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Generalized thermoelastic wave propagation in circumferential direction of transversely isotropic cylindrical curved plates

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Abstract

The propagation of thermoelastic waves along circumferential direction in homogeneous, transversely isotropic, cylindrical curved plates has been investigated in the context of theories of thermoelasticity. This type of study is important for ultrasonic non-destructive inspection of large-diameter pipes, which helps in the health monitoring of ailing infrastructure. Longitudinal stress-corrosion cracks are usually temperature dependent and can be detected more efficiently by inducing circumferential waves; hence the study of generalized thermoelastic wave propagation in the circumferential direction in a pipe wall is essential. Mathematical modeling of the problem of obtaining dispersion curves for curved transversely isotropic thermally conducting elastic plates leads to coupled differential equations. The model has been simplified by using the Helmholtz decomposition technique and the resulting equations have been solved by using separation of variable method to obtain the secular equations in isolated mathematical conditions for the plates with stress-free or rigidly fixed, thermally insulated and isothermal boundary surfaces. The closed form solutions are also obtained under different situations and conditions. The longitudinal shear motion and axially symmetric shear vibration modes get decoupled from the rest of the motion and are not affected by thermal variations, whereas for the non-axially symmetric case of plane strain vibrations, these modes remain coupled and are affected by temperature changes. Moreover, these vibration modes are found to be dispersive and dissipative in character. In order to illustrate theoretical development, numerical solutions are obtained and presented graphically for a zinc plate. The obtained results are also compared with those available in the literature in case of waves in cylindrical shell/circular annulus in the absence of thermomechanical coupling and thermal relaxation times.

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1. Introduction

Victorov [1] established the fundamental mathematical modeling of the problem for isotropic material properties. He introduced the concept of angular wave number and derived, decomposed and solved the governing differential equations. He found the solutions for convex and concave cylindrical surfaces by considering only one curved surface. Qu et al. [2] obtained the results for guided waves in isotropic curved plates by adding the boundary conditions for the second surface. Many investigators [3–8] have solved elastic wave propagation problems in homogeneous anisotropic and multilayered solids in the longitudinal direction along one or multiple curved surfaces to analyze the different aspects. Nayfeh [9] limited his analysis to the flat plate case and Armenakas and Reitz [10] have studied waves propagating in the axial direction of the cylinder. Recently, Towfighi et al. [11] studied the elastic wave propagation in circumferential direction in anisotropic cylindrical curved plates with the help of the Fourier series technique. They obtained and presented dispersion curves for anisotropic curved plates of different curvature. Mathematical modeling of wave propagation in the axial direction of a cylinder has been studied extensively. However, for wave propagation in the circumferential direction, which is essential for nondestructive testing (NDT) of large-diameter pipes, only fewer investigations exist in the literature [11]. The theory of elastic wave propagation in anisotropic solids is well known [12,13]. In case of suddenly applied thermal loading, thermal deformation and the role of inertia become greater. Since thermal stresses change very rapidly, the static analysis cannot capture its behavior. This dynamic thermoelastic stress response is significant and leads to the propagation of elastic stress waves in solids. The theory of thermoelasticity is well established [14]. The governing field equations in classical dynamic coupled thermoelasticity (CT) are wave-type (hyperbolic) equations of motion and a diffusion-type (parabolic) equation of heat conduction. Therefore, it is seen that part of the solution of the energy equation extends to infinity, implying that if a homogeneous isotropic elastic medium is subjected to thermal or mechanical disturbances, the effect of temperature and displacement fields is felt at an infinite distance from the source of disturbance. This shows that part of the disturbance has an infinite velocity of propagation, which is physically impossible. With this drawback in mind, some researchers, such as Lord and Shulman [15] and Green and Lindsay [16], modified Fourier law of heat conduction and constitutive relations so as to get a hyperbolic equation for heat conduction. These works include the time needed for the acceleration of heat flow and take into account the coupling between temperature and strain fields for isotropic materials. Banerjee and Pao [17] investigated the propagation of plane harmonic waves in infinitely extended anisotropic solids, taking into account the thermal relaxation time. Dhaliwal and Sherief [18] extended the generalized thermoelasticity [15] to anisotropic elastic bodies. Chandrasekharaiah [19] referred to a wave-like thermal disturbance as ‘second sound’. These theories are also supported by experiments [20–22] exhibiting the actual occurrence of second sound at low temperatures and small intervals of time. The investigators [23–28] studied the propagation of plane harmonic waves in homogeneous anisotropic heat-conducting elastic materials. Recently, Sharma [29] presented an exact analysis of the free vibrations of a simply supported, homogeneous, transversely isotropic cylindrical panel based on the three-dimensional coupled thermoelasticity.

