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Short Communication

# Error in the finite difference based probabilistic dynamic analysis: analytical evaluation

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## 1. Introduction

In his careful review about the stochastic mechanics Grigoriu [1] mentions: "Recent developments of efficient numerical algorithms for solving general stochastic mechanics problems are based on Monte Carlo simulations, stochastic finite element and boundary element, finite differences, stochastic Green functions, and other methods". The finite element method (FEM) in the stochastic setting attracted numerous investigators. In additional to several reviews there are the monographs by Nakagiri and Hisada [2], Ghanem and Spanos [3], Kleiber and Hien [4], Qiu and Liu [5], Haldar and Mahadevan [6], Elishakoff and Ren [7].

There are also several papers devoted to the stochastic analysis by the finite difference method (FDM). The relevant studies are those by Grigoriu and Khater [8], Grigoriu et al. [9], and by To [10].

The question arises on how accurate the stochastic versions of the FDM or FEM are. The accuracy of these methods in deterministic calculations has been studied in numerous investigations. Therefore, at the first glance, there is no need in studying their accuracy in stochastic applications, for stochastic mechanics uses deterministic differential equations, describing the pertinent phenomena, except treating the load and/or inner parameters to be random.

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On the other hand, stochastic mechanics is looking both for the stochastic characteristics of the output quantities, and, more importantly, it attempts to evaluate the reliability—probability that the structure will perform its mission—or its complement, the probability of failure. For the sensible exploitation of the structure, reliability must be extremely close to unity, or, in other words, probability of failure must be extremely small. The latter quantity can be expressed as $10^{-\alpha}$, where $\alpha$ may take values $3, 4, 5, 6$ or 7, or even more, depending on circumstances.

Therefore, the question whether the FDM or FEM can predict such small probabilities of failure with sufficient accuracy is of paramount importance. In this study we are investigating some eigenvalue problems associated with bars in the stochastic setting. For simplicity, the stochastic variables will be treated as continuous random variables.

## 2. Longitudinal vibrations of bar

Gantmacher [11] notes in his book: "Lagrange [12] demonstrated how it is possible using...[analytical] formulas and, in a limit passage, to obtain the free oscillation [frequencies] of a homogeneous string with fixed ends, the mass of which is no longer concentrated in $n$ points but is distributed uniformly along the string, which has a [given] density".

This implies that historically the vibrations of distributed systems started as their approximations as discrete lamped masses connected with springs. Lagrange used the limit analysis to derive the natural frequencies of the continuous string. The vibration of a string stretched along the $x$-axis with its ends fastened at $x = 0$ and $L$ and with $N$ beads, each of mass $m$ was studied by Gould [13].

FDM and FEM attempt to do exactly the opposite. With these methods, we try, using the approximations of different kind, to reduce the continuous system to the discrete one, and analyze the simpler system.

## 3. Analysis using first-order finite difference method

The governing differential equation for longitudinal vibration of a uniform bar reads

$$E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

in which $E$ is the bar modulus of elasticity, $\rho$ the density, $x$ the axial coordinate, $t$ the time and $u$ the axial displacement.

For free harmonic vibrations, we set $u(x,t) = U(x)\sin\omega t$; thus, Eq. (1) can be rewritten as

$$\frac{d^2U}{dx^2} + \frac{\rho}{E}\omega^2 U = 0. \tag{2}$$

Using the first-order central difference method, Eq. (2) is replaced by the equivalent equation, for any nodal point $i$, for uniform nodal points spacing, by

$$U_{i-1} + \left(\frac{\rho}{E}h^2\omega^2 - 2\right)U_i + U_{i+1} = 0 \tag{3}$$

in which $i$ is an arbitrary nodal point within the bar and $h$ is the uniform nodal spacing, obtained by dividing the total length of the bar $L$ into the number $N$ of segments.

