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Short Communication

Determination of the steady-state response of viscoelastically supported cantilever beam under sinusoidal base excitation

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1. Introduction

Steady-state response with respect to base movement of a viscoelastically supported cantilever beam is analyzed. The principle of relative motion is used to investigate the steady-state response of a viscoelastically supported cantilever beam. The Lagrange equations are used to examine the free vibration characteristics of an elastically supported cantilever beam and the steady-state response to a sinusoidally varying base excitation of a viscoelastically supported Bernoulli–Euler cantilever beam. The constraint condition against rotation of the supported end is taken into account by using Lagrange multipliers. In the study, for applying the Lagrange equations, the trial function denoting the deflection of the beam is expressed in polynomial form. By using the Lagrange equations, the problem is reduced to the solution of a system of algebraic equations. The influence of the damping and stiffness parameters on the steady-state response of the viscoelastically supported cantilever beam is investigated numerically for sinusoidally varying base movement for various damping and stiffness parameters. The results are given for the first two natural frequency ranges. Convergence studies are made. The validity of the obtained results is demonstrated by comparing them with exact solutions based on the Bernoulli–Euler beam theory obtained for the special cases of the investigated problem.

This problem is of considerable interest to the engineers designing structural and mechanical systems such as chimneys of plants, communication towers, manipulator arms and many others. As it is known, vibration damping is very important in engineering practice. In some cases,

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the damping treatments at the boundary supports can be an alternative solution to surface damping treatments with viscoelastic materials for beams and plates: For example, Fan et al. [1] proposed a method of analysis for the forced vibration of a beam with viscoelastic boundary supports based on complex normal mode analysis. Jacquot [2] developed a method to predict the stationary random response of a beam which has been modified by the attachment of a damped, lumped assembly of linear mechanical elements. Jacquot [3] investigated the forced random vibration of a cantilever beam with either a viscous damper or a damped dynamic vibration absorber attached at the tip to provide energy dissipation to suppress the randomly excited motions. Jacquot [4] studied optimal damper location for randomly forced cantilever beams. The steady-state response to a sinusoidally varying force was determined for a viscoelastically point-supported square or rectangular plate by Yamada et al. [5] by using the generalized Galerkin method. A generalization of this study to orthotropic rectangular plates was investigated by Kocatürk [6]. The steady-state response of a viscoelastically point-supported specially orthotropic square or rectangular plate was determined by Kocatürk and Altıntaş [7] by using an energy-based finite difference method. Vibration of orthotropic rectangular plates having viscoelastic point supports at the corners under the effect of sinusoidally varying concentrated moment is analyzed by Kocatürk et al. [8] by using the Lagrange equations.

In the present study, the problem is analyzed by using the Lagrange equations with the trial function in the polynomial form denoting the deflection of the plate for determining the peak values of the displacements of the tip of the viscoelastically supported cantilever beam excited by the base movement. The constraint condition against rotation of the supported end is taken into account by using Lagrange multipliers.

The problem considered is solved within the framework of the Bernoulli–Euler beam theory. The convergence study is based on the numerical values obtained for various numbers of polynomial terms. In the numerical examples, the steady-state responses to a sinusoidally varying force are determined for the first two peaks of the tip displacements and support reactions. The accuracy of the results is partially established by comparison with previously published accurate results for the special cases of the considered problem.

2. Analysis

Consider a viscoelastically supported elastic cantilever beam of length L and cross-section area A under base excitation effect at the end constrained against rotation as shown in Fig. 1, where k is the spring constant, c is the damping coefficient. The coordinate axis OX_1 oriented along the axis of the beam with the origin at O. Because the horizontal part of the beam is shown only to describe that the beam is constrained against rotation at the lower end, the mass of this part is not taken into account. There are no friction forces between the base and the beam system. The constraint condition against rotation of the supported end is taken into account by using Lagrange multipliers. Under the above-mentioned conditions, the steady-state responses of the viscoelastically supported cantilever beam to a sinusoidally varying base movement for various damping and stiffness values will be determined by