The problem of generalized thermoelastic wave propagation in the circumferential direction of homogeneous, transversely isotropic, curved plates has not been analyzed earlier and is considered for the first time in this paper. The mathematical model has been simplified by using the Helmholtz decomposition technique and the secular equations for different mechanical situations and thermal conditions have been obtained and discussed. The results obtained have also been compared and reduced to those available in the literature at appropriate stages of this work. The theoretical developments have been verified numerically and illustrated graphically for a zinc crystal plate.

2. Formulation of the problem

We consider a homogeneous, transversely isotropic, thermally conducting elastic cylindrical plate having inner and outer radii 'a' and 'b' respectively. The plate is assumed initially at uniform temperature T_0 in the undisturbed state. The geometry of the problem is shown in Fig. 1 and we consider the problem of wave propagation in the direction of the curvature. We will interchangeably call the wave carrier a curved plate, cylinder, pipe segment or simply pipe, all meaning the same thing. But we are interested in analyzing the dispersive waves in the curved plate for waves propagating from section S_1 to S_2 (see Fig. 1). Wave speed is proportional to radius of curvature.

This analysis does not include the reflected guided waves from the plate boundary. The considered geometry of the problem can be a segment of a cylinder or a complete cylinder. In cylindrical coordinates, the governing field equations of motion and heat conduction in the absence of body forces and heat sources are

$$\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}\,\sigma_{r\theta,\theta} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \ddot{u}_r,\\
&\sigma_{r\theta,r} + \frac{1}{r}\,\sigma_{\theta\theta,\theta} + \sigma_{\theta z,z} + \frac{2\sigma_{r\theta}}{r} = \rho \ddot{u}_\theta,\\
&\sigma_{rz,r} + \frac{1}{r}\,\sigma_{\theta z,\theta} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r} = \rho \ddot{u}_z, \qquad (1)
\end{aligned}$$

$$\begin{aligned}
&K_1\left(T_{,rr} + \frac{1}{r}\,T_{,r} + \frac{1}{r^2}\,T_{,\theta\theta}\right) + K_3 T_{,zz} - \rho C_e(\dot{T} + t_0 \ddot{T})\\
&\quad = T_0\left(\frac{\partial}{\partial t} + \delta_{1k} t_0\,\frac{\partial^2}{\partial t^2}\right)[\beta_1(e_{rr} + e_{\theta\theta}) + \beta_3 e_{zz}],
\end{aligned}$$

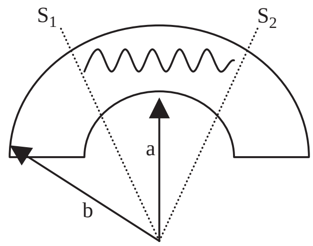

Fig. 1. Geometry of the problem.

where

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} - \beta_1(T + t_1\delta_{2k}\dot{T}), \quad \sigma_{r\theta} = c_{66}e_{r\theta},$$
$$\sigma_{\theta\theta} = c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} - \beta_1(T + t_1\delta_{2k}\dot{T}), \quad \sigma_{\theta z} = c_{44}e_{\theta z},$$
$$\sigma_{zz} = c_{13}e_{rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz} - \beta_3(T + t_1\delta_{2k}\dot{T}), \quad \sigma_{rz} = c_{44}e_{rz}, \tag{2}$$

$$e_{rr} = u_{r},_{r}, \quad e_{\theta\theta} = \frac{u_{\theta},_{\theta}}{r} + \frac{u_r}{r}, \quad e_{zz} = u_{z},_{z}, \quad e_{r\theta} = \frac{u_{r},_{\theta}}{r} + u_{\theta},_{r} - \frac{u_\theta}{r}, \quad e_{rz} = u_{r},_{z} + u_{z},_{r},$$

$$e_{\theta z} = u_{\theta},_{z} + \frac{1}{r}u_{z},_{\theta}, \quad c_{66} = \frac{c_{11} - c_{12}}{2}, \tag{3}$$

$$\beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3, \quad \beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3. \tag{4}$$

Here $\mathbf{u} = (u_r, u_\theta, u_z)$ is the displacement vector; $T(r, \theta, z, t)$ is the temperature change; $c_{11}, c_{12}, c_{13}, c_{33}$ and c_{44} are five independent isothermal elastic parameters; α_3, α_1 and K_3, K_1 are, respectively, the coefficients of linear thermal expansion and thermal conductivities along and perpendicular to the axis of symmetry; ρ and C_e are, respectively, the density and specific heat at constant strain and t_0 and t_1 are thermal relaxation times. The comma notation is used for spatial derivatives and the superposed dot represents time differentiation. δ_{1k} is Kronecker's delta in which $k = 1$ for Lord–Shulman (LS) theory and $k = 2$ for Green–Lindsay (GL) theory of thermoelasticity. It can be proved thermodynamically [23] that $K_1 > 0, K_3 > 0$ and of course $\rho > 0, T_0 > 0$. We assume in addition that $C_e > 0$ and that the parameters of isothermal linear elasticity are components of a positive definite fourth-order tensor. The necessary and sufficient conditions for the satisfaction of latter requirements are

$$c_{11} > 0, \quad c_{11} > c_{12}, \quad c_{11}^2 > c_{12}^2, \quad c_{44} > 0, \quad c_{33}(c_{11} + c_{12}) > c_{13}^2. \tag{5}$$

