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Journal of Sound and Vibration 281 (2005) 1157–1162
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Accurate higher-order approximations to frequencies of nonlinear oscillators with fractional powers

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Received 20 January 2004; received in revised form 30 March 2004; accepted 5 April 2004
Available online 28 October 2004

Abstract

An improved harmonic balance (HB) method is been presented to construct more accurate approximations to frequencies of oscillators with non-linear fractional powers. Unlike the classical HB method, linearization is carried out prior to harmonic balancing thus resulting in simple linear algebraic equations instead of complicated nonlinear algebraic equations. Hence, we are able to establish the approximate frequencies for the oscillators more directly. These approximate results are valid for various fractional powers including the limiting case of vanishing power. Comparing with previous approximate solutions, the approximate solutions derived here are more accurate with respect to established exact solutions.

In a previous paper, Mickens [1] studied a class of nonlinear, one-dimensional oscillators with a single-term nonlinearity involving an inverse odd-integer power. By raising each side of the force equation to that power and letting the coefficients of the resulting lowest-order harmonic be zero, he obtained approximate frequencies of the oscillators. Recently, Gottlieb [2] studied frequencies of oscillators with fractional-power nonlinearities. He obtained the corresponding direct first-order harmonic balance (HB) approximate result by utilizing the first Fourier coefficient. He

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.04.030

further explored the effects on the accuracy of the HB approach for successive manipulations of the underlying acceleration equation, such as that made in Ref. [1]. It was concluded that application of the HB method becomes much easier if more preliminary manipulations are made on the original differential equation before proceeding. However, the results will be significantly less accurate and the maximum relative error is about 10%. With the direct HB approximation, the errors of the resulting frequencies are reduced and the maximum relative error is less than 1.6%. The limiting cases of vanishing power were also investigated. The purpose of this paper is to construct higher-order approximations to the frequencies of nonlinear oscillators with fractional powers.

The HB method [3,4] is a procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. Although this method can be applied to nonlinear oscillatory problems where the nonlinear terms are not small and no perturbation parameter is required, most of the time it is very difficult to construct higher-order analytical approximations to the solution using this approach. This is because the HB method requires solving analytical solutions of sets of algebraic equations with very complex nonlinearities. Wu and Li [5] presented an approach which combines linearization of the nonlinear oscillation equation with the method of HB. This approach has been successfully generalized to a few classes of nonlinear oscillators [6–10]. We advance the harmonic balance method by first imposing linearization of the governing equation in order to avoid the shortcoming of having to solve numerically sets of equations with very complex nonlinearities. The most significant features of this new approach are its simplicity and its excellent accuracy for all fractional powers.

Consider the differential equation with single-term positive-power nonlinearity

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathrm{sign}\,(x)|x|^p = 0 \tag{1}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0. \tag{2}$$

Attention here is restricted primarily to rational powers less than unity. Eq. (1) states that the restoring force is an odd function of x.

Introducing a new independent variable, $\tau = \omega t$, we can rewrite (1) and (2) in the forms

$$\omega^2 x'' + \mathrm{sign}\,(x)|x|^p = 0 \tag{3}$$

and

$$x(0) = A, \quad x'(0) = 0, \tag{4}$$

where a prime denotes differentiation with respect to τ. The new independent variable τ is chosen in such a way that the solution of Eq. (3), which satisfies the assigned initial conditions in Eq. (4), is a periodic function of τ with a period of 2π. The corresponding period of the nonlinear oscillation is given by $T = 2\pi/\omega$. Here, both the periodic solution $x(\tau)$ and frequency ω (thus period T) depend on A. In view of the fact that the restoring force $f(x) = -\mathrm{sign}\,(x)|x|^p$ is an odd function of x the periodic solution $x(\tau)$ has the following Fourier series representation:

$$x(\tau) = \sum_{n=0}^{\infty} h_{2n+1} \cos[(2n+1)]\tau \tag{5}$$

which contains only odd multiples of τ.

Following the lowest-order HB method [3,4], a reasonable and simple initial approximation satisfying conditions in Eq. (4) can be taken as

$$x_0(\tau) = A \cos \tau. \tag{6}$$

Here, $x_0(\tau)$ is a periodic function of τ with a period of 2π. Linearizing the governing equations (3) and (4) with respect to the correction $y(\tau)$ at $x = x_0(\tau)$ leads to

$$\omega^2 x_0'' + \mathrm{sign}\,(x_0)|x_0|^p + \omega^2 y'' + p\left[\mathrm{sign}\,(\mathrm{x}_0)x_0\right]^{p-1} y = 0 \tag{7}$$

and

$$y(0) = 0, \quad y'(0) = 0 \tag{8}$$

where $y(\tau)$ is a periodic function of τ with a period of 2π to be determined later. From physics point of view, the idea is to express the periodic solution of Eq. (3) with the assigned conditions in Eq. (4) in the form of $x_0(\tau) + y(\tau)$ which is composed of the harmonics of the motion. Here, $x_0(\tau)$ is the main part satisfying initial conditions in Eq. (4), and $y(\tau)$ is the correction part. Then $y(\tau)$ is seen to satisfy, via linearization of the governing equation (3), a forced Mathieu-type second order differential equation with homogeneous initial conditions as given in Eq. (8). Solving the resulting system of linear equations (7) and (8) in $y(\tau)$ by the HB method may achieve the approximate frequency and periodic solution. This method can be viewed as an improvement of the HB method.

