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Short Communication

Elimination of vibration localization in mistuned periodic structures

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Abstract

In this note, mistuned periodic structures are considered. Due to mistunings, some components of such structures may vibrate with small amplitudes, while some other components may vibrate with significantly large amplitudes. Such a behavior is known as vibration localization and is undesirable. To have a means of determining the occurrence of vibration localization, a sensitivity matrix is defined. This matrix and its singular values are computable. It is argued that if some of the singular values of the sensitivity matrix are large, then vibration localization can possibly occur. More importantly, an effective passive technique is proposed that eliminates vibration localization in mistuned periodic structures. The technique is to add small components between the structure components. Using the sensitivity matrix, it is shown that the added components indeed eliminate vibration localization.

1. Introduction

Periodic structures consist of coupled identical components. There are many examples of periodic structures; for instance, blades used in turbomachinery, combs of comb-drives in micro electro-mechanical systems (MEMS), disks stacked on shafts of hard-disk drives, and large-scale satellite antennas, to name a few. Components of periodic structures are designed and manufactured to have the same geometry and material properties. In reality, however, this is

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not the case: components of periodic structures are not exactly identical and differ slightly from each other. In this case, the structure is said to be *mistuned*. The differences between components of a mistuned periodic structure and a desired component are called *mistunings*. Mistunings are typically due to slight differences in the geometry and material properties of the structure components which are mainly introduced during manufacturing.

If there were no mistuning in a periodic structure, then its components would have had the same dynamics. Small mistunings, however, can cause significant differences in the dynamics of structure components. For instance, consider a mistuned structure under a harmonic input. In such a structure, at resonance frequencies, some components may vibrate with small amplitudes, while some other components may vibrate with significantly large amplitudes. This behavior is known as *vibration localization* and has been studied by numerous researchers; see, e.g., Refs. [1–5] and references therein. Roughly speaking, vibration localization in a periodic structure is due to mistunings in its components, small damping in its components, and weak coupling between its components.

Vibration localization in a periodic structure is undesirable since it causes large vibrations and stresses and possible damage in some of the structure components. Several researchers have devised means of reducing vibration localization; see, e.g., Refs. [4,5] and references therein. There is, however, no effective technique by which vibration localization can be eliminated. Therefore, new techniques that successfully eliminate vibration localization are welcome; in particular, passive techniques.

In this note, a novel and effective passive technique is proposed that eliminates vibration localization in mistuned periodic structures. The organization of the note is as follows. In Section 2, a mathematical model of mistuned periodic structures is presented. In Sections 3 and 4, it is explained how to determine the possibility of the occurrence of vibration localization in such structures. In Section 5, it is shown that when a mistuned structure is augmented by small components, vibration localization in the structure is eliminated. Examples are given throughout the note.

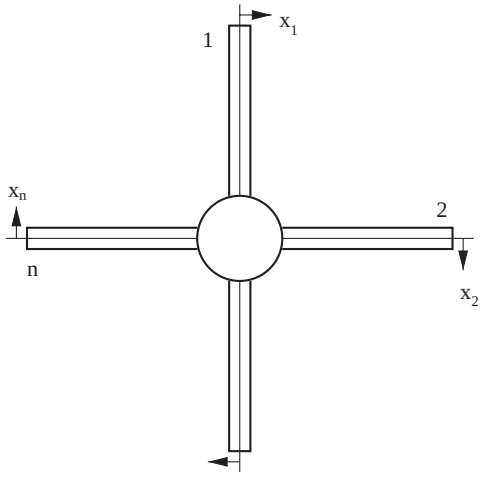

Fig. 1. A typical periodic structure having n components. Displacement of a component for an $i = 1, 2, \ldots, n$ is denoted by x_i.