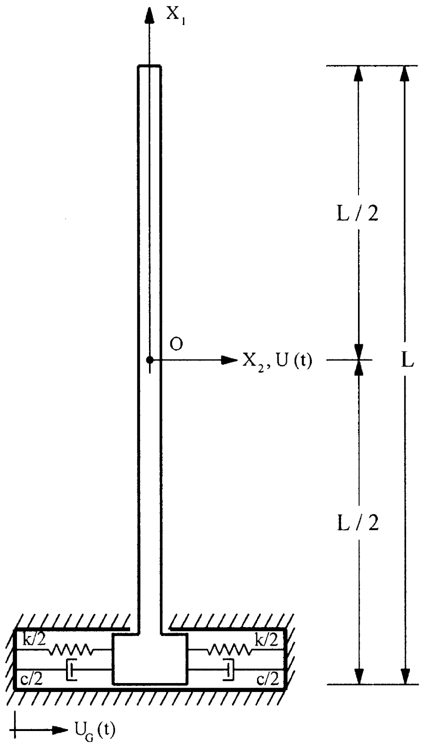

Fig. 1. Viscoelastically supported elastic cantilever beam under base excitation effect at the end constrained against rotation.

using the Lagrange equations. The relative displacement is defined as follows:

$$W(X_1, t) = U_T(X_1, t) - U_G(t), \tag{1}$$

where $U_T(X_1, t)$ is total displacement, $U_G(t)$ is base displacement. For a beam undergoing sinusoidally varying base movement, it can be defined that $U_G(t) = \bar{U}_G e^{i\omega t}$, where ω is radian frequency, $\bar{U}_G$ is the amplitude of the base displacement. The strain energy of bending in Cartesian coordinates is given by

$$U = \frac{EI}{2} \int_{-L/2}^{L/2} \left(\frac{\partial^2 W}{\partial X_1^2} \right)^2 dX_1, \tag{2}$$

where E is Young's modulus, I is the second moment of area of the cross-section of the beam, W is the complex amplitude of the steady-state response relative to the excitation movement $U_G(t)$. With rotary inertia neglected, the kinetic energy of the vibrating plate is

$$T = \frac{1}{2} \int_{-L/2}^{L/2} m \left(\frac{\partial U_T}{\partial t} \right)^2 dX_1, \tag{3}$$

where m is the mass of the beam per unit length. The additive strain energy and dissipation function of viscoelastic support are

$$F_s = \tfrac{1}{2}k_s(U_{TS} - U_G)^2, \quad D = \tfrac{1}{2}c_s(\dot{U}_{TS} - \dot{U}_G)^2, \tag{4}$$

where U_{TS}, $\dot{U}_{TS}$ are total displacement and total velocity of the supported end of the beam. The functional of the problem is

$$J = T - (U + F_S). \tag{5}$$

Introducing the following nondimensional parameters:

$$x_1 = \frac{X_1}{L}, \quad \bar{w}(x_1, t) = W/L, \quad \bar{u}_T(x_1, t) = U_T/L, \quad \bar{u}_G(t) = U_G/L, \tag{6}$$

the above energy expressions can be written at any time t as

$$U = \frac{EI}{2L}\int_{-1/2}^{1/2}\left(\frac{\partial^2 \bar{w}}{\partial x_1^2}\right)^2 \mathrm{d}x_1, \tag{7a}$$

$$U = \frac{mL^3}{2}\int_{-1/2}^{1/2}\left(\frac{\partial \bar{u}_T}{\partial t}\right)^2 \mathrm{d}x_1, \tag{7b}$$

$$F_s = \frac{k_s L^2}{2}(\bar{u}_{TS} - \bar{u}_G)^2, \quad D = \frac{c_s L^2}{2}\left(\dot{\bar{u}}_{TS} - \dot{\bar{u}}_G\right)^2. \tag{7c,d}$$