3. Boundary conditions

Let us consider following types of boundary conditions. The lower and upper surfaces $r = a$ and $r = b$ of the plate are assumed to be

(i) stress free, which leads to

$$\sigma_{rr} = 0, \quad \sigma_{rz} = 0, \quad \sigma_{r\theta} = 0, \tag{6a}$$

(ii) rigidly fixed, which implies that

$$u_r = 0, \quad u_z = 0, \quad u_\theta = 0, \tag{6b}$$

(iii) thermal conditions

$$T_{,r} + hT = 0, \tag{6c}$$

where h is Biot's heat transfer coefficient. Here $h \to 0$ refers to thermally insulated boundaries and $h \to \infty$ corresponds to isothermal surfaces.

4. Solution of the problem

In order to solve Eq. (1), we assume [29]

$$u_r = \frac{1}{r}\psi_{,\theta} - \phi_{,r}, \quad u_\theta = -\frac{1}{r}\phi_{,\theta} - \psi_{,r}, \quad u_z = w. \tag{7}$$

Upon using Eq. (7) in Eq. (1) and keeping in view that the various physical quantities are independent of axial coordinate (z), we find that the functions ϕ, T, ψ and w satisfy the non-dimensional equations

$$\left(\nabla_1^2 - \frac{\partial^2}{\partial t^2}\right)\phi = -(T + t_1\delta_{2k}\dot{T}), \tag{8a}$$

$$\nabla_1^2 T - (\dot{T} + t_0\ddot{T}) = -\varepsilon\left(\frac{\partial}{\partial t} + t_0\delta_{1k}\frac{\partial^2}{\partial t^2}\right)\nabla_1^2\phi, \tag{8b}$$

$$\left(\nabla_1^2 - \frac{1}{c_4}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{9}$$

$$\left(\nabla_1^2 - \frac{1}{c_2}\frac{\partial^2}{\partial t^2}\right)w = 0, \tag{10}$$

where

$$u_i' = \frac{\rho\omega^* v_1}{\beta_1 T_0}u_i, \quad r' = \frac{\omega^*}{v_1}r, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\beta_1 T_0}, \quad \omega^* = \frac{C_e c_{11}}{K_1}, \quad \epsilon = \frac{\beta_1^2 T_0}{\rho C_e c_{11}}, \quad a' = \frac{\omega^*}{v_1}a,$$
$$c_1 = \frac{c_{33}}{c_{11}}, \quad c_2 = \frac{c_{44}}{c_{11}}, \quad c_3 = \frac{c_{14} + c_{13}}{c_{11}}, \quad c_4 = \frac{c_{11} - c_{12}}{2c_{11}}, \quad \bar{\beta} = \frac{\beta_3}{\beta_1}, \quad \bar{K} = \frac{K_3}{K_1}, \quad b' = \frac{\omega^*}{v_1}b,$$
$$t' = \omega^* t, \quad t_0' = \omega^* t_0, \quad t_1' = \omega^* t_1, \quad T' = \frac{T}{T_0}, \quad v_1^2 = \frac{c_{11}}{\rho}, \quad c' = \frac{c}{v_1}, \quad \omega' = \frac{\omega}{\omega^*}, \tag{11}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Here dashes are ignored for convenience, ω^* is the characteristic frequency of the plate, v_1 is the longitudinal wave velocity in the medium and ε is the thermoelastic coupling constant. Eqs. (9) and (10) in ψ and w give purely transverse waves, which are not affected by temperature changes.

These waves are polarized in planes along and perpendicular to the z-axis and may be referred to as the shear horizontal (SH) and shear vertical (SV) waves, respectively.

We assume a solution of the form [17,23–29]

$$\phi(r,\theta,t) = \bar{\phi}(r)e^{i(p\theta-\omega t)},$$

$$T(r,\theta,t) = \bar{T}(r)e^{i(p\theta-\omega t)},$$

$$\psi(r,\theta,t) = \bar{\psi}(r)e^{i(p\theta-\omega t)},$$

$$w(r,\theta,t) = \bar{w}(r)e^{i(p\theta-\omega t)}, \tag{12}$$

where ω is the angular frequency and p is the angular wave-number. Towfighi et al. [11] pointed out that in cylindrical geometry, the generation of surface waves in the circumferential direction with a plane wave front requires the circumferential wave speed to be a function of the radial distance. We also adopt the same formulation here and hence assume that the phase velocity is not constant but changes with radius. The phase velocity at a point having radius r is given by

$$v_{ph}(r) = c_b r/b, \tag{13a}$$

where c_b is the phase velocity at the outer surface having a radius b. In the case of flat plate, the wave number ξ is defined as ω/v_{ph}, because curvature does not change, although for a curved plate the same definition would be r dependent. Therefore, the angular wavenumber p, which is independent of r, is defined as [11]

