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New improved series expansion for solving the moving oscillator problem

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Abstract

A new method able to evaluate the dynamic stress response of an elastic beam subject to moving oscillators is presented. The proposed procedure improves the convergence and accuracy of the conventional eigenfunction series expansion of beam response taking into account the gravitational, inertial and damping effects due to the moving oscillators. The improvement of the conventional solution is obtained by means of an extension to continuous systems of the dynamic correction method, originally proposed for discretized structures. The proposed method is able to account for the truncated higher order eigenfunctions by adding a pseudo-static term to the conventional series expansion. Numerical results are presented to demonstrate the capability of the method to accurately determine the discontinuity and jump in bending moment and shear force distributions, respectively.

1. Introduction

The vibrations caused by loads travelling along a distributed parameter system have long been an interesting topic in the field of civil engineering. In particular the dynamic response of beams subjected to these loads is a problem commonly encountered in many important engineering studies: as for example, in the design of railroads with high-speed trains and highway bridges with

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moving vehicles, etc. A comprehensive review of the history and literature concerning this topic can be found in Refs. [1,2].

As recently pointed out [2–6], three main types of problems have been studied in the literature: the moving force problem (see e.g. Refs. [3,7,8]), the moving mass problem (see e.g. Refs. [7,9,10]) and the moving oscillator (or moving vehicle) problem (see e.g. Refs. [2,3,6]). In the first problem the inertia of the moving system is neglected; and in the second one the inertia of the moving system is taken into account but the stiffness of the system is assumed infinite; when the stiffness of the moving system is considered finite the moving oscillator problem is obtained. The latter problem leads to a more realistic description of the interaction effects between moving loads and the beam. Moreover, once the solution of this problem is carried out, a very simple generalization makes possible the evaluation of the interaction effects in the case of multi-degree-of-freedom moving systems. To this aim, the coupled structural system is modelled in this paper as a combination of distributed and lumped parameter substructures, the beam and the vehicle, respectively and then, in the spirit of the component-mode synthesis method [11–14], the equations governing the dynamic response of the coupled beam–vehicle system are deduced from the equations of motion of the two substructures considered separately and appropriately selecting the constraint conditions at the interface nodes. The formulation can be easily extended to more realistic travelling vehicles.

The dynamic response of the moving oscillator problem is usually determined through the modal analysis which consists of a series expansion of the solution in terms of the eigenfunctions of the undamped and unloaded continuous system [1,2,7]. This conventional approach quickly converges to the solution when the calculation of the lateral displacement of the continuum structure is required. On the contrary, in the calculation of the bending moment and the shear force along the continuous structure, since higher order derivatives of the series are required, the expansion series converges poorly and cannot capture the jump in the shear force. To overcome the limitation of the conventional method, Pesterev and Bergman [5] recently introduced the so-called ‘‘improved series expansion’’ for the case of the moving load and oscillator problems. In this approach the response of the continuous system is considered as the sum of two contributions: the conventional modal expansion and the quasi-static solution of motion equations, which takes into account the deflection of the system due to the gravitational effects induced by moving oscillators. A further improvement has been recently proposed by Pesterev et al. [6].

The previously described approaches, which can be successfully applied for both moving force and oscillator systems, do not provide very accurate solutions when the vehicle possesses heavy mass. This is due to the fact that the inertial effects due to the motion of the vehicle are not taken into account in the quasi-static solution.

Recently, in the framework of the moving mass problem, the authors [15] proposed two different methods able to capture, with different levels of accuracy, the beam discontinuities in the bending moment and shear force laws, taking into account also the inertia effects. The first method can be thought of as a generalization of the classical modal analysis in which the eigensolutions of the undamped continuous system are evaluated by taking into account moving masses, while in the second one the response is evaluated by considering the particular solution of the differential equation governing the problem of moving masses. This method can be seen as an extension of the dynamic correction method (DCM) to continuous systems, originally proposed for discretized structural systems [16]. According to this method the improvement in the response is obtained by adding to the conventional series expansion(CSE) a correction term retaining information on the truncated higher order series terms.

In this paper the last method is reformulated for the more general case of a coupled continuous and discretized moving system. It is also shown that in such a way, contrary to the Pesterev and Bergman improved series expansion [5], it is possible to capture the discontinuities in the bending moment and shear force along the continuous system due to gravitational, inertial and damping effects of the moving oscillator.

In the numerical applications the best accuracy of the proposed approach with respect to the others presented in the literature is shown.

2. Motion equations of the coupled beam-oscillator system

Consider a spatially one-dimensional elastic beam of length l crossed by N moving oscillators with constant velocity v (Fig. 1). Assuming a linear-elastic behaviour of the beam and oscillators, and by neglecting the effect of the rotational inertia and shear strain on the flexural motion of the beam, the dynamic behaviour of the beam and oscillators, separately considered, are governed by the following differential equations, respectively:

$$\rho A\ddot{w}(x,t) + b\dot{w}(x,t) + EIw^{IV}(x,t) = \sum_{i=1}^{N} \chi_i(t) r_i(t)\delta(x - \xi_i(t)),$$
$$m_i\big(\ddot{u}_i(t) + \chi_i(t)\ddot{u}_{b,i}(t)\big) + c_i\dot{u}_i(t) + k_i u_i(t) = m_i g, \quad i = 1, \ldots, N, \tag{1a, b}$$

where the prime and dot over a variable denote space and time derivative, respectively.

In Eq. (1a) $w(x,t)$ is the lateral displacement of the beam; ρ, b, and E denote the mass density, the coefficient of viscous damping and the Young's modulus of the material, respectively; A and I are the cross-sectional area and the moment of inertia, respectively. For the sake of simplicity, all the previous quantities are here supposed constant on the whole length of the beam. Further $\delta(x - \xi_i(t))$ is the Dirac's delta function; $r_i(t)$ represents the interaction force, located at the instantaneous position $\xi_i(t)$ on the beam of the ith moving oscillator, transmitted to the beam by the oscillator, given by the following relationship:

$$r_i(t) = (c_i\dot{u}_i(t) + k_i u_i(t)) = m_i\left[g - \left(\ddot{u}_i(t) + \chi_i(t)\ddot{u}_{b,i}(t)\right)\right]. \tag{2}$$

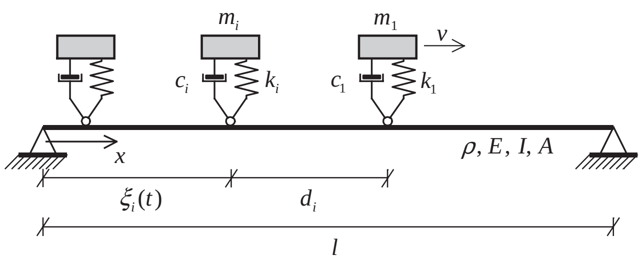

Fig. 1. Structural system: beam crossed by N moving oscillators.

The instantaneous position on the beam $\xi_i(t)$ of the ith moving oscillator, in the considered case of uniform motion, is given by the following relationship:

$$\xi_i(t) = vt - d_i, \tag{3}$$

where v is the constant velocity of the moving oscillators and d_i is the distance between the ith and the first oscillator, being obviously $d_1 = 0$ (see Fig. 1). Moreover, in Eq. (1a) $\chi_i(t)$ is the window function of the ith oscillator, defined as follows:

$$\chi_i(t) = \begin{cases} 1 & \text{for } 0 \leqslant \xi_i(t) \leqslant l, \\ 0 & \text{for } \xi_i(t) < 0 \text{ or } \xi_i(t) > l. \end{cases} \tag{4}$$

In Eq. (1b) $u_i(t)$ is the relative displacement of the ith moving oscillator with respect to the beam, diminished of the static displacement $u_{s,i}(t) = k_i^{-1} m_i g$; $u_{b,i}(t)$ is the displacement of the ith interface node, located at the instantaneous position $\xi_i(t)$, where the beam and the ith oscillator are connected (see Fig. 1). Moreover m_i, c_i and k_i are the mass, coefficient of viscous damping and stiffness of the ith oscillator, respectively, and g is the acceleration of gravity.