Using Eq. (6), we obtain the following Fourier series expansions:

$$\mathrm{sign}\,(x_0)|x_0|^p = A^p\left(a_{1p} \cos \tau + a_{3p} \cos 3\tau + a_{5p} \cos 5\tau + \cdots\right), \tag{9}$$

where

$$a_{1p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos \tau)^p \cos \tau \, \mathrm{d}\tau, \tag{10a}$$

$$a_{3p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos \tau)^p \cos 3\tau \, \mathrm{d}\tau, \tag{10b}$$

$$a_{5p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos \tau)^p \cos 5\tau \, \mathrm{d}\tau. \tag{10c}$$

We first set $y(\tau) = 0$, i.e., no correction to $x_0(\tau)$ takes place. Substituting $y(\tau) = 0$, Eqs. (6) and (9) into Eq. (7), and setting the coefficient of the resulting term $\cos \tau$ to zero yields

$$a_{1p}A^p - A\omega^2 = 0 \tag{11}$$

which can be solved for ω as a function of A, as

$$\omega = \omega_{1p}(A) = \sqrt{a_{1p}} \Big/ A^{(1-p)/2}. \tag{12}$$

The approximate frequency has previously been obtained, see Eq. (3.3) of Ref. [2]. The first approximation to the periodic solution is given by

$$u_1(\tau) = A \cos \tau, \quad \tau = \omega_{1p}(A)t. \tag{13}$$

In view of Eq. (5), the second approximation to $y(\tau)$ in Eq. (7) is taken of the form

$$y(\tau) = c_1(\cos \tau - \cos 3\tau) \tag{14}$$

which satisfies Eq. (8) automatically. From Eqs. (6), (9) and (14), we obtain

$$\begin{aligned} p[\mathrm{sign}\,(x_0)x_0]^{p-1}y &= 2pc_1A^{-1}[\mathrm{sign}\,(x_0)|x_0|^p](1-\cos 2\tau) \\ &= pc_1A^{p-1}[(a_{1p}-a_{3p})\cos \tau + (-a_{1p}+2a_{3p}-a_{5p})\cos 3\tau + \cdots]. \end{aligned} \tag{15}$$

Substituting Eqs. (6), (9), (14) and (15) into Eq. (7), and setting the coefficients of the resulting items $\cos \tau$ and $\cos 3\tau$ to zeros, respectively, lead to

$$-A\omega^2 + a_{1p}A^p + [pA^{p-1}(a_{1p}-a_{3p}) - \omega^2]c_1 = 0, \tag{16}$$

$$a_{3p}A^p + [9\omega^2 - pA^{p-1}(a_{1p}-2a_{3p}+a_{5p})]c_1 = 0. \tag{17}$$

From Eq. (17) it can be derived that

$$c_1 = -a_{3p}A^p \big/ [9\omega^2 - pA^{p-1}(a_{1p}-2a_{3p}+a_{5p})]. \tag{18}$$

Substitution of Eq. (18) into Eq. (16) yields

$$9A^2\omega^4 - A^{1+p}b_p\omega^2 - A^{2p}c_p = 0 \tag{19}$$

which can be solved for ω as function of A, as

$$\omega_{2p}(A) = \sqrt{\frac{b_p + \sqrt{b_p^2 + 36c_p}}{18}} \Bigg/ A^{(1-p)/2}, \tag{20}$$

where

$$b_p = (9+p)a_{1p} + (1-2p)a_{3p} + pa_{5p}, \tag{21}$$

$$c_p = p(-a_{1p}^2 + a_{1p}a_{3p} - a_{1p}a_{5p} + a_{3p}^2). \tag{22}$$

The sign "+" preceding $\sqrt{b_p^2 + 36c_p}$ in Eq. (20) has been determined by the condition that the ratio ω_{2p}/ω_{1p} tends to 1. Furthermore, c_1 in Eq. (18) can be obtained by substituting ω with $\omega_{2p}(A)$ in Eq. (20) which results in

$$\frac{c_{1p}(A)}{A} = -2a_{3p} \Bigg/ \left[b_p + \sqrt{b_p^2 + 36c_p} + 2p\left(-a_{1p} + 2a_{3p} - a_{5p}\right)\right]. \tag{23}$$

Table 1
Comparison of some values of the frequencies of the nonlinear oscillators with various fractional powers