The solution of this difference equation with constant coefficients is obtained by letting

$$U_i = A\lambda^i. \tag{4}$$

Substituting Eq. (4) into Eq. (3), one obtains the following equation for $\lambda$:

$$\frac{1}{\lambda} + \left(\frac{\rho}{E}h^2\omega^2 - 2\right) + \lambda = 0 \tag{5}$$

which has the solutions

$$\lambda_{1,2} = 1 - \frac{h^2\rho\omega^2}{2E} \pm i\sqrt{1 - \left(1 - \frac{h^2\rho\omega^2}{2E}\right)^2}. \tag{6}$$

The general solution for $U_i$ can be written as

$$U_i = A_1 \cos i\vartheta + A_2 \sin i\vartheta \tag{7}$$

in which $A_1$ and $A_2$ are arbitrary constants of integration and

$$\vartheta = \cos^{-1}\left(1 - \frac{h^2\rho\omega^2}{2E}\right). \tag{8}$$

The constants of integration are determined by boundary conditions, one at each end of the bar. For a bar that is clamped at one end and free at the other the following boundary condition hold:

$$U_0 = 0, \quad U_{N+1} = U_{N-1}. \tag{9}$$

For a clamped bar at both ends we have

$$U_0 = 0, \quad U_N = 0. \tag{10}$$

For a bar that is free at both ends the boundary conditions are

$$U_1 = U_{-1}, \quad U_{N+1} = U_{N-1}. \tag{11}$$

In the case of a clamped–free bar, the satisfaction of boundary conditions (9) yields, in view of Eq. (7),

$$A_1 = 0, \quad -A_1 \sin\vartheta \sin N\vartheta + A_2 \sin\vartheta \cos N\vartheta = 0. \tag{12}$$

Since the equations are homogeneous, the determinant of the coefficient of $A_1$ and $A_2$ must vanish.

This condition is expressed by

$$\cos N\vartheta = 0 \tag{13}$$

which has the solution

$$N\vartheta = k\pi/2, \quad k = 1, 2, 3, \ldots . \tag{14}$$

Evaluation of $\cos\vartheta$ yields

$$\cos\vartheta = \cos\frac{k\pi}{2N} = 1 - \frac{\rho h^2}{2E}\omega^2. \tag{15}$$

After some algebraic manipulations we obtain

$$\frac{\rho h^2}{E}\omega^2 = 4\sin^2\frac{k\pi}{4N} \tag{16}$$

in which $k$ should be set equal to unity for the first natural frequency. Keeping in mind that $h = L/N$, the bar's fundamental frequency may be expressed as

$$\omega_1^2 = \frac{\pi^2 E}{4\rho L^2}\left(\frac{\sin \pi/4N}{\pi/4N}\right)^2. \tag{17}$$

When $N$ tends to infinity we obtain the exact solution, as expected.

Now let us treat the bar's elastic modulus is a continuous random variable with given probability density function (PDF) assuming that the other parameters are deterministic quantities.

To avoid resonance phenomenon, the natural frequency of the bar must be less than an excitation frequency $\omega_0$:

$$\omega_1^2 < \omega_0^2. \tag{18}$$

Due to the randomness of $E$, the left hand side of Eq. (17) is also a random variable. If $E$ takes values in the interval $(0, \infty)$, not for all possible values of the elastic modulus inequality (18) will hold. We are interested in the fraction of the structures that fulfil Eq. (18).

The reliability $R$ is defined as the probability of the event specified in Eq. (18):

$$R = \mathrm{Prob}(\omega_1^2 < \omega_0^2). \tag{19}$$

Since usually we evaluate natural frequencies by approximate methods it make sense to resort to Eq. (17) to describe this usual situation. Bearing in mind the expression of the approximate natural frequency in Eq. (17) and substituting it in Eq. (19), the reliability is obtained as

$$R = \mathrm{Prob}\left[\frac{\pi^2 E}{4\rho L^2}\left(\frac{\sin \pi/4N}{\pi/4N}\right)^2 < \omega_0^2\right] \tag{20}$$

or

$$R = F_E\left[\frac{4\omega_0^2 \rho L^2}{\pi^2}\left(\frac{\pi/4N}{\sin \pi/4N}\right)^2\right]. \tag{21}$$

With the known expression of $R$ it is possible to solve the design problem of the bar, noting that the structure performs satisfactorily if the reliability is not less than a codified reliability $r_0$:

$$R \geqslant r_0, \quad 0 < r_0 \leqslant 1. \tag{22}$$

Alternatively, one can recast the problem in terms of the unreliability of the structure, defined as the probability of failure $P_f \leqslant p_0$ where $p_0$ is the tolerable level of probability of failure. The main objective in the design of a structure is to keep the probability of failure extremely small. Once we know the PDF for the random variable $E$, we obtain an expression of a design parameter—taken here as the length of the bar $L$—depending on number of elements $N$ and on the value of $r_0$.