2. A model of mistuned periodic structures

In this section, a mathematical model of mistuned periodic structures with n coupled components is presented; see Fig. 1 for a typical periodic structure. This model is obtained using the methodology in Ref. [3], and is as follows:

$$\begin{bmatrix} \alpha_1^3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2^3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_3^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}^3 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_n^3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \\ \ddot{\theta}_3(t) \\ \vdots \\ \ddot{\theta}_{n-1}(t) \\ \ddot{\theta}_n(t) \end{bmatrix} + \gamma \begin{bmatrix} \alpha_1^3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2^3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_3^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}^3 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_n^3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \dot{\theta}_3(t) \\ \vdots \\ \dot{\theta}_{n-1}(t) \\ \dot{\theta}_n(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 1+2\beta & -\beta & 0 & 0 & \cdots & 0 & 0 & -\beta \\ -\beta & 1+2\beta & -\beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\beta & 1+2\beta & -\beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & -\beta & 1+2\beta & -\beta \\ -\beta & 0 & 0 & 0 & \cdots & 0 & -\beta & 1+2\beta \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \\ \vdots \\ \theta_{n-1}(t) \\ \theta_n(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} f(t), \quad (1)$$

for all $t \geqslant 0$. In Eq. (1), the vector of angular displacements

$$[\,\theta_1(t) \quad \theta_2(t) \quad \cdots \quad \theta_n(t)\,]^{\mathrm{T}} =: \theta(t) \in \mathbb{R}^n, \quad (2)$$

for all $t \geqslant 0$; the vectors of initial displacements and initial velocities are, respectively, $\theta(0) = \mathbf{0}_n$ and $\dot{\theta}(0) = \mathbf{0}_n$, where $\mathbf{0}_n$ denotes the zero vector in $\mathbb{R}^n$; the positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the normalized lengths of the structure components, the positive real number γ denotes the normalized damping coefficient in the structure components, and the positive real number β denotes the normalized coupling parameter of the structure components; the input (influence) vector

$$[\,\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n\,]^{\mathrm{T}} =: b_f \in \mathbb{R}^n, \quad (3)$$

through which the scalar-valued input $t \mapsto f(t)$ is applied to the structure.

In a periodic structure, the lengths $\alpha_1, \alpha_2, \ldots, \alpha_n$ should be equal to a desired value α_d. In reality, however, these lengths are different from each other and α_d, even only slightly. That is, in reality, periodic structures are mistuned.

Let the coefficient matrices of $\ddot{\theta}(\cdot)$ and $\theta(\cdot)$ in Eq. (1) be denoted by M and K, respectively. Then, Eq. (1) can be written as

$$M\ddot{\theta}(t) + \gamma M\dot{\theta}(t) + K\theta(t) = b_f f(t), \quad \theta(0) = \mathbf{0}_n, \quad \dot{\theta}(0) = \mathbf{0}_n, \quad (4)$$

for all $t \geqslant 0$. The state-space representation of system (1) (equivalently, system (4)), which will be used in computations, is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix} = A\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix} + bf(t), \quad \begin{bmatrix}\theta(0)\\ \dot{\theta}(0)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_n\\ \mathbf{0}_n\end{bmatrix}, \tag{5a}$$

$$x(t) := [\, x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\,]^{\mathrm{T}} = C\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix}, \tag{5b}$$

for all $t \geqslant 0$, where

$$A = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -\gamma I_n \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \quad b = \begin{bmatrix} \mathbf{0}_n \\ M^{-1}b_f \end{bmatrix} \in \mathbb{R}^{2n}, \tag{6a}$$

$$C = [\, \mathrm{diag}\,[\alpha_1, \alpha_2, \ldots, \alpha_n] \quad 0\,] \in \mathbb{R}^{n\times 2n}, \tag{6b}$$

and I_n denotes the $n \times n$ identity matrix; the system output $x(t) \in \mathbb{R}^n$ is a vector consisting of the normalized displacements of the tips of the structure components, $x_i(t) = \alpha_i\theta_i(t)$, where $i = 1, 2, \ldots, n$.

Vibration localization in system (1) can be studied via the system representation in Eq. (5).