It has to be noted that the differential equations (1) govern the evolution of the structural response of the so-called moving oscillators problem. Furthermore, by analysing Eqs. (1) and (2), it is easy to show that both the moving forces and moving masses problems can be obtained from the moving oscillators problem, by neglecting the inertial effects and assuming the stiffness of the oscillators equal to infinite, respectively. Indeed, in the first case $r_i(t) = m_i g$ and in the second one $u_i(t) = 0$.

In what follows it is assumed that the beam is simply supported at two ends. Furthermore, zero initial conditions are assumed, implying that the beam is at rest at the time $t = 0$ when the first oscillator enters into the left end of the structure. Notice that this assumption merely simplifies the notation and does not affect the generality of the proposed procedure.

According to the conventional modal analysis of continuous structures [1,7], the function $w(x,t)$, representing the lateral displacement of the beam, can be expressed in an approximate way through a series expansion in terms of the first n eigenfunctions $\phi_j(x)$ of the beam as follows:

$$w_{\mathrm{CSE}}(x,t) = \sum_{j=1}^{n} \phi_j(x) y_j(t) = \boldsymbol{\phi}(x)^{\mathrm{T}} \mathbf{y}(t), \tag{5}$$

where the apex T means transpose, while the subscript CSE denotes that the lateral displacement $w_{\mathrm{CSE}}(x,t)$ is evaluated by means of the CSE. Moreover, in Eq. (5) $\mathbf{y}(t)$ is a vector of order n listing the generalized displacements $y_j(t)$ and $\boldsymbol{\phi}(x)$ is a vector function collecting the first n eigenfunctions $\phi_j(x)$ of the unloaded and undamped beam, evaluated as the solution of the following eigenproblem:

$$EI\boldsymbol{\phi}^{IV}(x) - \omega^2 \rho A \boldsymbol{\phi}(x) = 0. \tag{6}$$

The eigenfunctions $\phi_j(x)$ satisfy the boundary conditions and the following orthogonality conditions:

$$\rho A \int_0^l \boldsymbol{\phi}(x)\boldsymbol{\phi}(x)^{\mathrm{T}} \mathrm{d}x = \mathbf{I}_n, \quad EI \int_0^l \boldsymbol{\phi}(x)\boldsymbol{\phi}^{IV}(x)^{\mathrm{T}} \mathrm{d}x = \boldsymbol{\Omega}^2, \tag{7a, b}$$

where $\mathbf{I}_n$ is the identity matrix of order n and $\mathbf{\Omega}$ is the $n \times n$ diagonal matrix collecting the first frequencies ω_j of the beam.

By substituting the modal expansion (5) into Eq. (1a), pre-multiplying both sides by $\boldsymbol{\phi}(x)$ and integrating over the length l of the beam, the motion equations of the beam in terms of the generalized displacements $\mathbf{y}(t)$ are obtained in the following form:

$$\ddot{\mathbf{y}}(t) + \mathbf{\Xi}\dot{\mathbf{y}}(t) + \mathbf{\Omega}^2\mathbf{y}(t) = \mathbf{\Phi}(\xi(t))\mathbf{X}(t)\mathbf{r}(t). \tag{8}$$

In Eq. (8) $\mathbf{\Xi}$ is the modal damping matrix of the beam, defined as follows:

$$\mathbf{\Xi} = b\int_0^l \boldsymbol{\phi}(x)\boldsymbol{\phi}(x)^{\mathrm{T}}\mathrm{d}x = \frac{b}{\rho A}\mathbf{I}_n. \tag{9}$$

$\mathbf{r}(t)$ is a vector collecting the N interaction forces $r_i(t)$ transmitted to the beam by the moving oscillators; $\mathbf{X}(t)$ is the $N \times N$ diagonal matrix listing the window functions $\chi_i(t)$; finally $\mathbf{\Phi}(\xi(t))$ is the following $n \times N$ matrix:

$$\mathbf{\Phi}(\xi(t)) = \begin{bmatrix} \boldsymbol{\phi}(\xi_1(t)) & \boldsymbol{\phi}(\xi_2(t)) & \cdots & \boldsymbol{\phi}(\xi_N(t)) \end{bmatrix} \tag{10}$$

with $\boldsymbol{\xi}(t)$ being the vector listing the instantaneous positions $\xi_i(t)$ of the N moving oscillators.

In the spirit of the component-mode synthesis method [11–14], the equations of motion of the coupled beam–oscillators system are obtained starting from the equations of motion (8) and (1b) by imposing the following condition at the interface nodes

$$u_{b,i}(t) \equiv w_{\mathrm{CSE}}(\xi_i(t), t) = \boldsymbol{\phi}(\xi_i(t))^{\mathrm{T}}\mathbf{y}(t), \quad i = 1, \ldots, N. \tag{11}$$

By differentiating Eq. (11) and taking into account that, for the case under study of uniform motion of the oscillators, it results in

$$\begin{aligned} \boldsymbol{\phi}(\xi_i(t)) &= \dot{\xi}_i(t)\frac{\mathrm{d}\boldsymbol{\phi}(\xi_i(t))}{\mathrm{d}\xi_i} = v\boldsymbol{\phi}^I(\xi_i(t)), \\ \ddot{\boldsymbol{\phi}}(\xi_i(t)) &= \dot{\xi}_i(t)\frac{\mathrm{d}\dot{\boldsymbol{\phi}}(\xi_i(t))}{\mathrm{d}\xi_i} = v^2\boldsymbol{\phi}^{II}(\xi_i(t)); \end{aligned} \tag{12a, b}$$

the acceleration $\ddot{u}_{b,i}(t)$ of the ith interface node is obtained as follows

$$\ddot{u}_{b,i}(t) = v^2\boldsymbol{\phi}^{II}(\xi_i(t))^{\mathrm{T}}\mathbf{y}(t) + 2v\boldsymbol{\phi}^I(\xi_i(t))^{\mathrm{T}}\dot{\mathbf{y}}(t) + \boldsymbol{\phi}(\xi_i(t))^{\mathrm{T}}\ddot{\mathbf{y}}(t). \tag{13}$$

By substituting Eq. (2) and (13) into Eq. (1b) and (8), the equations of motion of the coupled beam–oscillators system, after standard mathematical manipulations, are obtained in the following compact form:

$$\mathbf{M}(t)\ddot{\mathbf{u}}(t) + \mathbf{C}(t)\dot{\mathbf{u}}(t) + \mathbf{K}(t)\mathbf{u}(t) = \boldsymbol{\tau}(t)g. \tag{14}$$

In Eq. (14) the matrices $\mathbf{M}(t)$, $\mathbf{C}(t)$ and $\mathbf{K}(t)$ and the vectors $\mathbf{u}(t)$ and $\tau(t)$ are defined as follows:

$$\mathbf{M}(t)=\begin{bmatrix}\mathbf{I}_n+\Delta\mathbf{M}_b(t) & \mathbf{M}_b(t)^{\mathrm{T}}\\ \mathbf{M}_b(t) & \mathbf{M}_v\end{bmatrix},\quad \mathbf{C}(t)=\begin{bmatrix}\xi+\Delta\mathbf{C}_b(t) & 0\\ \mathbf{C}_b(t) & \mathbf{C}_v\end{bmatrix},$$
$$\mathbf{K}(t)=\begin{bmatrix}\boldsymbol{\Omega}^2+\Delta\mathbf{K}_b(t) & 0\\ \mathbf{K}_b(t) & \mathbf{K}_v\end{bmatrix},$$
$$\mathbf{u}(t)=\begin{Bmatrix}\mathbf{y}(t)\\ \mathbf{u}_v(t)\end{Bmatrix},\quad \tau(t)=\begin{Bmatrix}\boldsymbol{\Phi}(\xi(t))\mathbf{M}_v\mathbf{X}(t)\gamma\\ \mathbf{M}_v\gamma\end{Bmatrix}. \tag{15a–e}$$