$$p = \omega/(v_{ph}(r)/r) = \omega b/c_b. \tag{13b}$$

Substitution of Eqs. (12) into Eqs. (8)–(10) gives us

$$(\nabla_2^2 + \omega^2)\bar{\phi} = i\omega\tau_1\bar{T}, \tag{14}$$

$$(\nabla_2^2 + \omega^2\tau_0)\bar{T} = \varepsilon\omega^2\tau_0'\nabla_2^2\bar{\phi}, \tag{15}$$

$$\left(\nabla_2^2 + \frac{\omega^2}{c_4}\right)\bar{\psi} = 0, \tag{16}$$

$$\left(\nabla_2^2 + \frac{\omega^2}{c_2}\right)\bar{w} = 0, \tag{17}$$

where

$$\nabla_2^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{p^2}{r^2},$$

$$\tau_1 = t_1\delta_{2k} + i\omega^{-1}, \quad \tau_0' = t_0\delta_{1k} + i\omega^{-1}, \quad \tau_0 = t_0 + i\omega^{-1}. \tag{18}$$

The solution of Eqs. (14)–(17) is written as

$$\bar{\phi} = \sum_{k=1}^{2}[A_k J_p(\omega a_k r) + B_k Y_p(\omega a_k r)],$$

$$\bar{T} = \mathrm{i}\omega\tau_1^{-1}\sum_{k=1}^{2}(a_k^2 - 1)[A_k J_p(\omega a_k r) + B_k Y_p(\omega a_k r)],$$

$$\bar{w} = A_3 J_p\left(\frac{\omega}{\sqrt{c_2}}\,r\right) + B_3 Y_p\left(\frac{\omega}{\sqrt{c_2}}\,r\right),$$

$$\bar{\psi} = A_4 J_p\left(\frac{\omega}{\sqrt{c_4}}\,r\right) + B_4 Y_p\left(\frac{\omega}{\sqrt{c_4}}\,r\right), \tag{19}$$

where $A_k, B_k,\ k = 1, 2,\ A_3, B_3, A_4$ and B_4 are arbitrary constants and

$$a_k^2 = \tfrac{1}{2}[(1 + \tau_0 - \mathrm{i}\omega\varepsilon\tau_0'\tau_1) \pm \{(1 - \tau_0 - \mathrm{i}\omega\varepsilon\tau_0'\tau_1)^2 - 4\mathrm{i}\omega\varepsilon\tau_0\tau_1\tau_0'\}^{1/2}], \quad k = 1, 2.$$

Here J_p and Y_p are, respectively, the Bessel functions of first and second kind and of order p.

5. Secular equation

The displacements, temperature and stresses are obtained as

$$u_r = \left(-\bar{\phi}' + \frac{\mathrm{i}p\bar{\psi}}{r}\right)\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{20}$$

$$u_\theta = \left(-\bar{\psi}' - \frac{\mathrm{i}p\bar{\phi}}{r}\right)\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{21}$$

$$u_z = \bar{w}\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{22}$$

$$T = \bar{T}\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{23}$$

$$\sigma_{rr} = 2c_4\left[\frac{\bar{\phi}'}{r} + \left(\frac{\omega^2}{2c_4} - \frac{p^2}{r^2}\right)\phi + \frac{\mathrm{i}p}{r}\left(\bar{\psi}' - \frac{1}{r}\,\bar{\psi}\right)\right]\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{24}$$

$$\sigma_{rz} = c_2\bar{w}'\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{25}$$

$$\sigma_{r\theta} = 2c_4\left[\frac{\bar{\psi}'}{r} + \left(\frac{\omega^2}{2c_4} - \frac{p^2}{r^2}\right)\bar{\psi} - \frac{\mathrm{i}p}{r}\left(\bar{\phi}' - \frac{\bar{\phi}}{r}\right)\right]\mathrm{e}^{\mathrm{i}(p\theta-\omega t)}, \tag{26}$$

where prime denotes differentiation with respect to radial coordinate r.

5.1. Stress-free plate

Invoking the stress-free and thermal boundary conditions (6a) and (6c) at the lower and upper surfaces $r = a, b$ of the plate and using Eqs. (23)–(26), we obtain the following secular equations:

$$J'_p\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)Y'_p\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right) - J'_p\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right)Y'_p\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right) = 0, \tag{27}$$