3. Analysis based on transfer functions

In studying vibration localization in mistuned periodic structures, in general, transfer functions from the applied input to structure displacements play an important role, as it will be shown in this note. Let $h_i(s)$ denote the transfer function from the input $f(\cdot)$ to the normalized displacement $x_i(\cdot) = \alpha_i\theta_i(\cdot)$ for an $i = 1, 2, \ldots, n$. Using Eqs. (4) and (5), it is concluded that

$$\begin{bmatrix} h_1(s) \\ h_2(s) \\ \vdots \\ h_n(s) \end{bmatrix} = \mathrm{diag}\,[\alpha_1, \alpha_2, \ldots, \alpha_n](Ms^2 + \gamma Ms + K)^{-1}b_f = C(sI_{2n} - A)^{-1}b, \tag{7}$$

where A, b, and C are given by in Eq. (6). To each transfer function $h_i(s)$ in Eq. (7), there corresponds an H_∞-norm defined by

$$\|h_i\|_\infty := \max_{\omega\in\mathbb{R}} |h_i(\mathrm{j}\omega)|, \tag{8}$$

where $\mathrm{j} = \sqrt{-1}$. The norm $\|h_i\|_\infty$ corresponds to the global maximum of the Bode magnitude plot of the transfer function $h_i(s)$. It is remarked that by using the state-space representation in Eq. (5), the transfer functions in Eq. (7) and their corresponding H_∞-norms can be conveniently computed by MATLAB programs for any n; see Ref. [6].

The occurrence of vibration localization due to mistunings is easily determined when $\|h_1\|_\infty, \|h_2\|_\infty, \ldots, \|h_n\|_\infty$ are known. If $\|h_i\|_\infty$ for at least one $i = 1, 2, \ldots, n$ is larger than the

H_∞-norms of other transfer functions, then vibration localization occurs. In other words, if $\|h_1\|_\infty, \|h_2\|_\infty, \ldots, \|h_n\|_\infty$ do not differ much from each other, then vibration localization does not occur.

To illustrate vibration localization due to mistunings, an example is now given.

Example 3.1. In system (1), let $n = 4$, the coupling coefficient $\beta = 0.005$, and the damping coefficient $\gamma = 0.01$. With this setup, several studies are conducted:

Study 1. *No mistuning*: Let $\alpha_i = 1$ for $i = 1, 2, 3, 4$. The H_∞-norms of transfer functions $h_1(s), h_2(s), h_3(s)$, and $h_4(s)$, obtained via Eq. (7), are

$$\|h_1\|_\infty = \|h_2\|_\infty = \|h_3\|_\infty = \|h_4\|_\infty = 100.05. \tag{9}$$

Since there is no mistuning, a same value for the H_∞-norms is expected: no mistuning implies no vibration localization. The Bode magnitude plots of $h_1(s), h_2(s), h_3(s)$, and $h_4(s)$ are depicted in Fig. 2. These plots overlap as expected.

Study 2. *Effect of mistunings*: Let there be mistunings and for instance

$$\alpha_1 = 0.98, \quad \alpha_2 = 1.04, \quad \alpha_3 = 1.05, \quad \alpha_4 = 1.03. \tag{10}$$

It should be remarked that in reality $\alpha_1, \alpha_2, \alpha_3$, and α_4 are not exactly known; it is only known that they are close to the desired value $\alpha_d = 1$. The H_∞-norms of transfer functions $h_1(s), h_2(s), h_3(s)$, and $h_4(s)$ are

$$\|h_1\|_\infty = 92.68, \quad \|h_2\|_\infty = 85.60, \quad \|h_3\|_\infty = 122.65, \quad \|h_4\|_\infty = 95.95. \tag{11}$$

It is clear that due to small mistunings, the H_∞-norms are appreciably different from each other; in particular, $\|h_3\|_\infty$ is larger than the other norms. Thus, vibration localization occurs. The Bode magnitude plots of $h_1(s), h_2(s), h_3(s)$, and $h_4(s)$ are depicted in Fig. 3. These plots are different from each other due to mistunings.