In the latter equations the following matrices have been defined:

$$\Delta\mathbf{M}_b(t)=\boldsymbol{\Phi}(\boldsymbol{\xi}(t))\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{\xi}(t)),\quad \mathbf{M}_b(t)=\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{\xi}(t)),$$
$$\Delta\mathbf{C}_b(t)=2_v\boldsymbol{\Phi}(\boldsymbol{\xi}(t))\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}_1^{\mathrm{T}}(\boldsymbol{\xi}(t)),\quad \mathbf{C}_b(t)=2v\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}_1^{\mathrm{T}}(\boldsymbol{\xi}(t)),$$
$$\Delta\mathbf{K}_b(t)=v^2\boldsymbol{\Phi}(\boldsymbol{\xi}(t))\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}_2^{\mathrm{T}}(\boldsymbol{\xi}(t)),\quad \mathbf{K}_b(t)=v^2\mathbf{M}_v\mathbf{X}(t)\boldsymbol{\Phi}_2^{\mathrm{T}}(\boldsymbol{\xi}(t)) \tag{16a–f}$$

with $\mathbf{M}_v$, $\mathbf{C}_v$ and $\mathbf{K}_v$ being the $N\times N$ diagonal matrices listing the masses m_i, coefficients of viscous damping c_i and stiffness k_i of the moving oscillators, respectively; $\mathbf{u}_v(t)$ a vector listing the N relative displacements $u_i(t)$ of the oscillators ; γ a vector whose N components are all equal to 1; $\boldsymbol{\Phi}_1(\boldsymbol{\xi}(t))$ and $\boldsymbol{\Phi}_2(\boldsymbol{\xi}(t))$ the following $n\times N$ matrices:

$$\boldsymbol{\Phi}_1(\boldsymbol{\xi}(t))=\begin{bmatrix}\boldsymbol{\phi}^{I}(\xi_1(t)) & \boldsymbol{\phi}^{I}(\xi_2(t)) & \cdots & \boldsymbol{\phi}^{I}(\xi_N(t))\end{bmatrix},$$
$$\boldsymbol{\Phi}_2(\boldsymbol{\xi}(t))=\begin{bmatrix}\boldsymbol{\phi}^{II}(\xi_1(t)) & \boldsymbol{\phi}^{II}(\xi_2(t)) & \cdots & \boldsymbol{\phi}^{II}(\xi_N(t))\end{bmatrix}, \tag{17a, b}$$

where the apex on the vector function $\boldsymbol{\phi}(\xi_i(t))$ denotes the partial derivative of $\boldsymbol{\phi}(x)$ with respect to x evaluated at $x=\xi_i(t)$.

It has to be noted that due to the time dependence of the instantaneous position $\xi_i(t)$ of the interface nodes where the beam and oscillators are connected, in Eq. (14) $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are time-dependent not symmetric matrices. Notice that, by neglecting the time derivatives of the position functions $\xi_i(t)$, the differentiation of Eq. (11) leads to the following simplified form for the accelerations of the interface nodes:

$$\ddot{u}_{b,i}(t)=\boldsymbol{\phi}(\xi_i(t))^{\mathrm{T}}\ddot{\mathbf{y}}(t),\quad i=1,\ldots,N. \tag{18}$$

By substituting Eq. (2) and (18) into Eq. (1b) and (8) the following simplified equations of motion of the coupled beam–oscillators system are obtained:

$$\mathbf{M}(t)\ddot{\tilde{\mathbf{u}}}(t)+\tilde{\mathbf{C}}\dot{\tilde{\mathbf{u}}}(t)+\tilde{\mathbf{K}}\tilde{\mathbf{u}}(t)=\boldsymbol{\tau}(t)g, \tag{19}$$

where $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ are symmetric and not time-dependent matrices, defined as follows:

$$\tilde{\mathbf{C}}=\begin{bmatrix}\boldsymbol{\Xi} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_v\end{bmatrix},\quad \tilde{\mathbf{K}}=\begin{bmatrix}\boldsymbol{\Omega}^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_v\end{bmatrix}. \tag{20a, b}$$

The accuracy of the simplified solution $\tilde{\mathbf{u}}(t)$ with respect to the complete one $\mathbf{u}(t)$ will be discussed in the numerical application section.

It has to be noted that due to the time dependence of the coefficient matrices of differential equations (14) and (19), the equations of motion of the coupled beam–oscillators system can be solved by means of the well-known step-by-step integration procedures [17,18], by assuming the coefficient matrices constant in each time step.

3. Methods for the evaluation of bending moment and shear force

3.1. Conventional series expansion

Once the set of ordinary differential equations (14) is solved, the bending moment and shear force laws along the beam can be obtained by differentiating the series expansion (5) with respect to the spatial coordinate x. Specifically, for an uniform beam the bending moment and shear force laws are given, respectively, by

$$\begin{aligned} M_{\mathrm{CSE}}(x,t) &= -EI w^{II}_{\mathrm{CSE}}(x,t) = -EI \boldsymbol{\phi}^{II}(x)^{\mathrm{T}} \mathbf{y}(t), \\ V_{\mathrm{CSE}}(x,t) &= -EI w^{III}_{\mathrm{CSE}}(x,t) = -EI \boldsymbol{\phi}^{III}(x)^{\mathrm{T}} \mathbf{y}(t), \end{aligned} \tag{21a, b}$$

where the subscript CSE denotes that these functions are evaluated by means of the CSE (5). Since the eigenfunctions of the beam are smooth functions, Eq. (21) are not able to capture the discontinuity in the bending moment and shear force laws due to the interaction forces $r_i(t)$ transmitted to the beam by the moving oscillators at the interface node. Then Eq. (21) converge very poorly and the Gibbs phenomenon [19] has been evidenced when the shear force is approximate by means of a large number of terms in the series expansion (5).

3.2. Improved series expansion

To overcome the bad convergence of the CSE (5) in the evaluation of bending moment and shear force, Pesterev and Bergman [5] proposed, for both cases of moving forces and oscillators, the so-called improved series expansion. This approach can be seen as the extension to continuous systems of the well-known mode-acceleration method (*MAM*) proposed in the literature [20,21] for discretized structural systems.

According to this method, the response of the beam is evaluated as the sum of the CSE (5) and a quasi-static response, which takes into account the quasi-static displacements of the beam under the quasi-static interaction forces $r_{s,i}$ transmitted by the moving oscillators associated with the truncated higher order eigenfunctions. The quasi-static response is evaluated as the difference of the quasi-static displacement function $w_s(x,t)$ of the beam under the interaction forces $r_{s,i}$, obtained as the solution of Eq. (1a) by neglecting the damping and inertial effects, and the quasi-static modal solution $\mathbf{y}_s(t)$ rewritten in the nodal space. Hence the improved solution can be written as follows:

$$w_{\mathrm{MAM}}(x,t) = w_{\mathrm{CSE}}(x,t) + w_s(x,t) - \boldsymbol{\phi}^{\mathrm{T}}(x)\mathbf{y}_s(t), \tag{22}$$

where the subscript MAM denotes that the lateral displacement of the beam is evaluated, according to the improved series expansion, by the extension to continuous systems of the MAM.

In the latter equation the quasi-static solutions $w_s(x,t)$ and $\mathbf{y}_s(t)$ are given respectively as

$$w_s(x,t) = \sum_{i=1}^{N} \chi_i(t) r_{s,i}(t) G(x, \xi_i(t)) = \mathbf{G}^{\mathbf{T}}(x, \boldsymbol{\xi}(t)) \mathbf{X}(t) \mathbf{r}_s,$$
$$\mathbf{y}_s(t) = \boldsymbol{\Omega}^{-2} \boldsymbol{\Phi}(\boldsymbol{\xi}(t)) \mathbf{X}(t) \mathbf{r}_s \tag{23a, b}$$

with $\mathbf{r}_s$ and $\mathbf{G}(x, \boldsymbol{\xi}(t))$ being the vectors of order N listing the quasi-static interaction forces $r_{s,i}$ and the Green's function $\mathbf{G}(x, \xi_i(t))$, evaluated at the locations $\xi_i(t)$ of the moving oscillators, respectively. The Green's function, also known in the literature as influence function, describes the deflection of the beam at the point x under an unit static force applied to the point $\xi_i(t)$ [5]. Analytical closed-form expressions of this function can be easily deduced for the beam governed by Eq. (1a). Notice that the quasi-static interaction forces $r_{s,i}$ transmitted to the beam by the oscillators can be obtained from Eq. (1b) by neglecting the inertial and damping effects as follows

$$r_{s,i} = k_i u_{s,i} = m_i g \tag{24}$$

with $u_{s,i} = k_i^{-1} m_i g$ being the quasi-static response of the oscillators, and then the forces $r_{s,i}$ are coincident with the weight of the oscillators. It follows that the quasi-static displacement functions $w_s(x,t)$ and $\mathbf{y}_s(t)$ take into account only the gravitational effects due to the moving oscillators, while the damping and inertial effects induced by the motion of the oscillators are neglected.