P	$\omega_e A^{(1-p)/2}$	$\omega_{Mp} A^{(1-p)/2}$ (Error)	$\omega_{1p} A^{(1-p)/2}$ (Error)	$\omega_{2p} A^{(1-p)/2}$ (Error)	$c_{1p}(A)/A$ (%)
3/4	1.02496	1.01559(−0.91%)	1.02567(0.07%)	1.02484(−0.01%)	0.82
5/7	1.02866	1.00958(−1.86%)	1.02961(0.09%)	1.02850(−0.02%)	0.95
2/3	1.03365	1.02085(−1.24%)	1.03498(0.13%)	1.03343(−0.02%)	1.11
3/5	1.04075	1.01840(−2.15%)	1.04273(0.19%)	1.04041(−0.03%)	1.36
1/2	1.05164	1.06225(1.01%)	1.05491(0.31%)	1.05106(−0.05%)	1.73
3/7	1.05960	1.02282(−3.47)	1.06405(0.42%)	1.05880(−0.08%)	2.01
1/3	1.07045	1.04912(−1.99%)	1.07685(0.60%)	1.06928(−0.11%)	2.39
1/4	1.08018	1.06839(−1.09%)	1.08868(0.79%)	1.07860(−0.15%)	2.70
1/5	1.08613	1.04812(−3.50%)	1.09609(0.92%)	1.08426(−0.17%)	2.95
1/6	1.09013	1.06109(−2.66%)	1.10170(1.06%)	1.08805(−0.19%)	3.09
1/7	1.09302	1.04405(−4.48%)	1.10487(1.08%)	1.09077(−0.21%)	3.20
1/8	1.09519	1.05424(−3.74%)	1.10768(1.14%)	1.09282(−0.22%)	3.28
1/9	1.09689	1.04017(−5.17%)	1.10989(1.19%)	1.09442(−0.23%)	3.34
1/10	1.09826	1.04867(−4.52%)	1.11168(1.22%)	1.09569(−0.23%)	3.39
1/11	1.09938	1.03684(−5.69%)	1.11315(1.25%)	1.09674(−0.24%)	3.43
$1/\infty$	1.11072	1(−9.97%)	1.12838(1.59%)	1.10729(−0.31%)	3.85

The second approximate periodic solution is then given by

$$u_2(\tau) = [A + c_{1p}(A)]\cos\tau - c_{1p}(A)\cos 3\tau, \quad \tau = \omega_{2p}(A)t. \tag{24}$$

Note that, as $p \to 0$, we have from Eqs. (10a)–(10c)

$$\lim_{p\to 0} a_{1p} = \frac{4}{\pi}, \quad \lim_{p\to 0} a_{3p} = -\frac{4}{3\pi}, \quad \lim_{p\to 0} a_{5p} = \frac{4}{5\pi}. \tag{25}$$

The use of Eqs. (21) and (22) yields

$$\lim_{p\to 0} b_p = \frac{104}{3\pi}, \quad \lim_{p\to 0} c_p = 0. \tag{26}$$

Therefore, making use of Eqs. (20), (23), (25) and (26) yields

$$\lim_{p\to 0} \omega_{2p}(A) = \sqrt{\frac{104}{27\pi}} \Big/ A^{1/2} \approx 1.10729\big/A^{1/2}, \quad \lim_{p\to 0} \frac{c_{1p}(A)}{A} = \frac{1}{26} \approx 3.85\%. \tag{27}$$

The comparison of some numerical results with respect to the exact frequency ω_e of Gottlieb [2] is presented in Table 1. The numerical results include the approximate frequency ω_{Mp} obtained from Eqs. (4.3), (4.6), (4.9), (4.12) and (4.14) of Gottlieb [2] with the Micken's method [1], the approximate frequency ω_{1p} calculated from Eq. (3.3) of Gottlieb [2] which is equivalent to the first approximation presented in Eq. (12) here, and the second approximate frequency ω_{2p} derived in Eq. (20). In addition, the corresponding values of the ratio $c_{1p}(A)/A$ are also shown in the last column.

From the last column of Table1, it can be observed that the ratio $c_{1p}(A)/A$ of amplitude of correction term to amplitude of oscillation is less than 3.85% for various fractional powers, which accounts for rationality of the linearization of governing equations (3) and (4) with respect to the correction $y(\tau)$. Table 1 also indicates that the second approximate frequency in Eq. (20) is more accurate than the first approximate frequency in Eq. (12) for the various fractional powers. The last row in Table 1 corresponds to the limit of frequency for fractional power as $p \to 0$. It is concluded that the method of Mickens [1] underestimates the frequency by 9.97%; the method of Gottlieb [2] overestimates the frequency by 1.59%; while the method presented here improves the approximation to an error with an upper bound of 0.31%.

In summary, an improved HB method has been presented to construct more accurate approximation to frequencies of oscillators with nonlinear fractional powers. The present method introduces linearization of governing equation to the HB method. Unlike the classical HB method, linearization is carried out prior to harmonic balancing thus resulting in simple linear algebraic equations instead of complicated nonlinear algebraic equations. Hence, we are able to establish the approximate frequencies for the oscillators more directly. These approximate results are valid for various fractional powers including the limiting case of vanishing power. In addition, it does not require the presence of a small perturbation parameter necessary in the perturbation method and thus it is valid for all amplitudes of oscillation. The approximate solutions derived here are the best frequency approximation results as compared with the previous ones, and the maximum relative error has been significantly reduced.

Acknowledgements

The work described in this paper was supported by a grant from City University of Hong Kong [Project No. 7001534 (BC)].

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