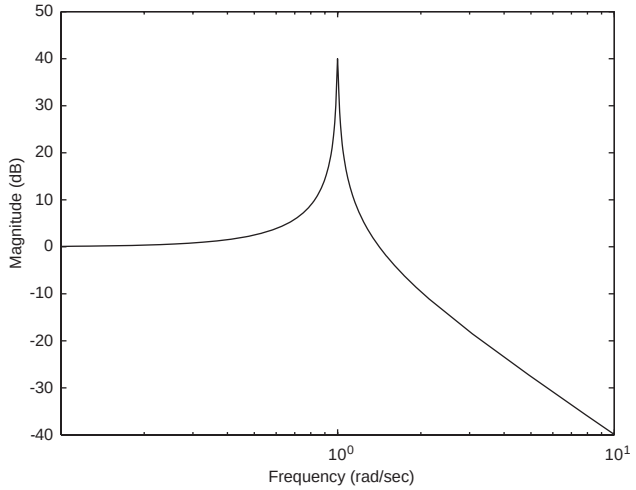

Fig. 2. The Bode magnitude plots of $h_1(s)$, $h_2(2)$, $h_3(s)$, and $h_4(s)$ in Example 3.1, when there is no mistuning in the structure.

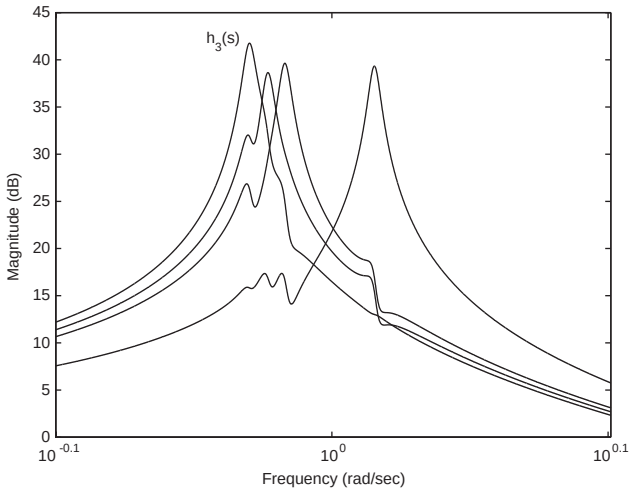

Fig. 3. The Bode magnitude plots of $h_1(s)$, $h_2(2)$, $h_3(s)$, and $h_4(s)$ in Example 3.1, when there are mistunings in the structure.

4. Sensitivity of norms to mistunings

In a mistuned structure, the lengths $\alpha_1, \alpha_2, \ldots, \alpha_n$ differ from each other and a desired value α_d. If these lengths were exactly known, then $\|h_1\|_\infty, \|h_2\|_\infty, \ldots, \|h_n\|_\infty$ could have been computed, by which the occurrence of vibration localization could have been determined. In reality, however, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not exactly known; it is only known that they are close to α_d. That is, $\|h_1\|_\infty, \|h_2\|_\infty, \ldots, \|h_n\|_\infty$ cannot be computed. Therefore, there should be a means of determining the occurrence of vibration localization without computing $\|h_1\|_\infty, \|h_2\|_\infty, \ldots, \|h_n\|_\infty$. In this section, it is shown that such a means is indeed available.

For an $i = 1, 2, \ldots, n$, the transfer function $h_i(s)$, which is obtained via Eq. (7), is a function of $\alpha_1, \alpha_2, \ldots, \alpha_n$; so is $\|h_i\|_\infty$. This norm as a function is thus denoted by $(\alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto \|h_i\|_\infty(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Suppose that this function is computed at $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$. To determine how much $\|h_i\|_\infty(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is different from $\|h_i\|_\infty(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ when α_i is slightly different from $\bar{\alpha}_i$ for some $i = 1, 2, \ldots, n$, the following matrix, to be called the *sensitivity matrix*, is computed

$$S(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) = \begin{bmatrix} \dfrac{\partial\|h_1\|_\infty}{\partial\alpha_1} & \dfrac{\partial\|h_1\|_\infty}{\partial\alpha_2} & \cdots & \dfrac{\partial\|h_1\|_\infty}{\partial\alpha_n} \\ \dfrac{\partial\|h_2\|_\infty}{\partial\alpha_1} & \dfrac{\partial\|h_2\|_\infty}{\partial\alpha_2} & \cdots & \dfrac{\partial\|h_2\|_\infty}{\partial\alpha_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial\|h_n\|_\infty}{\partial\alpha_1} & \dfrac{\partial\|h_n\|_\infty}{\partial\alpha_2} & \cdots & \dfrac{\partial\|h_n\|_\infty}{\partial\alpha_n} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{12}$$