In our special circumstances we know the exact expression for the natural frequency. Thus, one is able to evaluate the exact reliability. One can pose the following question: What is the exact reliability for the design value of $L$ obtained from approximate analysis?

Substitution of the approximate parameter $L = L(N, r_0)$ into the expression of the exact value of frequency for a clamped–free bar, yields a general expression for the "*actual*" reliability, according to parameters $N$ and $r_0$. Comparison of the actual reliability and the required $r_0$ enables us to evaluate the accuracy of FDM, in the stochastic setting.

## 4. Exponentially distributed elastic modulus

Let us specify an exponential distribution for the random modulus of elasticity:

$$f_E(e) = \begin{cases} 0, & e<0, \\ a \times \exp[-ae], & e \geqslant 0, \quad a>0, \end{cases} \tag{23}$$

where the mathematical expectation and the variance are, respectively, $M[E] = 1/a$ and $\mathrm{Var}[E] = 1/a^2$. The approximate reliability has the expression

$$R_{\mathrm{approx}} = 1 - \exp\left[-\frac{1}{M[E]}\frac{4\rho\omega_0^2L^2}{\pi^2}\left(\frac{\pi/4N}{\sin \pi/4N}\right)^2\right]. \tag{24}$$

Demanding the level of the codified reliability value $r_0$ to be achieved of $R_{\mathrm{approx}} = r_0$, we obtain for the length of the bar the expression

$$L_{\mathrm{approx}} = L(N, r_0) = \frac{\pi}{2\omega_0}\sqrt{\frac{M[E]}{\rho}\ln\frac{1}{1-r_0}}\left(\frac{\sin \pi/4N}{\pi/4N}\right). \tag{25}$$

Bearing in mind the expression of the exact solution for the natural frequency for the clamped–free bar, the exact reliability has the following form:

$$R_{\mathrm{exact}} = \mathrm{Prob}\left(\frac{\pi^2E}{4\rho L^2} < \omega_0^2\right). \tag{26}$$

With Eq. (25) taken into account,

$$R_{\mathrm{actual}} = R_{\mathrm{actual}}(N, r_0) = R_{\mathrm{exact}}|_{L=L_{\mathrm{approx}}} = 1 - \exp\left[-\frac{1}{M[E]}\frac{4\rho\omega_0^2L_{\mathrm{approx}}^2}{\pi^2}\right]. \tag{27}$$

The substitution of Eq. (25) in Eq. (27) yields

$$R_{\mathrm{actual}} = 1 - \exp\left[\left(\frac{\sin \pi/4N}{\pi/4N}\right)^2\ln(1-r_0)\right]. \tag{28}$$