It is known that some expressions satisfying the geometrical boundary conditions are chosen for $\bar{w}(x_1, t)$ and by using the Lagrange equations, the natural boundary conditions are also satisfied. For applying the Lagrange equations, the trial function $\bar{w}(x_1, t)$ is approximated by space-dependent polynomial terms $x_1^0, x_1^1, x_1^2, \ldots, x_1^N$ and time-dependent generalized displacement coordinates $\bar{a}_n(t)$. Thus,

$$\bar{w}(x_1, t) = \sum_{n=0}^{N} \bar{a}_n(t) x_1^n, \tag{8}$$

where $\bar{w}(x_1, t)$ is the relative steady-state response (the relative transverse deflection) of the beam to a sinusoidally varying base motion

$$U_G(t) = \bar{u}_G L = u_G \mathrm{e}^{\mathrm{i}\omega t} L. \tag{9}$$

Each term x_1^n must satisfy the geometrical boundary conditions. The constraint condition of the support is satisfied by using the Lagrange multipliers. Therefore, it is not necessary at first for these functions to satisfy the geometrical boundary conditions. As it is known, there is no need for these functions to satisfy the natural boundary conditions. However, if the natural boundary conditions were also satisfied when selecting the functions, then the rate of convergence would be high. The only constraint condition for the considered problem is

$$\beta W'(X_{1S}) = 0, \tag{10}$$

where X_{1S} denotes the location of the support, prime denotes the derivative with respect to X_1. In Eq. (10), β is the Lagrange multiplier and in the considered problem it is support moment reacting against the rotation of the supported end of the beam. The Lagrange multiplier formulation of the considered problem requires us to construct the Lagrangian functional

$$J^* = J + \beta W'(X_S) \tag{11}$$

which attains its stationary value at the solution (W, β). Then, after introducing

$$\bar{a}_{N+1} = \beta \tag{12}$$

application of the Lagrange equations,

$$\frac{\partial J^*}{\partial \bar{a}_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial J^*}{\partial \dot{\bar{a}}_k} + Q_D = 0, \quad k = 0, 1, 2, 3, \ldots, N+1, \tag{13}$$

where the overdot stands for the partial derivative with respect to time, Q_D is the generalized damping force expressed as

$$Q_D = -\frac{\partial D}{\partial \dot{\bar{a}}_k}, \quad k = 0, 1, 2, 3, \ldots, N+1, \tag{14}$$

yields a set of linear algebraic equations. Introducing the following nondimensional parameters:

$$\kappa = \frac{kL^3}{EI}, \quad \gamma = \frac{cL}{EI}, \quad \lambda^2 = \frac{m\omega^2 L^4}{EI} \tag{15}$$

and considering that when the base motion is expressed as in Eq. (9), then the time-dependent generalized functions can be expressed as follows:

$$\bar{a}_n(t) = a_n \mathrm{e}^{\mathrm{i}\omega t}. \tag{16}$$

In Eq. (16), a_n is a complex variable containing a phase angle. The dimensionless complex amplitude of the displacement of a point of the beam can be expressed as

$$w(x_1) = \sum_{n=0}^{N} a_n x_1^n. \tag{17}$$

Using Eq. (13) by taking into account Eq. (1), a set of linear algebraic equations is obtained which can be expressed in the following matrix form:

$$[A]\{a_n\} + \mathrm{i}\lambda\gamma[B]\{a_n\} - \lambda^2[C]\{a_n\} = \{q\}, \quad n = 0, 1, 2, 3, \ldots, N+1, \tag{18}$$

where $[A]$, $[B]$ and $[C]$ are coefficient matrices obtained by using Eq. (13), $\{a\}$ is the vector representation of the elements of a_n, and the elements of the generalized force $\{q\}$ are expressed as

$$q_k = u_G \lambda^2 \int_{-0.5}^{0.5} x_1^k \,\mathrm{d}x_1, \quad k = 0, 1, 2, 3, \ldots, N+1. \tag{19}$$