where

$$\frac{\partial \|h_k\|_\infty}{\partial \alpha_l} = \frac{\|h_k\|_\infty(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_l + \delta\alpha_l, \ldots, \bar{\alpha}_n) - \|h_k\|_\infty(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_l, \ldots, \bar{\alpha}_n)}{\delta\alpha_l}, \tag{13}$$

for all $k, l = 1, 2, \ldots, n$, with $0 < \delta\alpha_l \ll 1$. In Eq. (13), $\bar{\alpha}_i$ is known for all $i = 1, 2, \ldots, n$. Thus, by using a same value for $\delta\alpha_l$ for all $l = 1, 2, \ldots, n$, the right-hand side of Eq. (13) is readily computable; so is the matrix $S(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$.

Once the sensitivity matrix is computed, it can be decomposed by the singular value decomposition as

$$S(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) = U\Sigma V^{\mathrm{T}}, \tag{14}$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ are unitary matrices, and $\Sigma = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_n] \in \mathbb{R}^{n\times n}$ is a matrix the diagonal elements of which are the singular values of $S(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$, ordered as $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$; see, e.g., Refs. [7,8]. If some of the singular values are large, then $\|h_i\|_\infty(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is much different from $\|h_i\|_\infty(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n)$ for some $i = 1, 2, \ldots, n$.

For a periodic structure, $\bar{\alpha}_i = \alpha_d$ for all $i = 1, 2, \ldots, n$. Therefore, the sensitivity matrix to be computed is $S(\bar{\alpha}_1 = \alpha_d, \bar{\alpha}_2 = \alpha_d, \ldots, \bar{\alpha}_n = \alpha_d)$. If some of the singular values of this matrix are large, then mistunings (slight differences between α_i and α_d for some $i = 1, 2, \ldots, n$) can lead to large differences between $\|h_i\|_\infty$ for some $i = 1, 2, \ldots, n$; that is, vibration localization can occur.

An example is now given to illustrate the usefulness of the sensitivity matrix in determining the occurrence of vibration localization.

Example 4.1. Consider the structure in Example 3.1. The structure is periodic where the desired length of its components is $\alpha_d = 1$. However, the structure is mistuned. That is, $\alpha_1, \alpha_2, \alpha_3$, and α_4 are different from each other, are not exactly known, but are close to α_d. It is desirable to decide whether or not vibration localization can occur in the structure. The decision can be made by means of the singular values of the sensitivity matrix. This matrix is

$$S(1,1,1,1) = \begin{bmatrix} 10191.36 & -3826.15 & -4971.16 & -3826.15 \\ -3828.94 & 10188.82 & -3811.52 & -4973.94 \\ -4973.94 & -3828.94 & 10188.82 & -3828.94 \\ -3828.91 & -4973.91 & -3828.91 & 10188.85 \end{bmatrix}. \tag{15}$$

In computing $S(1,1,1,1)$ via Eqs. (12) and (13), $\delta\alpha_l = 0.001$ was used for all $l = 1, 2, 3, 4$. The singular values of $S(1,1,1,1)$ are

$$\sigma_1 = 15167.06, \quad \sigma_2 = 15158.36, \quad \sigma_3 = 12868.33, \quad \sigma_4 = 2435.88. \tag{16}$$

Since the singular values of the sensitivity matrix are large, vibration localization can occur. This conclusion was reached in Example 3.1, where it was assumed that the values of lengths of the components of the mistuned structure were known (see Eq. (10)); an assumption that does not hold in reality.

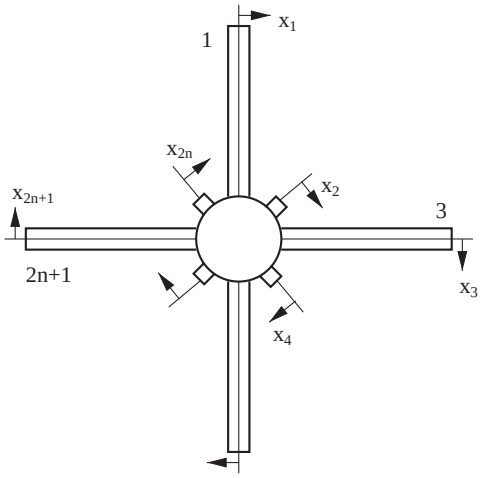

Fig. 4. A periodic structure is augmented by adding n small components between the structure components. The added components eliminate vibration localization.

