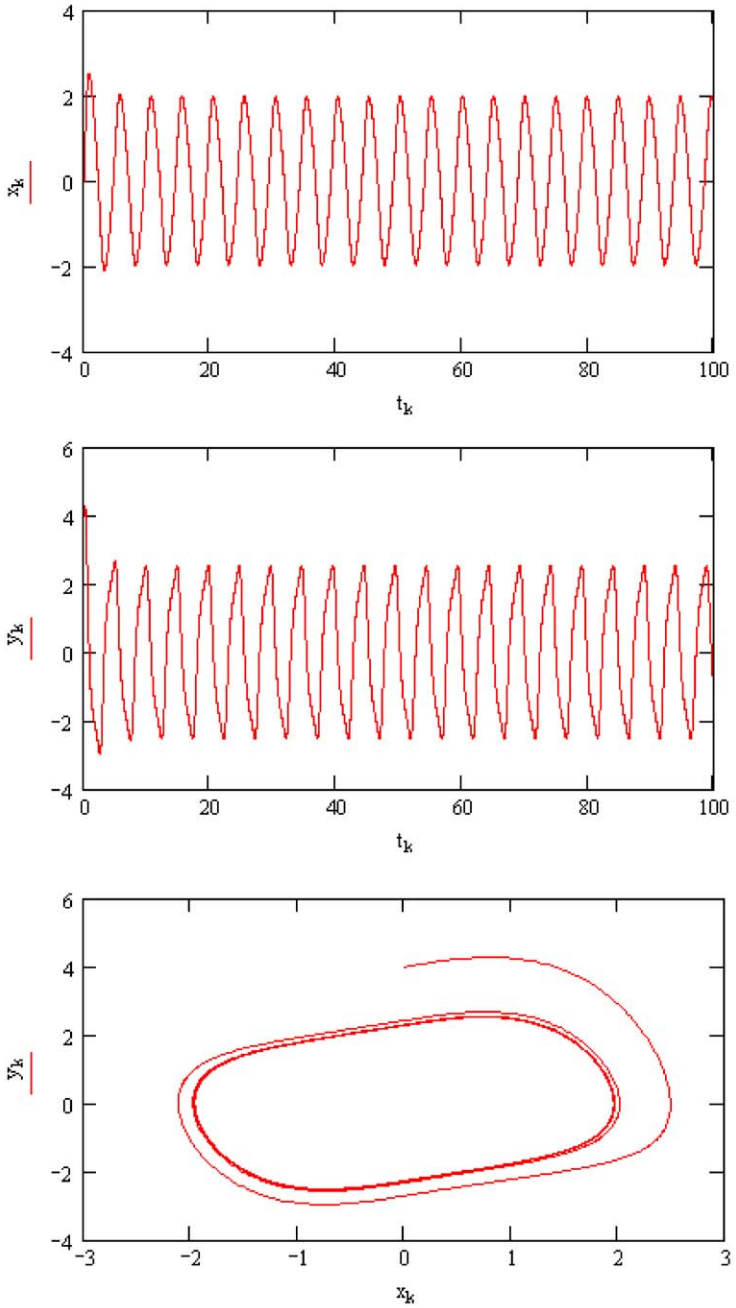

Fig. 2. Plots of x_k and y_k versus k, and x_k versus y_k for $\varepsilon = 0.5$, $\Delta t = 0.01$, and $(x_0, y_0) = (0.0, 4.0)$.

Applying the harmonic balance requirements, where the coefficients of $\cos \omega t$ and $\sin \omega t$ are equated to zero, it follows that

$$-A\omega^2 + \frac{8A^2}{3\pi} = 0, \quad 1 - \tfrac{1}{4}A^2 = 0, \tag{21}$$

and thus

$$A = 2, \quad \omega = \sqrt{\frac{16}{3\pi}}. \tag{22}$$

Therefore, an approximation to the limit-cycle solution is

$$x(t) \approx 2\cos\left(\sqrt{\frac{16}{3\pi}}t\right). \tag{23}$$

Finally, we use a nonstandard finite difference numerical integration method [3] to integrate Eq. (1); this scheme is

$$x_{k+1} = \psi x_k + \phi y_k, \tag{24a}$$

$$y_{k+1} = -x_k|x_{k+1}| + \varepsilon(1-(x_{k+1})^2)y_k, \tag{24b}$$

where the step size in time is Δt; x_k and y_k are, respectively, approximations to $x(t_k)$ and $y(t_k)$; and $t_k = (\Delta t)k$, where k is an integer; and

$$\psi = \cos(\Delta t), \quad \phi = \sin(\Delta t). \tag{25}$$

The initial conditions for Fig. 1 are: $(x_0, y_0) = (0, 0.1)$. This is a case in which the trajectory in the phase-plane spirals out to approach the limit-cycle. In Fig. 2, the initial conditions are $(x_0, y_0) = (0, 4)$ and this corresponds to the situation where the trajectories approach the limit-cycle from outside the limit-cycle. Both results obtained are consistent with the analytical results.

In summary, a nonlinear van der Pol-type equation has been investigated and the following conclusions were reached:

1. For $\varepsilon > 0$, there exists a unique periodic solution; this is the limit-cycle.
2. The method of harmonic balance gives an approximation to this solution and it takes the form

$$x(t) \simeq 2\cos\left[\sqrt{\frac{16}{3\pi}}t\right]. \tag{26}$$

A future problem will be to determine an approximation to the solution which includes the transients occurring before the limit-cycle behaviour is reached.

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