$$\left|E_{lj}\right| = 0, \quad l,j = 1,2,3,4,5,6, \tag{28}$$

where

$$\begin{aligned}
E_{11} &= a_1 J'_p(\omega a_1\eta_1) + \omega\eta_1\left(\frac{1}{2c_4} - \frac{p^2}{\omega^2\eta_1^2}\right)J_p(\omega a_1\eta_1),\\
E_{13} &= a_2 J'_p(\omega a_2\eta_1) + \omega\eta_1\left(\frac{1}{2c_4} - \frac{p^2}{\omega^2\eta_1^2}\right)J_p(\omega a_2\eta_1),\\
E_{15} &= \mathrm{i}p\left[\frac{1}{\sqrt{c_4}}J'_p\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right) - \frac{1}{\omega\eta_1}J_p\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right)\right],\\
E_{21} &= -\mathrm{i}a_1 p\left[J'_p(\omega a_1\eta_1) - \frac{1}{\omega a_1\eta_1}J_p(\omega a_1\eta_1)\right],\\
E_{23} &= -\mathrm{i}a_2 p\left[J'_p(\omega a_2\eta_1) - \frac{1}{\omega a_2\eta_1}J_p(\omega a_2\eta_1)\right],\\
E_{25} &= \frac{1}{\sqrt{c_4}}J'_p\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right) + \omega\eta_1\left(\frac{1}{2c_4} - \frac{p^2}{\omega^2\eta_1^2}\right)J_p\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right),\\
E_{31} &= \frac{\mathrm{i}\omega}{\tau_1}(a_1^2 - 1)[\omega a_1 J'_p(\omega a_1\eta_1) + hJ_p(\omega a_1\eta_1)],\\
E_{33} &= \frac{\mathrm{i}\omega}{\tau_1}(a_2^2 - 1)[\omega a_2 J'_p(\omega a_2\eta_1) + hJ_p(\omega a_2\eta_1)],\\
E_{35} &= 0.
\end{aligned} \tag{29}$$

Here $h \to 0$ corresponds to thermally insulated boundaries and $h \to \infty$ refers to that of the isothermal one. The elements E_{lj} $(j = 2, 4, 6)$ of the determinantal equation (28) can be obtained by just replacing the Bessel functions of the first kind in E_{lj} $(j = 1, 3, 5)$ with those of the second kind, while E_{lj}'s $(l = 4, 5, 6)$ are obtained by replacing η_1 in E_{lj} $(l = 1, 2, 3)$ with η_2, where $\eta_1 = a/R = 1 - \eta^*/2$, $\eta_2 = b/R = 1 + \eta^*/2$, $\eta^* = (b - a)/R$ is the thickness-to-mean radius ratio of the plate. Eq. (27) governs the motion corresponding to the case of longitudinal shear where only the u_z displacement occurs and is given by Eq. (25). These modes of vibrations are not affected by temperature change. Eq. (27) agrees with Graff [30] (cf. Eq. 8.26.2) for the case of waves in cylindrical shell in elastokinetics. For $p = 0$, i.e. for the motion independent of θ, Eq. (27) becomes

$$J_1\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)Y_1\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right) - J_1\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right)Y_1\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right) = 0. \tag{30}$$

The amplitude ratio in this case is given by

$$\frac{A_3}{B_3} = -Y_1\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)\bigg/J_1\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right). \tag{31}$$

Eq. (30) corresponds to axially symmetric shear modes and as in the case of solid rod the lowest torsional mode is non-dispersive. These modes are not affected by thermal variations. Eq. (28) corresponds to the case of plane-strain motion. For $p = 0$, i.e. for the motion independent of θ, it is found that under these conditions of axially symmetric vibrations the extensional and shear modes get decoupled from each other. The secular equation (28) for $p = 0$ provides us

$$E'_{25}E'_{56} - E'_{55}E'_{26} = 0, \tag{32}$$

$$\left|E'_{lj}\right| = 0, \quad l = 1,3,4,6, \quad j = 1,2,3,4, \tag{33}$$

where E'_{lj} can be written from E_{lj}, $l,j = 1,\ldots,6$ by setting $p = 0$.

5.2. Rigidly fixed plate

Invoking the rigidly fixed and thermal boundary conditions (6b) and (6c) at the lower and upper surfaces $r = a, b$ of the plate and using Eqs. (20)–(23), we obtain the secular equations as

$$J_p\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)Y_p\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right) - J_p\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right)Y_p\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right) = 0, \tag{34}$$

$$\left|F_{lj}\right| = 0, l, \quad j = 1,2,3,4,5,6, \tag{35}$$

where

$$\begin{aligned}
F_{11} &= \omega a_1\eta_1 J'_p(\omega a_1\eta_1),\\
F_{13} &= \omega a_2\eta_1 J'_p(\omega a_2\eta_1),\\
F_{15} &= -\mathrm{i}pJ_p(\omega\eta_1/\sqrt{c_4}),\\
F_{21} &= \mathrm{i}pJ_p(\omega a_1\eta_1),\\
F_{23} &= \mathrm{i}pJ_p(\omega a_2\eta_1),\\
F_{25} &= \omega\eta_1 J'_p(\omega\eta_1/\sqrt{c_4})/\sqrt{c_4},\\
F_{31} &= \frac{\mathrm{i}\omega}{\tau_1}(a_1^2-1)[\omega a_1 J'_p(\omega a_1\eta_1) + hJ_p(\omega a_1\eta_1)],\\
F_{33} &= \frac{\mathrm{i}\omega}{\tau_1}(a_2^2-1)[\omega a_2 J'_p(\omega a_2\eta_1) + hJ_p(\omega a_2\eta_1)],\\
F_{35} &= 0.
\end{aligned} \tag{36}$$