5. Elimination of vibration localization

In this section, a novel technique is proposed that eliminates vibration localization in mistuned periodic structures. The proposed technique desensitizes the H_∞-norms of the transfer functions of such structures to slight mistunings in the lengths of the structure components.

In a periodic structure, let there be n components numbered by odd numbers $1, 3, \ldots, 2n-1$; see Fig. 4. These components are called the *principal components*. The lengths of the principal components are $\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}$, and are desired to be equal to α_d.

Let the structure be augmented by adding n components between the principal components as shown in Fig. 4. The added components are numbered by even numbers $2, 4, \ldots, 2n$ and are called the *auxiliary components*. The normalized lengths of the auxiliary components are $\alpha_2, \alpha_4, \ldots, \alpha_{2n}$, and are desired to be equal to $\alpha_s \ll \alpha_d$. It is remarked that the auxiliary components are chosen much smaller than the principal components. Moreover, the auxiliary components should preserve the symmetry and periodicity of the structure.

The augmented structure is represented by

$$\hat{M}\ddot{\hat{\theta}}(t) + \gamma\hat{M}\dot{\hat{\theta}}(t) + \hat{K}\hat{\theta}(t) = \hat{b}_f f(t), \quad \hat{\theta}(0) = 0_{2n}, \quad \dot{\hat{\theta}}(0) = 0_{2n}, \tag{17}$$

for all $t \geqslant 0$, where

$$\hat{\theta}(t) := [\,\theta_1(t) \quad \theta_2(t) \quad \cdots \quad \theta_{2n-1}(t) \quad \theta_{2n}(t)\,]^{\mathrm{T}} \in \mathbb{R}^{2n}, \tag{18a}$$

$$\hat{b}_f = [\,\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{2n-1} \quad \alpha_{2n}\,]^{\mathrm{T}} \in \mathbb{R}^{2n}, \tag{18b}$$

$$\hat{M} = \mathrm{diag}\,[\alpha_1^3, \alpha_2^3, \ldots, \alpha_{3n-1}^3, \alpha_{2n}^3] \in \mathbb{R}^{2n\times 2n}, \tag{18c}$$

and the matrix $\hat{K} \in \mathbb{R}^{2n\times 2n}$ has the same structure as that of K in Eq. (1). The state-space representation of system (17) has the same form as that in Eq. (5), except that it has a different

state-vector and coefficient matrices. There are now $2n$ transfer functions from $f(\cdot)$ to $x_i(\cdot) = \alpha_i\theta_i(\cdot)$, denoted by $h_i(s)$ for an $i = 1, 2, \ldots, 2n$. These transfer functions are given by

$$\begin{bmatrix} h_1(s) \\ h_2(s) \\ \vdots \\ h_{2n}(s) \end{bmatrix} = \text{diag}\,[\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}](\hat{M}s^2 + \gamma\hat{M}s + \hat{K})^{-1}\hat{b}_f. \tag{19}$$

By adding the auxiliary components, the H_∞-norms of the transfer functions corresponding to the principal components, i.e., $\|h_{2i-1}\|_\infty(\alpha_d, \alpha_s, \ldots, \alpha_d, \alpha_s)$ for all $i = 1, 2, \ldots, n$, will become insensitive to mistunings in *both* principal and auxiliary components. The truth of this statement can be verified by computing the sensitivity matrix of the augmented structure, namely, $S(\alpha_d, \alpha_s, \ldots, \alpha_d, \alpha_s)$. Therefore, vibration localization is eliminated by adding the auxiliary components.

To show the efficacy of the auxiliary components in eliminating vibration localization an example is given.