For free vibration analysis, when the base motion and damping of the supports are zero in Eq. (19), this situation results in a set of linear homogeneous equations that can be expressed in the following matrix form:

$$[A]\{a_n\} - \lambda^2[C]\{a_n\} = \{0\}. \tag{20}$$

The maximum displacement of the tip of the beam is

$$w(0.5) = \sum_{n=0}^{N} a_n 0.5^n. \tag{21}$$

The dimensionless horizontal reaction force at the base is

$$R = (\kappa + \mathrm{i}\gamma\lambda)w(-0.5). \tag{22}$$

The number of unknown coefficients is $(N + 2)$. Again, the number of equations which can be written by using Eq. (13) is $(N + 2)$, which is given in matrix form by Eq. (18). Therefore, the total number of these equations is equivalent to the total number of unknown displacements and these unknowns can be determined by using the above-mentioned equations.

3. Numerical results

The steady-state response of a cantilever beam to a base motion $U_G(t) = u_G \mathrm{e}^{\mathrm{i}\omega t} L$, viscoelastically supported at the base, is calculated numerically. In the numerical calculations, dimensionless amplitude of the base movement is taken as $u_G = 0.1$.

As far as the author knows, there are no existing results on viscoelastically supported cantilever beam under the effect of sinusoidally varying base motion. Therefore a short investigation of the free vibration of a rigidly supported ($\kappa = \infty$) cantilever beam is made for comparing the obtained results with the existing exact results of free vibration characteristics of the cantilever beam. Also, the natural frequencies of an elastically supported beam are determined for various values of stiffness parameter. The natural frequencies of elastically supported cantilever beam are determined by calculating the eigenvalues λ of the frequency Eq. (20). It is possible to simulate infinite lateral support stiffness by setting the translational stiffness coefficient equal to 1×10^8 at the support for comparing the obtained results with the existing results of a rigidly supported cantilever beam. In Table 1, the calculated frequency parameters λ are compared with those of Timoshenko and Young [9] and the convergence is tested by taking the number of terms $(N + 1) = 3, 6, 9, 12$. It is seen that the present converged values show excellent agreement with those of Timoshenko and Young [9].

It is observed from Table 1 that, the frequency parameter decreases as the number of the polynomial terms increases. It means that the convergence is from above. Convergence study indicates that the calculated values are converged to within five significant figures.

From here on, in the calculation of the results of the present study, 12 terms of the polynomial series are used, namely the size of the determinant is 13×13.

When increasing the stiffness parameter κ, the frequency parameters increase monotonically and ultimately become the value of a rigidly supported cantilever beam.

For determining the mode shapes of the vibration, for the considered eigenvalue, a coefficient is taken as known in Eq. (20), then the other coefficients are determined according to this known coefficient. After that, by using Eq. (17), the mode shape of the considered vibration can be determined. The tip displacements of the cantilever beam are determined for various damping

Table 1
Convergence study of frequency parameters λ and comparison of the obtained results with the existing exact results for the special case ($\kappa=\infty$) of the problem

	Determinant size	Peak 1	Peak 2
	3×3	3.98604	11.90006
$\kappa=50$	6×6	3.25710	11.30717
	9×9	3.25710	11.30545
	12×12	3.25710	11.30545
	3×3	4.22448	15.87935
$\kappa=100$	6×6	3.38431	14.30891
	9×9	3.38430	14.30213
	12×12	3.38430	14.30213
	3×3	4.40997	30.42289
$\kappa=400$	6×6	3.48279	19.54710
	9×9	3.48278	19.49325
	12×12	3.48278	19.49325
	3×3	4.47214	—
$\kappa=\infty\approx10^8$	6×6	3.51602	22.15783
	9×9	3.51602	22.03449
	12×12	3.51602	22.03449
Timoshenko and Young [9]	(exact)	3.51602	22.03449
$\kappa=\infty$			

parameters γ for various values of κ by using Eqs. (17) and (18). The dimensionless reaction forces R are obtained for various damping parameters γ for various values of κ by using Eqs. (17), (18) and (22).