Here $h \to 0$ corresponds to thermally insulated boundaries of the plate and $h \to \infty$ to that of the isothermal one. The elements F_{lj} ($j = 2,4,6$) and F_{lj} ($l = 4,5,6$) can be rewritten by following the same analogy as applied to E_{lj}. Eq. (34) again governs the motion corresponding to the case of longitudinal shear in the rigidly fixed plate where only u_z displacement occurs and is given by Eq. (22). Also, Eq. (34) is the equivalent form of Eq. (27) in this case for the waves propagating in a cylindrical shell in elastokinetics. These modes of vibrations are not affected by thermal variations. For $p = 0$, Eq. (34) becomes

$$J_0\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)Y_0\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right) - J_0\left(\frac{\omega\eta_2}{\sqrt{c_2}}\right)Y_0\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right) = 0. \tag{37}$$

The amplitude ratio in this case is given by

$$\frac{A_3}{B_3} = -Y_0\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right)\Big/J_0\left(\frac{\omega\eta_1}{\sqrt{c_2}}\right). \tag{38}$$

Eq. (35) corresponds to the case of plane-strain motion. For $p = 0$, i.e. for the motion independent of θ, it is found that under these conditions of axially symmetric vibrations, the extensional and shear modes uncouple from each other. The secular equation for axially symmetric shear modes ($p = 0$) is given by

$$F'_{25}F'_{56} - F'_{55}F'_{26} = 0.$$

This implies that

$$J_1\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right)Y_1\left(\frac{\omega\eta_2}{\sqrt{c_4}}\right) - J_1\left(\frac{\omega\eta_2}{\sqrt{c_4}}\right)Y_1\left(\frac{\omega\eta_1}{\sqrt{c_4}}\right) = 0. \tag{39}$$

The secular equation for extensional modes becomes

$$\left|F'_{lj}\right| = 0, \quad l = 1,3,4,6, \quad j = 1,2,3,4, \tag{40}$$

where F'_{lj} can be obtained from F_{lj} by setting $p = 0$. The decoupled shear modes do not depend on thermal variations and are not affected due to change in temperature. For the non-axially symmetric ($p \neq 0$) case of plane-strain vibrations, these modes remain coupled and are affected by thermal variations. The vibrational modes are dispersive and dissipative in character. The above analysis reduces to homogeneous isotropic, cylindrical curved thermoelastic plate if we set

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{12} = c_{13} = \lambda, \quad c_{44} = \mu = c_{66}, \quad \beta_1 = \beta = \beta_3, \quad K_1 = K = K_3. \tag{41}$$

The analysis in case of coupled thermoelasticity (CT) can be obtained by setting $t_0 = 0 = t_1$ and for uncoupled thermoelasticity (UCT) by taking $\varepsilon = 0, t_0 = 0 = t_1$ in the present study. The secular equations and all other relevant results in case of LS and GL theories of dynamic generalized thermoelasticity can be obtained from the above analysis by taking $k = 1$ and $k = 2$, respectively, in expressions (18) for τ_0, τ_1 and τ'_0 and then using the resulting values of these parameters in different relations at various stages. Upon using parameters defined in Eq. (41) in the above analysis, the secular Eq. (28) in the absence of temperature effect viz. uncoupled thermoelasticity ($\varepsilon = 0$) reduces to the corresponding equation obtained by Liu and Qu [6] in case of guided circumferential waves in a circular annulus (cf. Eq. (17)) in dimensionless form here.

6. Numerical results and discussion

In order to illustrate theoretical results obtained in the preceding sections, we now present some numerical results. The material chosen for this purpose is single crystal of zinc, the physical data for which is given below [26]:

$$\rho = 7.14 \times 10^3\,\text{Kgm}^{-3}, \quad c_{11} = 1.628 \times 10^{11}\,\text{Nm}^{-2}, \quad c_{12} = 0.362 \times 10^{11}\,\text{Nm}^{-2},$$

$$c_{13} = 0.508 \times 10^{11}\,\text{Nm}^{-2}, \quad c_{33} = 0.627 \times 10^{11}\,\text{Nm}^{-2}, \quad c_{44} = 0.385 \times 10^{11}\,\text{Nm}^{-2},$$

$$\beta_1 = 5.75 \times 10^6\,\text{Nm}^{-2}\,\text{deg}^{-1}, \quad \beta_3 = 5.07 \times 10^6\,\text{Nm}^{-2}\,\text{deg}^{-1}, \quad C_e = 3.9 \times 10^2\,\text{JKgm}^{-1}\,\text{deg},$$

$$\omega^* = 5.01 \times 10^{11}\,\text{s}^{-1}, \quad T_0 = 296\,\text{K}, \quad \varepsilon = 0.0221.$$