By differentiating Eq. (22) with respect to the spatial coordinate x and taking into account Eqs. (23) and (24), the improved bending moment and shear force laws obtained along the beam are

$$M_{\mathrm{MAM}}(x,t) = M_{\mathrm{CSE}}(x,t) - EI\left[\mathbf{G}^{II}(x, \boldsymbol{\xi}(t))^{\mathrm{T}} - \boldsymbol{\phi}^{II}(x)^{\mathrm{T}} \boldsymbol{\Omega}^{-2} \boldsymbol{\Phi}(\boldsymbol{\xi}(t))\right] \mathbf{M}_v \mathbf{X}(t) \boldsymbol{\gamma} g,$$
$$V_{\mathrm{MAM}}(x,t) = V_{\mathrm{CSE}}(x,t) - EI\left[\mathbf{G}^{III}(x, \boldsymbol{\xi}(t))^{\mathrm{T}} - \boldsymbol{\phi}^{III}(x)^{\mathrm{T}} \boldsymbol{\Omega}^{-2} \boldsymbol{\Phi}(\boldsymbol{\xi}(t))\right] \mathbf{M}_v \mathbf{X}(t) \boldsymbol{\gamma} g. \tag{25a, b}$$

It has to be noted that because the quasi-static displacement functions $\mathbf{y}_s(t)$ and $w_s(x,t)$, defined in Eq. (23), are not able to take into account the damping and inertial effects due to the motion of the oscillators, in the improved solution given by Eq. (25) these effects are neglected. This limitation of the improved solution can be evidenced by evaluating the jump $\Delta V_{\mathrm{MAM}}(\xi_i(t), t)$ in the shear force at the abscissa $x = \xi_i(t)$ defining the instantaneous position of the ith interface node. For the simply supported beam the Green's function can be written in the following form [5]:

$$G(x, \xi_i(t)) = \frac{1}{6EI}\left[\frac{\xi_i(t) - l}{l}\left(x^2 - 2l\xi_i(t) + \xi_i^2(t)\right)x + (x - \xi_i(t))^3 U(x - \xi_i(t))\right], \tag{26}$$

where $U(x - \xi_i(t))$ is the unit step function. In view of the continuity of the functions $\phi_j(x)$ and $V_{\mathrm{CSE}}(x,t)$, the following expression of the jump $\Delta V_{\mathrm{MAM}}(\xi_i(t), t)$ is obtained:

$$\Delta V_{\mathrm{MAM}}(\xi_i(t), t) = V_{\mathrm{MAM}}\left(\xi_i^+(t), t\right) - V_{\mathrm{MAM}}\left(\xi_i^-(t), t\right) = -\chi_i(t) m_i g = -\chi_i(t) r_{s,i}(t) \tag{27}$$

with $\xi_i^+(t)$ and $\xi_i^-(t)$ being the abscissas at the right- and left-hand side of the instantaneous position $\xi_i(t)$ of the ith moving oscillator, respectively. Eq. (27) shows that by applying the improved series expansion, the jump in the shear force takes into account only the gravitational effects of the ith moving oscillator.

3.3. Proposed improved series expansion

As shown, the above described improved series expansion takes into account only the gravitational effects associated with moving oscillators. By operating in this manner, the discontinuities in the shear force law are equal to the scalar value of oscillators weight.

In this section, with the aim of accounting for the gravitational, damping and inertial effects due to the moving oscillators, although in approximate form, a new method is proposed. The main idea of the proposed method stems from the DCM, originally proposed by Borino and Muscolino [16] for discrete structural systems. According to the DCM, the dynamic response of the beam can be improved by superimposing on the CSE a term associated with the particular solution of Equation (1a), bearing information on the truncated higher order eigenfunctions. This term is evaluated as the difference of the particular solution $w_p(x,t)$ of Eq. (1a) and the modal particular solution $\mathbf{y}_p(t)$ rewritten in the nodal space. Hence the proposed improved solution of the beam can be written as follows:

$$w_{\mathrm{DCM}}(x,t) = w_{\mathrm{CSE}}(x,t) + w_p(x,t) - \boldsymbol{\phi}^{\mathrm{T}}(x)\mathbf{y}_p(t), \tag{28}$$

where the subscript DCM denotes that the lateral displacement of the beam is improved by applying an extension of the DCM.

The particular solutions $w_p(x,t)$ and $\mathbf{y}_p(t)$ can be evaluated in approximate form by using a step-by-step integration procedure under the hypothesis that the interaction forces $r_i(t)$ transmitted to the beam by the N moving oscillators are constant within each time step. For instance, let the time interval be subdivided into small steps of equal length Δt so that $t_k = k\Delta t$ is the kth sampling time instant. Moreover, since the beams are usually lightly damped, it is possible to evaluate the particular solution of the motion equation of the beam neglecting the damping term. This greatly simplifies the evaluation of $w_p(x,t)$, and $\mathbf{y}_p(t)$, the damping coefficient b, of the beam is assumed to be equal to zero.

Under the latter hypotheses $w_p(x,t)$, in the kth time step, is obtained as the particular solution of the following differential equations:

$$\rho A\ddot{w}(x,t) + EIw^{IV}(x,t) = \sum_{i=1}^{N} \chi_i(t) r_i(t_k)\delta(x - \xi_i(t)), \quad t_{k-1} \leqslant t \leqslant t_k, \tag{29}$$

where $r_i(t_k)$ is the assumed constant value of the ith interaction force within the kth time step given as

$$r_i(t_k) = c_i\dot{u}_i(t_k) + k_i u_i(t_k). \tag{30}$$

This particular solution $w_p(x,t)$ is related to the particular solution $M_p(x,t)$ written in terms of bending moment by means of the following relationship:

$$w_p(x,t) = -\int_0^l G^{II}(x,\zeta)M_p(\zeta,t)\mathrm{d}\zeta. \tag{31}$$

The solution $M_p(x,t)$ can be evaluated, according to the unconditionally convergent procedure described in Ref. [7], by using the following iterative relationship:

$$M_p^{(r)}(x,t) = M_s(x,t) + \rho AEI \int_0^l G^{II}(x,\zeta)\ddot{w}_p^{(r-1)}(\zeta,t)\mathrm{d}\zeta, \quad t_{k-1} \leqslant t \leqslant t_k, \tag{32}$$

where the apex (r) denotes the particular solution $M_p^{(r)}(x,t)$ at the ith iteration. In Eq. (32) $G(x,\zeta)$ is the Green's function and $M_s(x,t)$ is the quasi-static bending moment of the beam given as

$$M_s(x,t) = -EI\sum_{i=1}^{N}\chi_i(t)G^{II}(x,\xi_i(t))r_i(t_k). \tag{33}$$

By introducing the following functions:

$$H_M^{(r)}(x,\xi_i(t)) = \int_0^l \int_0^l G^{II}(x,\zeta)G^{II}(\zeta,z)\frac{\partial^2 H_M^{(r-1)}(z,\xi_i(t))}{\partial \xi_i^2}\mathrm{d}z\,\mathrm{d}\zeta,$$
$$H_M^{(0)}(x,\xi_i(t)) = G^{II}(x,\xi_i(t)) \tag{34a, b}$$

the particular solution $M_p(x,t)$ in terms of bending moment can be written as follows

$$M_p(x,t) = -EI\sum_{i=1}^{N}\chi_i(t)r_i(t_k)\sum_{r=0}^{\infty}\left(-\rho AEIv^2\right)^r H_M^{(r)}(x,\xi_i(t)), \quad t_{k-1}\leqslant t\leqslant t_k. \tag{35}$$