Figs. 2a–f shows the tip displacements for various values of γ, for $\kappa = 25, 50, 75, 100, 200, 400$, respectively. It is seen in Figs. 2a–f that resonant peaks appear and also antiresonant peaks or lowest values appear between adjacent peaks. In Figs. 2 and 5, the solid lines represent the response curve of a beam with undamped elastic support in which $\gamma = 0$, and the dotted lines represent the response curve of a beam with viscoelastic support in which $\gamma = 15$. The points of intersection of these two lines are fixed points, through which all the response curves pass, regardless of the damping parameters. By choosing a suitable value for the damping parameter γ, it is possible to reduce the peak values of the tip displacements and of the reaction forces to the values of the related quantities which correspond to the intersection points shown in Figs. 2a–f and 5a–f. Existence of such points is useful for an optimum design of a system by choosing appropriate damping parameter for each stiffness value. By choosing appropriate damping parameters, resonant peaks of the tip displacements and reaction forces disappear and the related peak quantities become small. Within a certain range of the frequencies, the tip displacements becomes very small, which indicates the possibility of vibration isolation.

In Table 2, the frequencies at which the peak values of the tip displacements for the considered frequency range occur are determined for various damping parameters γ for $\kappa = 25, 50, 75, 100, 200, 400$ by using Eqs. (18) and (17). The dash sign (-) in Table 2 shows that there is no resonant peak for the considered parameters.

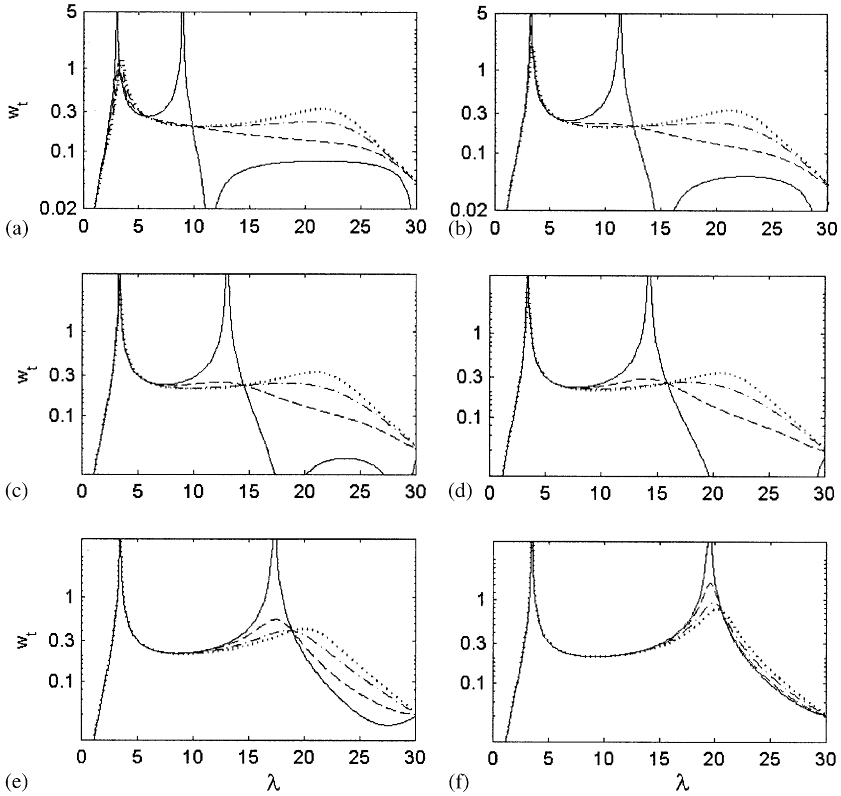

Fig. 2. The tip displacements with the variation of λ for various values of γ, for (a) $\kappa = 25$; (b) $\kappa = 50$; (c) $\kappa = 75$; (d) $\kappa = 100$; (e) $\kappa = 200$; (f) $\kappa = 400$. $\gamma = 0$ ——, $\gamma = 5$ – – – , $\gamma = 10$ –·–·– , $\gamma = 15$ · · · · · · · .