The phase velocity of various modes of wave propagation has been computed from secular Eqs. (28) and (35) for various values of wavenumber and for different boundary conditions. The numerical computations have been performed by taking $a = 0.1$ and $b = 1.0$. The dispersion curves corresponding to secular equations (28) and (35) for different modes are presented in Figs. 2a, 2b, 3a and 3b in the context of various theories of thermoelasticity viz. CT, LS and GL. Fig. 4 containing the dispersion curves in the absence of temperature field viz. in case of UCT is also added for comparison purpose here. From Figs. 2 and 3, it is observed that the phase velocities of different modes of wave propagation start from large values at vanishing wave number and then exhibit strong dispersion until the velocity flattens out to the value of the thermoelastic Rayleigh wave velocity of the material at higher wave numbers. The reason for this asymptotic approach is that for short wavelengths (or high frequencies) the material plate behaves increasingly like a thick slab and hence the coupling between upper and lower boundary surfaces is reduced and as a result the properties of symmetric and skew symmetric waves become more and more similar. In the limit for an infinitely thick slab, the motion at the upper surface is not confined to the lower surface, and the displacements become localized near the free boundaries, thus the Lamb wave dispersion curves asymptotically approach those of Rayleigh waves. Another investigation of the figures shows that the curves approach the Bleustein–Gulyaev (B–G) wave velocity or the shear

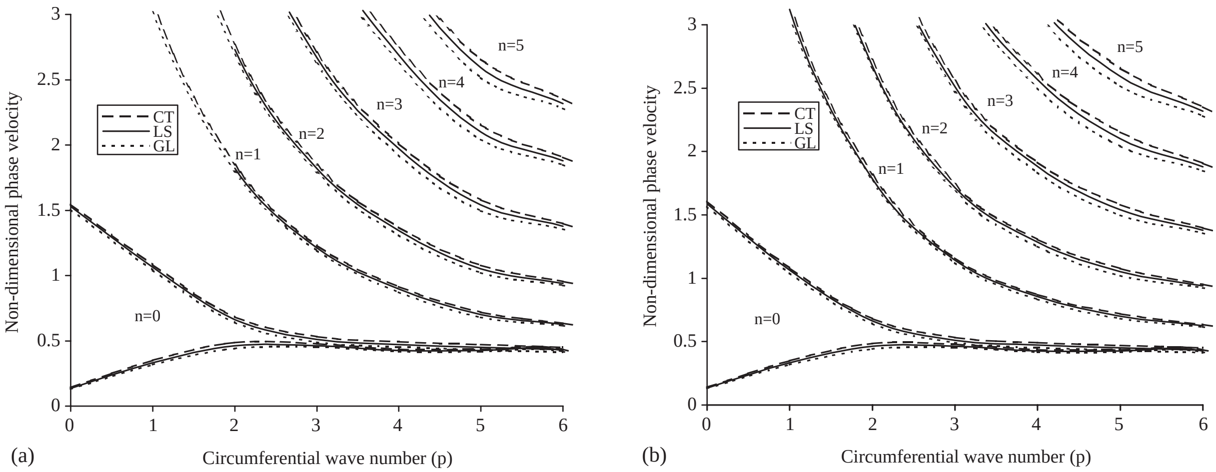

Fig. 2. (a) Phase velocity profile of wave modes in a stress-free, isothermal plate with circumferential wavenumber. (b) Phase velocity profile of wave modes in a stress-free, thermally insulated plate with circumferential wavenumber.