In the evaluation of this solution it has been taken into account that for the selected boundary conditions of the beam it results

$$\dot{\chi}_i(t)G^{II}(x,\xi_i(t)) = 0. \tag{36}$$

Starting from Eq. (35), the particular solution $w_p(x,t)$ can be obtained by means of Eq. (31) as follows:

$$w_p(x,t) = \sum_{i=1}^{N}\chi_i(t)r_i(t_k)\sum_{r=0}^{\infty}\left(-\rho Av^2\right)^r H_w^{(r)}(x,\xi_i(t)), \quad t_{k-1}\leqslant t\leqslant t_k, \tag{37}$$

where the following functions have been defined:

$$H_w^{(r)}(x,\xi_i(t)) = EI\int_0^l G^{II}(x,\zeta)H_M^{(r)}(\zeta,\xi_i(t))\mathrm{d}\zeta = \int_0^l G(x,\zeta)\frac{\partial^2 H_w^{(r-1)}(\zeta,\xi_i(t))}{\partial\xi_i^2}\mathrm{d}\zeta,$$

$$H_w^{(0)}(x,\xi_i(t)) \equiv G(x,\xi_i(t)). \tag{38a, b}$$

Contrary to $w_p(x,t)$, the modal particular solution $\mathbf{y}_p(t)$ can be evaluated in closed form by solving the following differential equation:

$$\ddot{\mathbf{y}}(t) = \boldsymbol{\Omega}^2\mathbf{y}(t) = \boldsymbol{\Phi}(\boldsymbol{\xi}(t))\mathbf{X}(t)\mathbf{r}(t_k), \quad t_{k-1}\leqslant t\leqslant t_k, \tag{39}$$

where $\mathbf{r}(t_k)$ is the vector collecting the N interaction forces $r_i(t_k)$ evaluated within the kth time step. It results in

$$\mathbf{y}_p(t) = \mathbf{W}^{-1}\mathbf{\Phi}(\boldsymbol{\xi}(t))\mathbf{X}(t)\mathbf{r}(t_k), \quad t_{k-1} \leqslant t \leqslant t_k, \tag{40}$$

with $\mathbf{W}$ being the $n \times n$ diagonal matrix defined as follows:

$$\mathbf{W} = \mathbf{\Omega}^2 - v^2\sqrt{\frac{\rho A}{EI}}\mathbf{\Omega}. \tag{41}$$

Once the particular solutions $w_p(x,t)$ and $\mathbf{y}_p(t)$ are evaluated, the bending moment and shear force laws within the kth time step, can be obtained by differentiating Eq. (28) as follows:

$$\begin{aligned} M_{\mathrm{DCM}}(x,t) &= M_{\mathrm{CSE}}(x,t) - EI\left[\mathbf{h}^{II}(x,\boldsymbol{\xi}(t))^{\mathrm{T}} - \boldsymbol{\phi}^{II}(x)^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{\Phi}(\boldsymbol{\xi}(t))\right]\mathbf{X}(t)\mathbf{r}(t_k), \\ V_{\mathrm{DCM}}(x,t) &= V_{\mathrm{CSE}}(x,t) - EI\left[\mathbf{h}^{III}(x,\boldsymbol{\xi}(t))^{\mathrm{T}} - \boldsymbol{\phi}^{III}(x)^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{\Phi}(\boldsymbol{\xi}(t))\right]\mathbf{X}(t)\mathbf{r}(t_k) \end{aligned} \tag{42a, b}$$

with $\mathbf{h}(x,\boldsymbol{\xi}(t))$ being the vector defined as

$$\mathbf{h}(x,\boldsymbol{\xi}(t)) = \begin{Bmatrix} h(x,\xi_1(t)) \\ h(x,\xi_2(t)) \\ \vdots \\ h(x,\xi_N(t)) \end{Bmatrix}, \quad h(x,\xi_i(t)) = \sum_{r=0}^{\infty}(-\rho A v^2)^r H_w^{(r)}(x,\xi_i(t)). \tag{43a, b}$$

The proposed improved solution is able to take into account gravitational, damping and inertial effects due to the moving oscillators. In fact, in view of the continuity of the functions $\phi_j(x)$ and $V_{\mathrm{CSE}}(x,t)$, the jump in the shear force at the abscissa $x = \xi_i(t_k)$ defining the instantaneous position of the ith moving oscillator at the instant $t = t_k$, is given by

$$\begin{aligned} \Delta V_{\mathrm{DCM}}(\xi_i(t_k),t_k) &= V_{\mathrm{DCM}}\left(\xi_i^{+}(t_k),t_k\right) - V_{\mathrm{DCM}}\left(\xi_i^{+}(t_k),t_k\right) \\ &= -\chi_i(t_k)(c_i\dot{u}_i(t_k) + k_i u_i(t_k)) \\ &= -\chi_i(t_k)m_i\left(g - \ddot{u}_i(t_k) - \ddot{u}_{b,i}(t_k)\right) = -\chi_i(t_k)r_i(t_k). \end{aligned} \tag{44}$$

Eq. (44) shows that, contrary to Eq. (27), by applying the proposed procedure the inertial, damping and gravitational effects due to the moving oscillators are included.

4. Numerical applications

The aim of numerical applications is to demonstrate the better accuracy of the proposed approach with respect to the CSE and the improved series expansion (MAM) in the evaluation of the bending moment and shear force distributions along the beam. The results obtained by using the proposed approach are denoted here by the subscript DCM since this method improves the CSE by means of an extension to the continuous systems of the DCM, originally proposed for discretized structures. This method, contrary to the CSE and the improved series expansion

(MAM), is able to take into account gravitational, inertial and damping effects induced on the beam by the moving oscillator. Moreover, the case of a fixed oscillator is first presented with the aim of providing comparisons between the solution evaluated by means of a finite element discretization (FE) of the coupled beam–oscillators system and CSE, MAM and DCM approaches. This comparison is not performed in the case of a moving oscillator in view of the poor accuracy in the solution of this problem provided by using the common finite element standard codes.

4.1. Simply supported beam with fixed oscillator

In this first example the simply supported beam connected to a single fixed oscillator subjected to a force $f(t)$, depicted in Fig. 2, is considered. The aim of this application is to compare the accuracy of the conventional and the two improved series expansion, described in the previous sections, with respect to the solution obtained by means of a FE discretization of the coupled beam–oscillators system, here considered as the referred solution.

In particular the following data have been assumed for the beam:

$$l = 10\ \mathrm{m},\quad \rho = 2.5t,\quad E = 3.0\times 10^7\ \mathrm{kN/m^2},\quad I = 3.12\times 10^{-3}\ \mathrm{m}^4,\quad A = 0.15\ \mathrm{m}^2,$$

while the oscillator, connected to the beam at the abscissa $\xi_0 = 5$ m, is characterized by the natural frequency ω and the adimensional mass parameter μ defined as follows:

$$\mu = \frac{m}{\rho A l}. \tag{45}$$

For sake of simplicity, and without losing generality, both beam and oscillator are supposed as not damped.

The example under study can be seen as a particularization of the problem of a beam subjected to a single moving oscillator with velocity v equal to zero, and then the equations of motion of the coupled beam–oscillators system can be written in the following form:

$$\mathbf{M}_0\ddot{\mathbf{q}}_0(t) + \mathbf{K}_0\mathbf{q}_0(t) = \boldsymbol{\tau}_0 f(t), \tag{46}$$

where the following matrices have been defined:

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{I}_n + m\boldsymbol{\phi}(\xi_0)\boldsymbol{\phi}^{\mathrm{T}}(\xi_0) & m\boldsymbol{\phi}(\xi_0) \\ m\boldsymbol{\phi}^{\mathrm{T}}(\xi_0) & m \end{bmatrix},\quad \mathbf{K}_0 = \begin{bmatrix} \boldsymbol{\Omega}^2 & 0 \\ 0 & k \end{bmatrix},\quad \tau_0 = \begin{Bmatrix} \boldsymbol{\phi}(\xi_0) \\ 1 \end{Bmatrix}. \tag{47a–c}$$

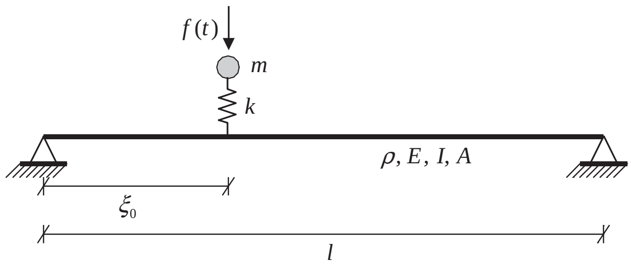

Fig. 2. Beam connected to a single fixed undamped oscillator.