In Table 3, the minimum peak values, the corresponding frequencies and damping values are given for $\kappa = 25, 50, 75, 100, 200, 400$ for two peaks of the tip displacements. In Table 4, the minimum peak values, the corresponding frequencies and damping values are given for $\kappa = 25, 50, 75, 100, 200, 400$ for two peaks of the support reactions. It is seen from Tables 3 and 4 that, the frequencies for peak values of tip displacements and reaction forces are a little different from each other. For very low and high values of γ, the frequencies for peak values of center tip displacements and reaction forces become equal to each other. It is interesting to note that the minimum peak values of reactions of peaks 1 and 2 are equal to each other and increase linearly with the increase of κ as can be seen from Table 4.

When γ and κ are both zero, then, it is obvious that the tip displacement is zero because of zero friction between the base and beam system. In the case of great κ values, the viscoelastically supported cantilever beam behaves like rigidly supported cantilever beam. It is seen from Figs. 2a–f that with the increase in the stiffness parameter κ, the optimum damping parameter γ increases also and therefore the effect of the damping parameter γ on the response curves decreases. This situation is more pronounced for the lower frequencies for tip displacements as it can be deduced from Figs. 2a–f. Although it is not shown explicitly in the figures, a similar

Table 2
The frequencies at which the peak values of the force transmissibilities occur for various values of dimensionless damping and stiffness values

Peaks	$\gamma = 0$	$\gamma = 5$	$\gamma = 10$	$\gamma = 15$	$\gamma = 20$	$\gamma = 25$	$\gamma = 30$
$\kappa = 25$							
First peak	3.022	3.157	3.353	3.438	3.472	3.488	3.496
Second peak	8.934	—	20.585	21.526	21.768	21.868	21.920
$\kappa = 50$							
First peak	3.257	3.282	3.338	3.392	3.431	3.456	3.472
Second peak	11.305	—	19.461	21.226	21.618	21.777	21.859
$\kappa = 75$							
First peak	3.341	3.349	3.371	3.397	3.422	3.443	3.458
Second peak	32.972	12.341	18.403	20.922	21.468	21.687	21.798
$\kappa = 100$							
First peak	3.384	3.388	3.398	3.412	3.427	3.441	3.454
Second peak	14.302	14.058	17.899.	20.637	21.321	21.597	21.737
$\kappa = 200$							
First peak	3.450	3.450	3.452	3.3454	3.457	3.460	3.464
Second peak	17.292	17.452	18.511	19.992	20.850	21.278	21.511
$\kappa = 400$							
First peak	3.483	3.483	3.483	3.483	3.484	3.484	3.485
Second peak	19.493	19.578	19.837	20.229	20.639	20.978	21.230

Table 3
The minimum peak values, corresponding frequencies and damping values for $\kappa = 25, 50, 75, 100, 200, 400$ for two peaks of the tip displacements

		$\kappa = 25$	$\kappa = 50$	$\kappa = 75$	$\kappa = 100$	$\kappa = 200$	$\kappa = 400$
Peak 1	λ	3.281	3.391	3.431	3.451	3.483	3.499
	γ	7.787	14.850	21.940	28.980	60.585	102.000
	w_t	1.0526	2.0749	3.1051	4.1362	8.187	16.4526
Peak 2	λ	—	12.566	14.650	15.620	18.930	20.550
	γ	—	8.330	8.100	8.352	11.150	18.875
	w_t	—	0.2098	0.2302	0.2565	0.3987	0.7493

situation is valid for large values of γ: When the value of γ is too large, then the effect of the stiffness coefficient κ is negligible on the behavior of the system.