Example 5.1. Consider the structure in Example 3.1. The (principal) components of the structure are now numbered by $1, 3, 5, 7$. Four auxiliary components are added to the structure and numbered by $2, 4, 6, 8$. The desired length of all principal components is $\alpha_d = 1$. The desired length of all auxiliary components is chosen $\alpha_s = 0.01$. For this structure, the sensitivity matrix is $S(1, 0.01, 1, 0.01, 1, 0.01, 1, 0.01)$ and is computed via Eq. (12). The singular values of this matrix, when $\delta\alpha_l = 0.001$ for $l = 1, 2, \ldots, 8$, are

$$\sigma_1 = 463.40, \quad \sigma_2 = 463.33, \quad \sigma_3 = 463.30, \quad \sigma_4 = 463.27, \tag{20a}$$

$$\sigma_5 = 213.23, \quad \sigma_6 = 112.39, \quad \sigma_7 = 80.38, \quad \sigma_8 = 62.66. \tag{20b}$$

Since the singular values are much smaller than those in Eq. (17), it is concluded that vibration localization does not occur.

To confirm the efficacy of the auxiliary components in desensitizing $\|h_1\|$, $\|h_3\|$, $\|h_5\|$, and $\|h_7\|$ to mistunings, and hence in eliminating vibration localization, let

$$\alpha_1 = 0.98, \quad \alpha_3 = 1.04, \quad \alpha_5 = 1.05, \quad \alpha_7 = 1.03, \tag{21a}$$

$$\alpha_2 = 0.011, \quad \alpha_4 = 0.012, \quad \alpha_6 = 0.0098, \quad \alpha_8 = 0.0105. \tag{21b}$$

The lengths of the principal components are those in Eq. (10), and those of the auxiliary components differ from the desired length $\alpha_s = 0.01$. The H_∞-norms of transfer functions $h_1(s), h_2(s), \ldots, h_8(s)$ are

$$\|h_1\|_\infty = 98.54, \quad \|h_3\|_\infty = 101.39, \quad \|h_5\|_\infty = 102.15, \quad \|h_7\|_\infty = 101.02, \tag{22a}$$

$$\|h_2\|_\infty = 10.44, \quad \|h_4\|_\infty = 10.91, \quad \|h_6\|_\infty = 9.86, \quad \|h_8\|_\infty = 10.20. \tag{22b}$$

It is clear that the H_∞-norms of the transfer functions corresponding to the principal components are almost equal; so are those of the transfer functions corresponding to the auxiliary components. That is, even though there are mistunings in all components of the augmented structure, the principal components have the same dynamics. This implies that vibration