Once the equations of motion (46) have been solved, by applying the MAM improved series expansion the following expression for lateral displacement, bending moment and shear force of the beam are obtained:

$$\begin{aligned} w_{\mathrm{MAM}}(x,t) &= w_{\mathrm{CSE}}(x,t) + \left[G(x,\xi_0) - \boldsymbol{\phi}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]f(t), \\ M_{\mathrm{MAM}}(x,t) &= M_{\mathrm{CSE}}(x,t) - EI\left[G^{II}(x,\xi_0) - \boldsymbol{\phi}^{II}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]f(t), \\ V_{\mathrm{MAM}}(x,t) &= V_{\mathrm{CSE}}(x,t) - EI\left[G^{III}(x,\xi_0) - \boldsymbol{\phi}^{III}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]f(t). \end{aligned} \tag{48a−c}$$

Instead, a generalization in the case of velocity v of the oscillator equal to zero of the proposed improved series expansion (DCM) leads to the following equations:

$$\begin{aligned} w_{\mathrm{DCM}}(x,t) &= w_{\mathrm{CSE}}(x,t) + \left[G(x,\xi_0) - \boldsymbol{\phi}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]ku(t_k), \\ M_{\mathrm{DCM}}(x,t) &= M_{\mathrm{CSE}}(x,t) - EI\left[G^{II}(x,\xi_0) - \boldsymbol{\phi}^{II}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]ku(t_k), \\ V_{\mathrm{DCM}}(x,t) &= V_{\mathrm{CSE}}(x,t) - EI\left[G^{III}(x,\xi_0) - \boldsymbol{\phi}^{III}(x)^{\mathrm{T}}\boldsymbol{\Omega}^{-2}\boldsymbol{\phi}(\xi_0)\right]ku(t_k), \end{aligned} \tag{49a−c}$$

The numerical results are here evaluated in the case of an oscillator characterized by an adimensional mass parameter $\mu = 0.2$ and natural frequency $\omega = 30$ rad/s and subject to the harmonic force $f(t) = f_0 \sin(\omega_f t)$ with $f_0 = 10$ kN and $\omega_f = 15$ rad/s. In Figs. 3 and 4 the dynamic response of the beam, in terms of bending moment $M(x,t)$ and shear force $V(x,t)$ distributions, respectively, evaluated by applying the CSE and the two improved ones, are reported in comparison with the response obtained by means of the FE discretization. The bending moment and shear force distributions depicted in Figs. 3 and 4 are evaluated at the time $t = 0.6$ s by using 2 and 5 eigenfunctions of the beam. These figures evidence that the conventional series expansion, termed in figure as CSE, is not able to capture the discontinuities in the bending moment and shear force laws. These discontinuities are partially captured by the MAM approach, which considers only the effects due to the force $f(t)$ acting on the oscillator. Instead, both $f(t)$ and inertial effects, due to the motion of the oscillator, are taken into account by using the DCM approach, and the obtained bending moment and shear force laws almost coincide with the ones evaluated by using the FE discretization.

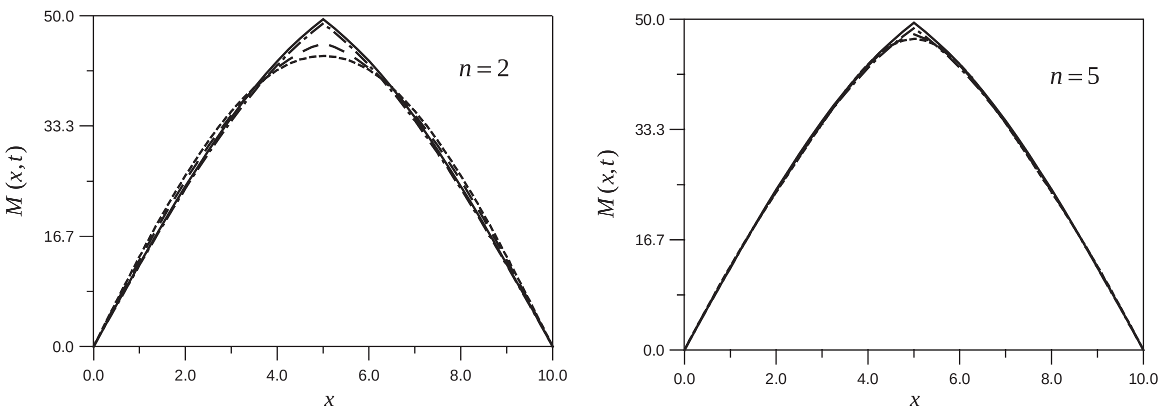

Fig. 3. Bending moment distributions for the case of fixed oscillator at the instant $t = 0.6$ s by considering n eigenfunctions. ——, *FE*; -----, *CSE*; – –, *MAM*; – - –, *DCM*.

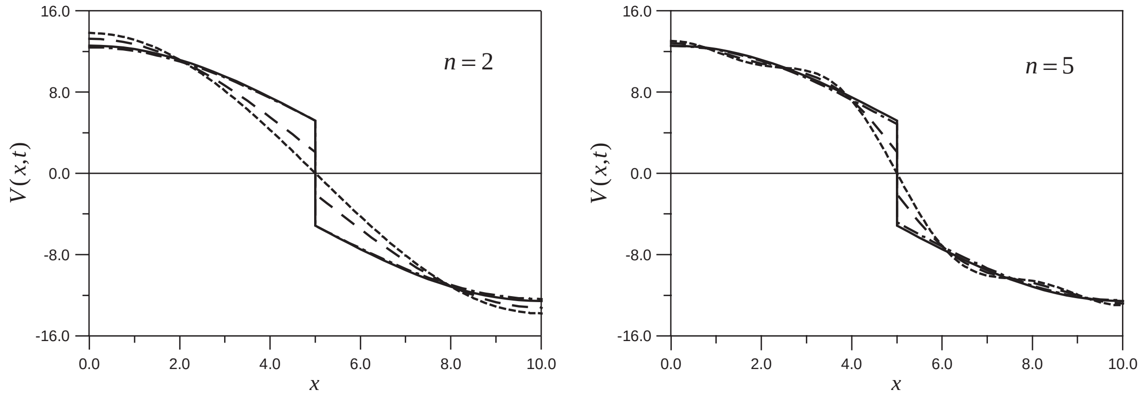

Fig. 4. Shear force distributions for the case of fixed oscillator at the instant $t = 0.6$ s by considering n eigenfunctions. ——, *FE*; -----, *CSE*; – –, *MAM*; – - –, *DCM*.

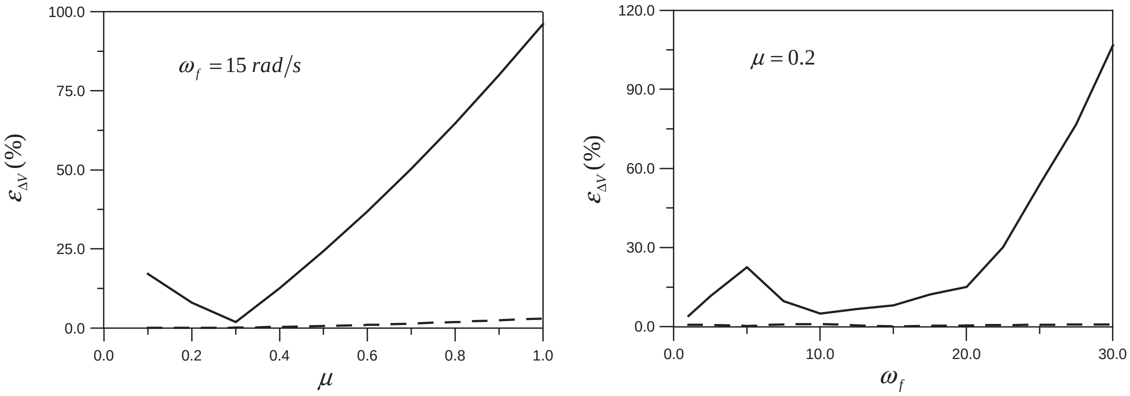

Fig. 5. Percentage errors of the jump in the shear force for the case of fixed oscillator by considering $n = 5$ eigenfunctions. ——, *MAM*; – –, *DCM*.