Figs. 3 and 4 show that with the variation of the damping parameter γ, a damping parameter can be obtained for which the first and second peak values of the tip displacements respectively are minimum. The peak values of the tip displacements occur at different values of λ while changing the damping parameter γ. However, the frequency parameter λ remains between the frequency parameters λ obtained for $\gamma = 0$ and $\gamma = \infty$. Therefore, in Figs. 3 and 4, while changing γ for obtaining minimum peak value of the force transmissibility for the considered mode, the

Table 4
The minimum peak values, corresponding frequencies and damping values for $\kappa = 25, 50, 75, 100, 200, 400$ for two peaks of the support reaction forces

		$\kappa = 25$	$\kappa = 50$	$\kappa = 75$	$\kappa = 100$	$\kappa = 200$	$\kappa = 400$
Peak 1	λ	3.257	3.385	3.419	3.440	3.480	3.499
	γ	7.621	14.847	21.222	27.051	61.323	87.473
	R	5.000	10.000	15.000	20.000	40.000	80.000
Peak 2	λ	11.306	14.302	16.109	17.292	19.493	20.753
	γ	2.215	3.497	4.660	5.785	10.265	19.300
	R	5.000	10.000	15.000	20.000	40.000	80.000

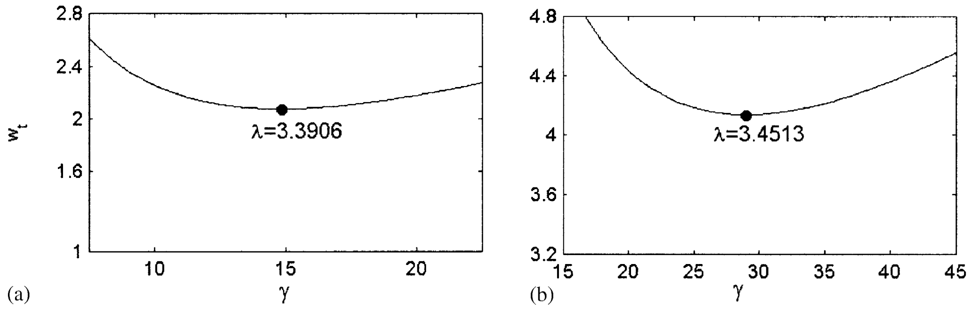

Fig. 3. The first peak tip displacements for various values of γ for (a) $\kappa = 50$, (b) $\kappa = 100$.

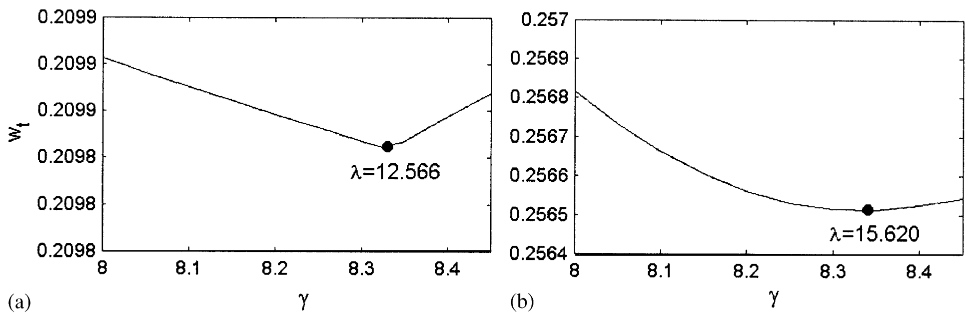

Fig. 4. The second peak tip displacements for various values of γ for (a) $\kappa = 50$, (b) $\kappa = 100$.

frequency parameter λ also changes a little. As it was explained before, λ changes between λ obtained for $\gamma = 0$ and λ obtained for $\gamma = \infty$. It is seen in Figs. 2a–f and 5a–f that, regardless of the damping parameters, there are some points of intersection of the transverse deflection curves of the tip of the beam and support reactions of the beam. The optimum values of γ for the tip displacements are shown in Figs. 3 and 4 only for $\kappa = 50$ and $\kappa = 100$. Similar figures can be obtained for the other values of κ for the tip displacements and for the reaction forces.