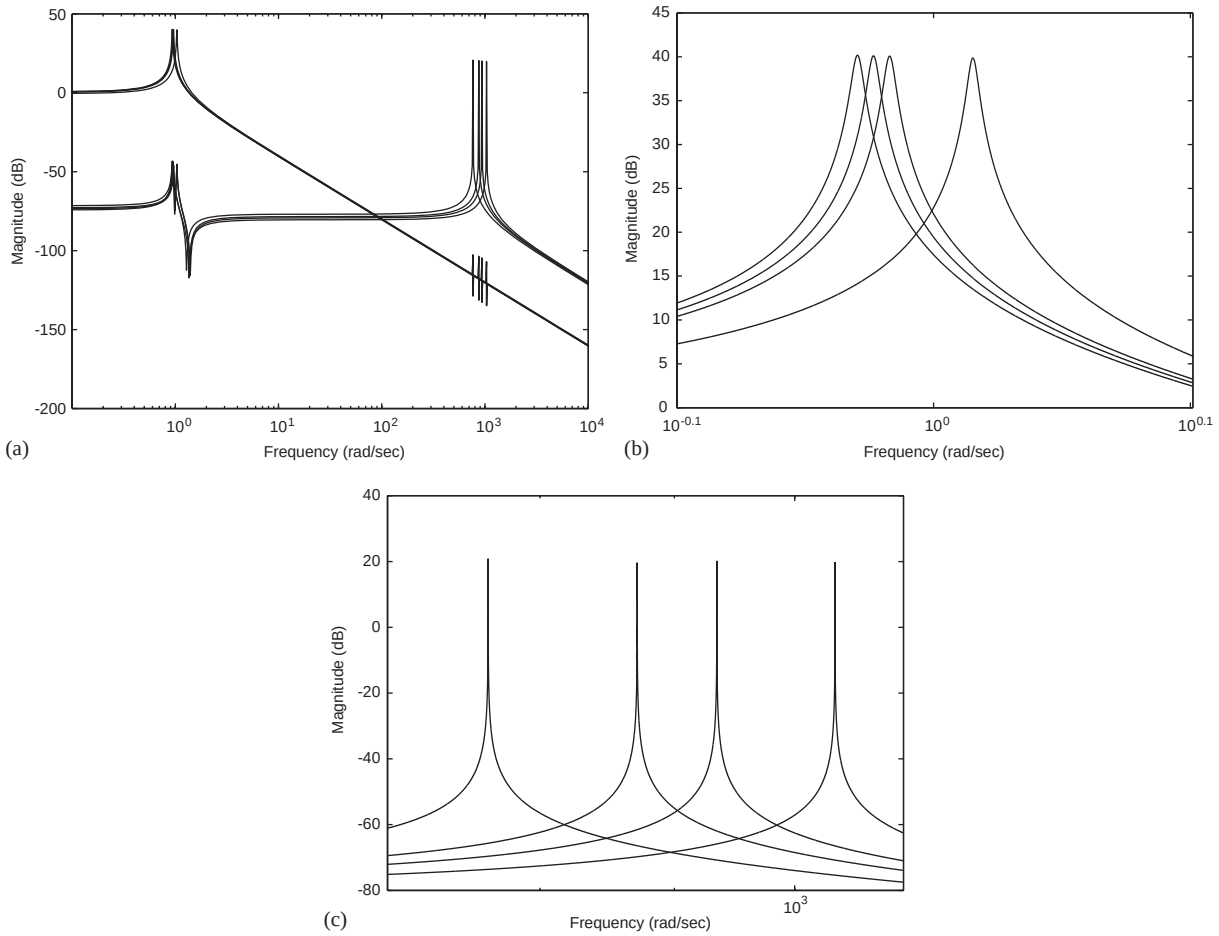

Fig. 5. (a) The Bode magnitude plots of the transfer functions of the augmented structure in Example 5.1. The plots corresponding to the transfer functions of the principal components, $h_1(s)$, $h_3(s)$, $h_5(s)$, and $h_7(s)$, peak close to $\omega = 1$. The plots corresponding to the transfer functions of the auxiliary components, $h_2(s)$, $h_4(s)$, $h_6(s)$, and $h_8(s)$, peak close to $\omega = 1000$. (b) Magnification of the plots corresponding to the transfer functions of the principal components. (c) Magnification of the plots corresponding to the transfer functions of the auxiliary components.

localization is eliminated. The Bode magnitude plots of the transfer functions of the augmented structure are depicted in Fig. 5(a). The plots corresponding to the transfer functions of the principal components, $h_1(s)$, $h_3(s)$, $h_5(s)$, and $h_7(s)$, peak close to $\omega = 1$; see also the magnified plots in Fig. 5(b). The plots corresponding to the transfer functions of the auxiliary components, $h_2(s)$, $h_4(s)$, $h_6(s)$, and $h_8(s)$, peak close to $\omega = 1000$; see also the magnified plots in Fig. 5(c).

6. Conclusions

In this note, vibration localization in mistuned periodic structures was studied using a mathematical model of such structures. This model is an n-degree of freedom system under a

scalar-valued input. There are n transfer functions corresponding to this structure that relate the structure displacements to the applied input. When mistunings in the structure are known, the H_∞-norms of the transfer functions can be computed, thereby the possibility of the occurrence of vibration localization can be determined. In reality, however, mistunings in the structure are not known. Thus, the H_∞-norms of the structure transfer functions cannot be computed, and consequently the occurrence of vibration localization cannot be determined. To have a means of determining the occurrence of vibration localization, a sensitivity matrix was defined. This matrix and its singular values are computable. If some of the singular values of the sensitivity matrix are large, then vibration localization can possibly occur.

More importantly, in this note an effective passive technique was proposed that eliminates vibration localization in mistuned periodic structures. The elimination is achieved by adding small components between the structure components. By using the sensitivity matrix, the efficacy of the added components in eliminating vibration localization was demonstrated. A rigorous mathematical proof as why the added components can eliminate vibration localization will be presented in the near future.

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