The main accuracy of the DCM approach with respect to the MAM is evidenced in Fig. 5 where the percentage errors $\varepsilon_{\Delta V}$ of the jump in the shear force versus the adimensional mass parameter μ and versus the frequency ω_f of the applied force $f(t)$, respectively, are reported. In these figures $\varepsilon_{\Delta V}$ is the percentage error of the jump in the shear force evaluated by using the following relationship:

$$\varepsilon_\theta = \frac{|\theta_R - \theta_A|}{|\theta_R|} \times 100 \tag{50}$$

θ_R and θ_A being the referred and approximate values of θ, respectively. In particular, in Fig. 5 it has been assumed that $\theta_R = \Delta V_{\mathrm{FE}}(\xi_0, t_1)$ and $\theta_A = \Delta V_{\mathrm{MAM}}(\xi_0, t_1)$ for the solid line and $\theta_A = \Delta V_{\mathrm{DCM}}(\xi_0, t_1)$ for the dashed line. The jump in the shear force $\Delta V_{\mathrm{MAM}}(\xi_0, t_1)$, $\Delta V_{\mathrm{DCM}}(\xi_0, t_1)$ and $\Delta V_{\mathrm{FE}}(\xi_0, t_1)$ are evaluated by using 5 eigenfunctions of the beam at the time t_1 in which the applied force $f(t)$ engages its maximum value f_0 for the first time. Fig. 5 shows that the percentage errors $\varepsilon_{\Delta V}$ of the jump in the shear force evaluated by using the MAM approach increase by increasing the mass of the oscillator and the frequency ω_f of the applied force. On the contrary, the errors $\varepsilon_{\Delta V}$ evaluated by using the DCM approach are almost independent from the mass of the oscillator and the frequency of the applied force, being very small in all the examined cases.

4.2. Simply supported beam subjected to moving oscillators

As a second numerical application the same beam studied in the previous section, but subjected to an undamped moving oscillator with constant velocity v, is considered.

The aim of this further numerical investigation is first to assess the accuracy of the solution $\tilde{\mathbf{u}}(t)$ of the simplified equations of motion of the coupled beam–oscillators system, given by Eq. (19), obtained by neglecting the time derivatives of the position $\xi(t)$ of the oscillator and having symmetric not time-dependent matrix $\tilde{\mathbf{K}}$, with respect to the solution $\mathbf{u}(t)$ of the complete equations of motion, given by Eq. (14). To this aim in Fig. 6, the percentage errors ε_w of the maximum displacement w_{MAX} of the beam, evaluated by using 5 eigenfunctions, versus the velocity v of the moving oscillator are reported. The percentage error ε_w is evaluated by means of Eq. (50) with $\theta_R = w_{\mathrm{MAX}}$ and $\theta_A = \tilde{w}_{\mathrm{MAX}}$, $\tilde{w}_{\mathrm{MAX}}$ being the maximum value of displacement $\tilde{w}(x,t)$ evaluated by solving the simplified equation of motion given by Eq. (19). Fig. 6 shows that, as expected, the percentage errors ε_w increase by increasing velocity v of the moving oscillator. In fact Eq. (19) is obtained by neglecting in Eq. (13) the terms $2v\boldsymbol{\phi}^I(\xi_i(t))^{\mathrm{T}}\dot{\mathbf{y}}(t)$ and $v^2\boldsymbol{\phi}^{II}(\xi_i(t))^{\mathrm{T}}\mathbf{y}(t)$, that are negligible only for low values of the velocity v of the moving oscillator. In particular, for the beam under study, the simplified equations of motion may give no accurate solution, with a maximum error of about 45%. It follows that all terms in the equations of motion are considered in these numerical applications.

In Figs. 7 and 8 the dynamic response of the beam, in terms of bending moment $M(x,t)$ and shear force $V(x,t)$ distributions evaluated by using the CSE and the two improved ones for $\mu = 0.2$ and $v = 10$ m/s, are shown. The bending moment and shear force distributions depicted in Figs. 7 and 8 are evaluated at the time $t = 0.5$ s, at which the oscillator is at the mid-span of the beam, by using 2 and 5 eigenfunctions of the beam. Like the case of a fixed oscillator, Figs. 7 and 8

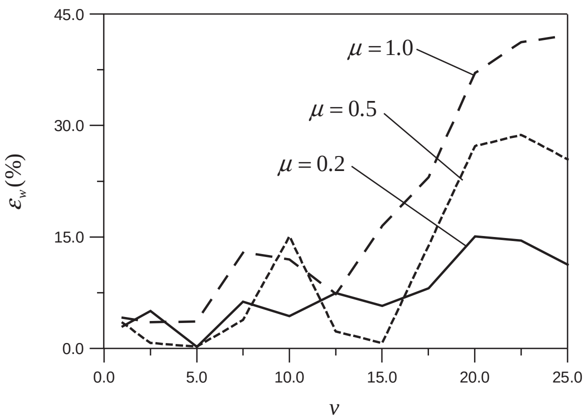

Fig. 6. Percentage errors of the maximum displacement of the beam versus the velocity v of the moving oscillator.

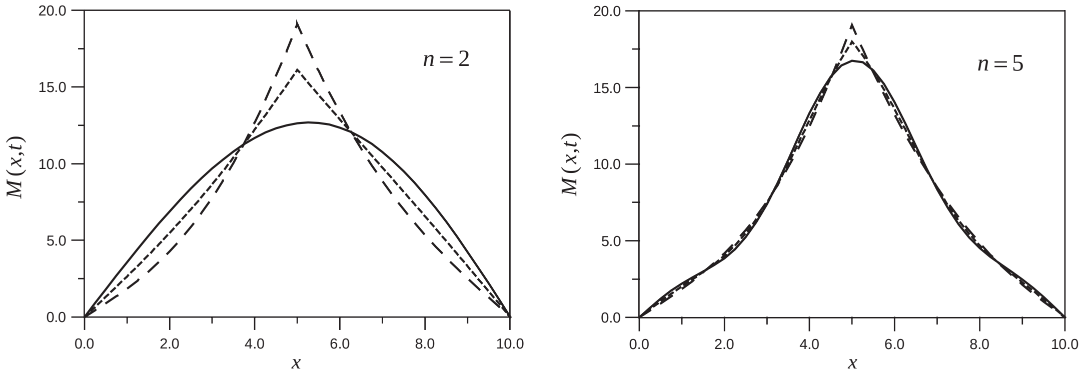

Fig. 7. Bending moment distributions for the case of moving oscillator at the instant $t = 0.5$ s by considering n eigenfunctions. ——, *CSE*; -----, *MAM*; — —, *DCM*.