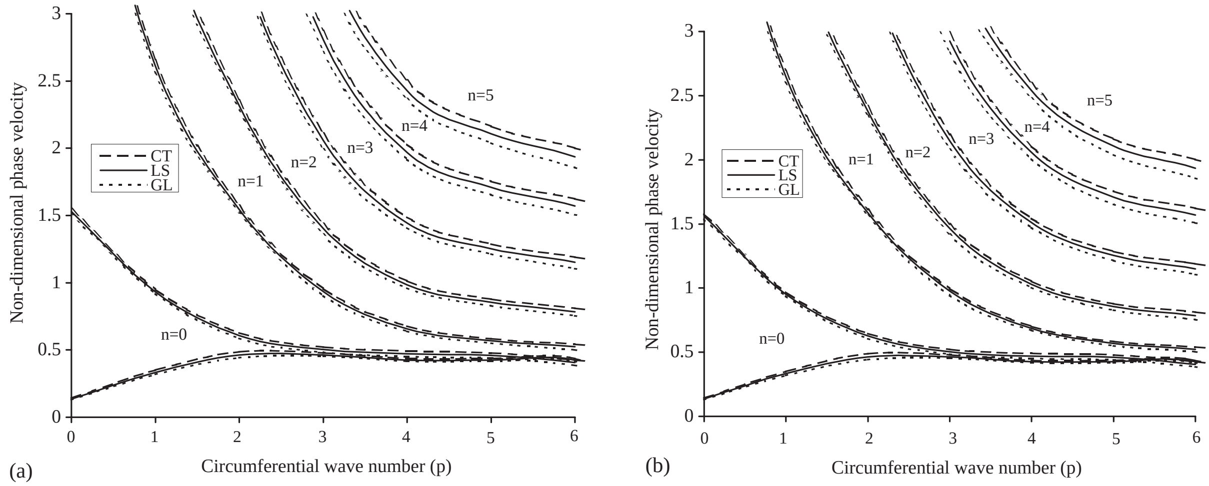

Fig. 3. (a) Phase velocity profile of wave modes in a rigidly fixed, isothermal plate with circumferential wavenumber. (b) Phase velocity profile of wave modes in a rigidly fixed, thermally insulated plate with circumferential wavenumber.

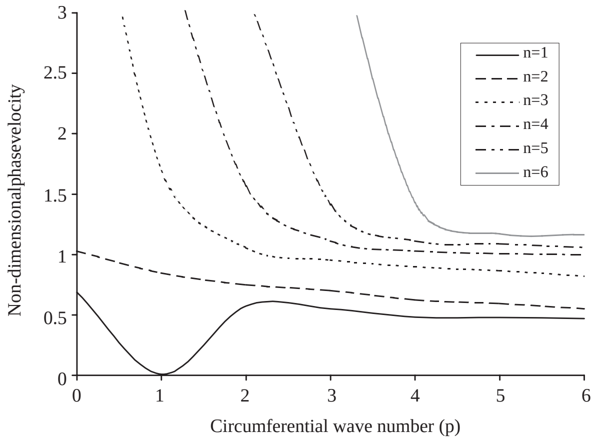

Fig. 4. Phase velocity profile of wave modes in a stress-free plate with circumferential wave number in uncoupled thermoelasticity.

horizontal (SH) wave velocity at higher wave number. The velocity of the first mode tends towards the B–G wave velocity, whereas the velocities of other modes tend towards the SH wave velocity. This is due to the fact that for the first mode the surface wave for the plate will become dominant when the wavelength is small as compared to the thickness of the plate. However, the B–G wave velocity and the SH wave velocity are almost the same in the current study. It also shows that the higher modes can only exist beyond certain values of the wavenumber, for example, the second mode begins around $p = 1.0$ in Fig. 2 for a plate with stress-free boundaries and in case of plate having rigidly fixed boundaries, the second mode appears around $p = 0.7$, as can be seen from Fig. 3. The dispersion curves for the first six modes in case of UCT have also been obtained and plotted in Fig. 4 for single crystal of zinc. The dispersion curves in Fig. 4 look similar to those of Qu et al. [6] for $\eta = 0.1$, except the modification due to anisotropic effects. It is once again seen that at higher wavenumbers the first mode is almost non-dispersive for the considered thickness of the plate and asymptotically approaches the Rayleigh wave velocity. Except the first three modes, all higher modes have phase velocity greater than the shear wave speed in the considered wavenumber range. Because the phase velocity c_b is greater than the shear wave speed for very thick plate, it is seen that the phase velocity of second mode here is also greater than the Rayleigh wave velocity and approaches the Rayleigh wave velocity as $p \to \infty$. This agrees with the corresponding fact noted by the earlier authors [1,6] in case of isotropic materials. The trends of variations of various dispersion curves are noted to be similar to those obtained by Towfighi et al. [11], except the modifications due to thermomechanical coupling and thermal relaxation times in addition to the application of the exact analytical technique. It is also observed that in the context of various theories of thermoelasticity (CT, LS and GL), the phase velocity in CT theory has higher value than in other theories. The value of phase velocity in LS and GL theories follows the one in CT theory of thermoelasticity.

7. Conclusions

In this paper, the Bessel functions with complex arguments have been directly used to study thermoelastic wave propagation problems in anisotropic cylindrical plates along the circumferential direction. Three displacement potential functions are introduced in order to simplify the equations of motion and heat conduction equation. It is noticed that the longitudinal shear motion and axially symmetric shear vibration modes get decoupled from the rest of the motion, which are not affected by thermal variations. For the non-axially symmetric case of plane strain vibrations, these modes remain coupled and are affected by temperature change and thermal relaxation times. However, in case of symmetric vibrations, extensional and shear modes of wave propagation get decoupled from each other. The latter modes are not affected due to the thermal variations and thermal relaxation times and vice versa. This study is important for ultrasonic non-destructive inspection of large diameter pipes, which helps in the health monitoring of ailing infrastructure. Longitudinal stress-corrosion cracks are detected more efficiently by inducing circumferential waves; hence, the study of generalized thermoelastic wave propagation in the circumferential direction in a pipe wall is essential. It can also be used to check the applicability of various kinds of two-dimensional simplified shell theory in elastokinetics and numerical methods such as FEM and BEM.

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