Although there can be found an optimum damping parameter for each stiffness parameter, as it can be seen from Tables 3 and 4, increase in the stiffness parameter causes increase in the

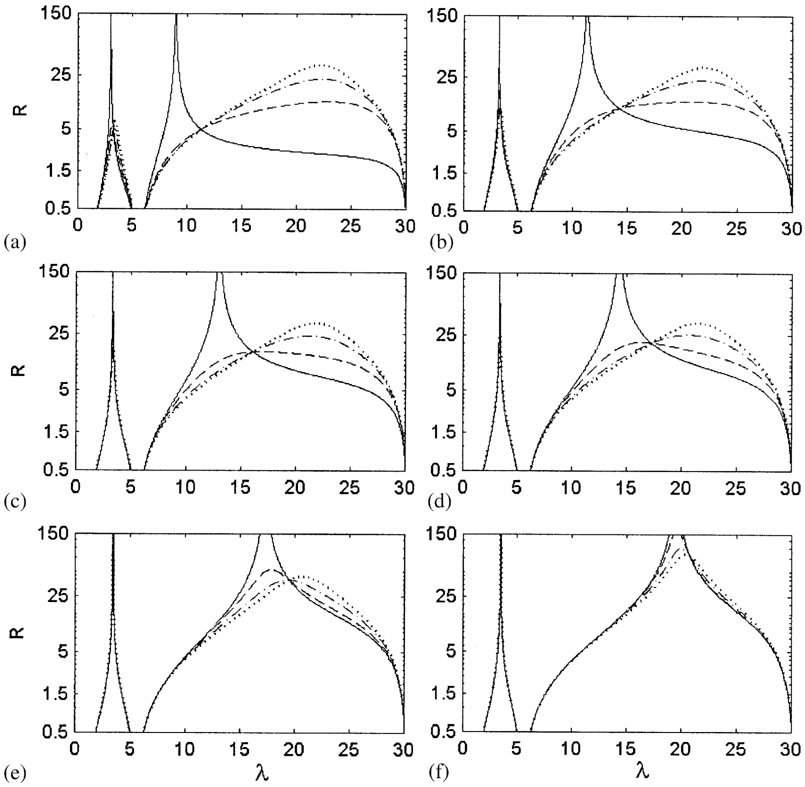

Fig. 5. The reaction forces with the variation of λ for various values of γ for (a) $\kappa = 25$, (b) $\kappa = 50$, (c) $\kappa = 75$, (d) $\kappa = 100$, (e) $\kappa = 200$, (f) $\kappa = 400$. $\gamma = 0$ ——, $\gamma = 5$ – – –, $\gamma = 10$ –·–·–, $\gamma = 15$ · · · · · · ·.

minimum peak values of the tip displacements. Therefore, in designing such systems, the stiffness parameter should be chosen as small as it could be by considering the permitted relative displacement of the lower end of the beam.

4. Conclusions

By using the Lagrange equations, the first two natural frequencies of elastically supported cantilever beams and the steady-state response of a viscoelastically supported cantilever beam to a sinusoidally varying base movement has been studied. The obtained natural frequencies for the rigidly supported cantilever beam are compared with the exact results. To use the Lagrange equations with the trial function in the polynomial form and to satisfy the constraint condition by the use of Lagrange multipliers is a very good way for studying the structural behavior of viscoelastically supported cantilever beams. For the same accuracy level, it needs considerably fewer degrees of freedom than the finite element method and energy-based finite difference method as it was demonstrated by Kocatürk et al. [8].

By the application of the above-mentioned solution technique, the first two values of the natural frequencies are determined, the convergence characteristics of the frequency parameters are investigated numerically. It is seen that the rate of convergence is very high. The response curves to a sinusoidally varying base movement are determined numerically for viscoelastically supported cantilever beams. The effect of the viscosity and stiffness of the support of the cantilever beam on response curves is investigated and shown in the figures and tables.

All of the obtained results are very accurate and may be useful for designing structural and mechanical systems subject to base motion.

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