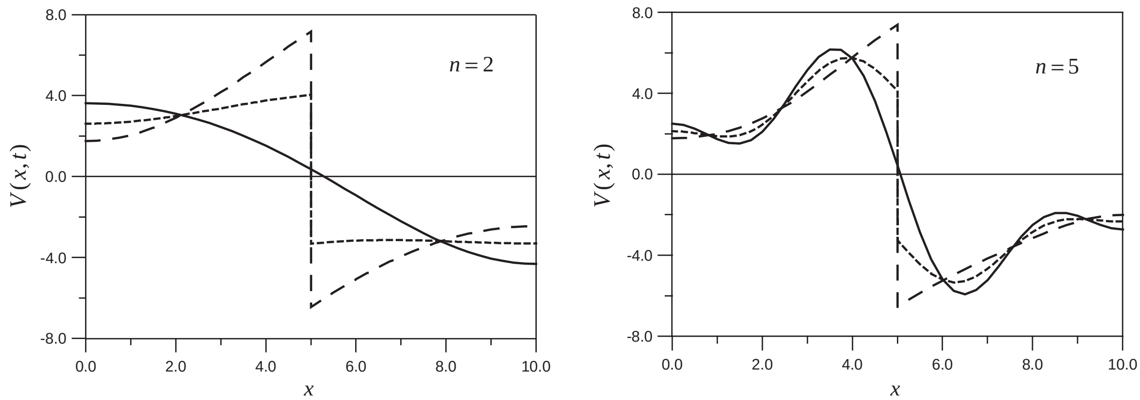

Fig. 8. Shear force distributions for the case of moving oscillator at the instant $t = 0.5$ s by considering n eigenfunctions. ——, *CSE*; -----, *MAM*; — —, *DCM*.

show that the CSE is not able to capture the discontinuities in the bending moment and shear force laws, while these discontinuities are captured, with different levels of accuracy, by the MAM and DCM approaches. In particular in the MAM approach the discontinuities in the bending moment and shear force laws are due to the gravitational effects of the oscillator only, while the DCM approach considers both the gravitational and inertial effects.

To evidence the differences of the two improved series expansion, in Fig. 9 the values of jump in the shear force evaluated by applying the MAM and DCM series expansion versus the adimensional mass parameter μ, for different values of the velocity v of the oscillator, are depicted. The jump in the shear force is evaluated at the time $t = 0.5$ s, at which the oscillator is at the mid-span of the beam, by using 5 eigenfunctions. Fig. 9 shows that by applying the MAM approach, the jump in the shear force, being coincident with the oscillator's weight, increases linearly with the mass of the oscillator and is independent of the velocity v. On the

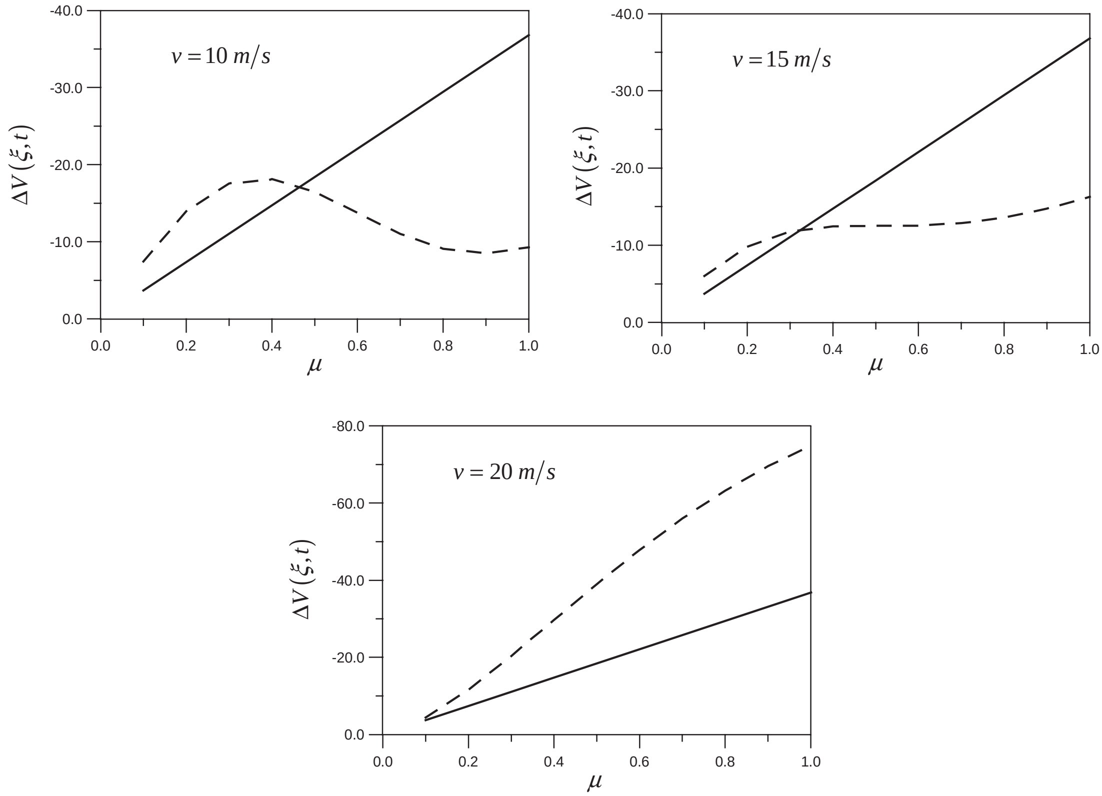

Fig. 9. Jump in the shear force for the case of moving oscillator. ——, *MAM*; – –, *DCM*.

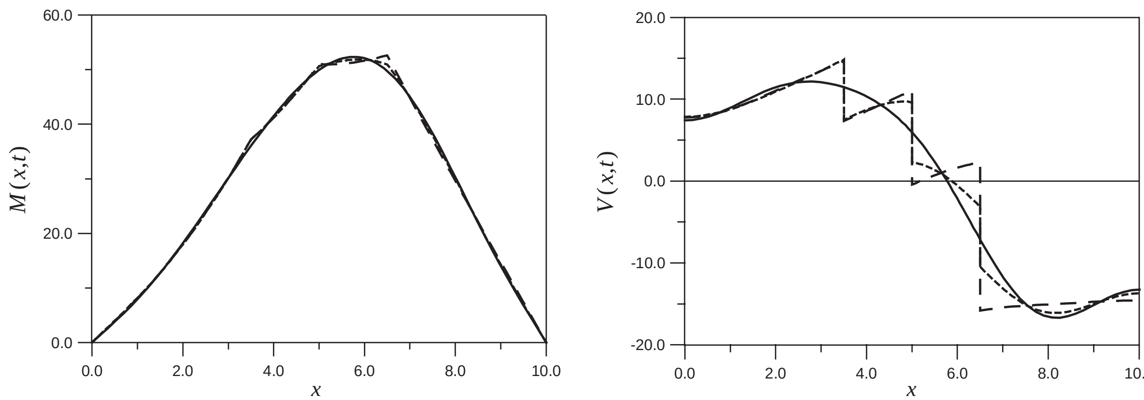

Fig. 10. Bending moment and shear force distributions for the case of three moving oscillators at the instant $t = 0.65$ s. ——, *CSE*; -----, *MAM*; — —, *DCM*.

contrary, by using the DCM approach, the jump in the shear force is strongly influenced by the velocity v of the moving oscillator, and its biggest value is not necessarily obtained for the heaviest oscillator.

Finally, as a third example the same beam previously studied and subjected to a set of $N = 3$ undamped moving oscillators with constant velocity $v = 10$ m/s is considered. All the oscillators are characterized by the values $\mu = 0.2$ and $\omega = 30$ rad/s and the adopted distance between two consecutive oscillators is $d = 1.5$ m. In Fig. 10 the dynamic response of the beam, in terms of bending moment $M(x, t)$ and shear force $V(x, t)$ distributions evaluated by using the CSE and the two improved ones is reported. The bending moment and shear force distributions are evaluated at the time $t = 0.65$ s, at which the second oscillator is at the mid-span of the beam, by using 5 eigenfunctions. Like the previously considered cases, Fig. 10 shows that the CSE is not able to capture the discontinuities in the bending moment and shear force laws, while these discontinuities are captured, with different levels of accuracy, by the MAM and DCM series expansions.

5. Conclusions

A new method able to calculate with considerable accuracy the bending moment and shear force distributions of an elastic beam carrying moving oscillators is presented. The proposed procedure improves the convergence and accuracy of the conventional eigenfunction series expansion of beam response by considering the particular solution of the differential equation, governing the problem, associated with the truncated terms of the eigenfunction expansion. The proposed method can be seen as an extension of the DCM, originally proposed for discretized structural systems.

It is also shown that the method is able to take into account gravitational, inertial and damping effects of the moving oscillators, contrary to a recently proposed improved series expansion where the correction term, evaluated according to the mode acceleration method, takes into account the gravitational effect only. In the numerical application the case of a beam with fixed oscillator is first studied to demonstrate the capability and accuracy of the proposed method to determine the discontinuities and jumps in the bending moment and shear force distributions, respectively.

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