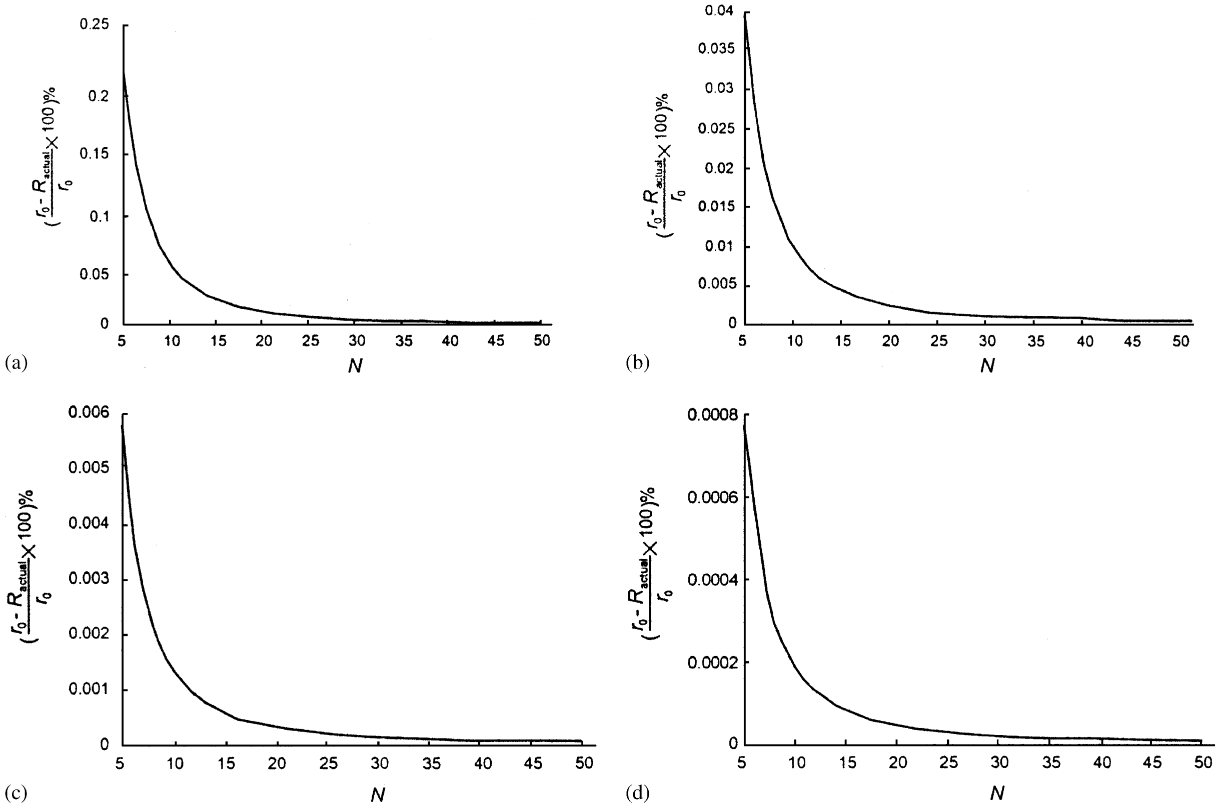


Fig. 1. Error in reliability: evaluation versus the discretization parameter $N$, for various codified reliabilities $r_0$: (a) $r_0 = 0.90$, (b) $r_0 = 0.99$, (c) $r_0 = 0.999$, (d) $r_0 = 0.9999$.

Evaluating $R_{actual}$ for increasing $N$, we obtain values that turn out to be less than $r_0$. Fig. 1 shows the percentage error between $R_{actual}$ and $r_0$ for increasing values of $N$, when $r_0$ is fixed at 0.90, 0.99, 0.999 and 0.9999, respectively.

Naturally, only the points on the figures corresponding to integer values of $N$ have a physical sense.

For $r_0 = 0.90$, for example, the error decreases from 0.2% for $N = 5$ ($R_{actual} = 0.898094$) to 0.053% for $N = 10$ ($R_{actual} = 0.899526$) to 0.023% for $N = 15$ ($R_{actual} = 0.899789$).

Analogously, it is possible to evaluate the *actual* probability of failure for fixed values of *allowed* probability of failure.

Fig. 2 portrays the percentage error between $P_{f,actual}$ and $p_0$ for increasing values of $N$, when $p_0$ is fixed, respectively, at 0.1, 0.01, 0.001 and 0.0001.

We observe that the error decreases as the codified value of reliability $r_0$ increases. That means the behaviour of the structure improves for higher values of reliability set in the design process. This is a remarkable, albeit intuitively a not expected result.

The "*actual*" probability of failure is related with the codified probability of failure $p_0$ by the relation $P_{f,actual} = p_0^{\delta}$, where $\delta = [\sin(\pi/4N)/\pi/4N]^2$. Because the exponent $\delta$ takes values smaller

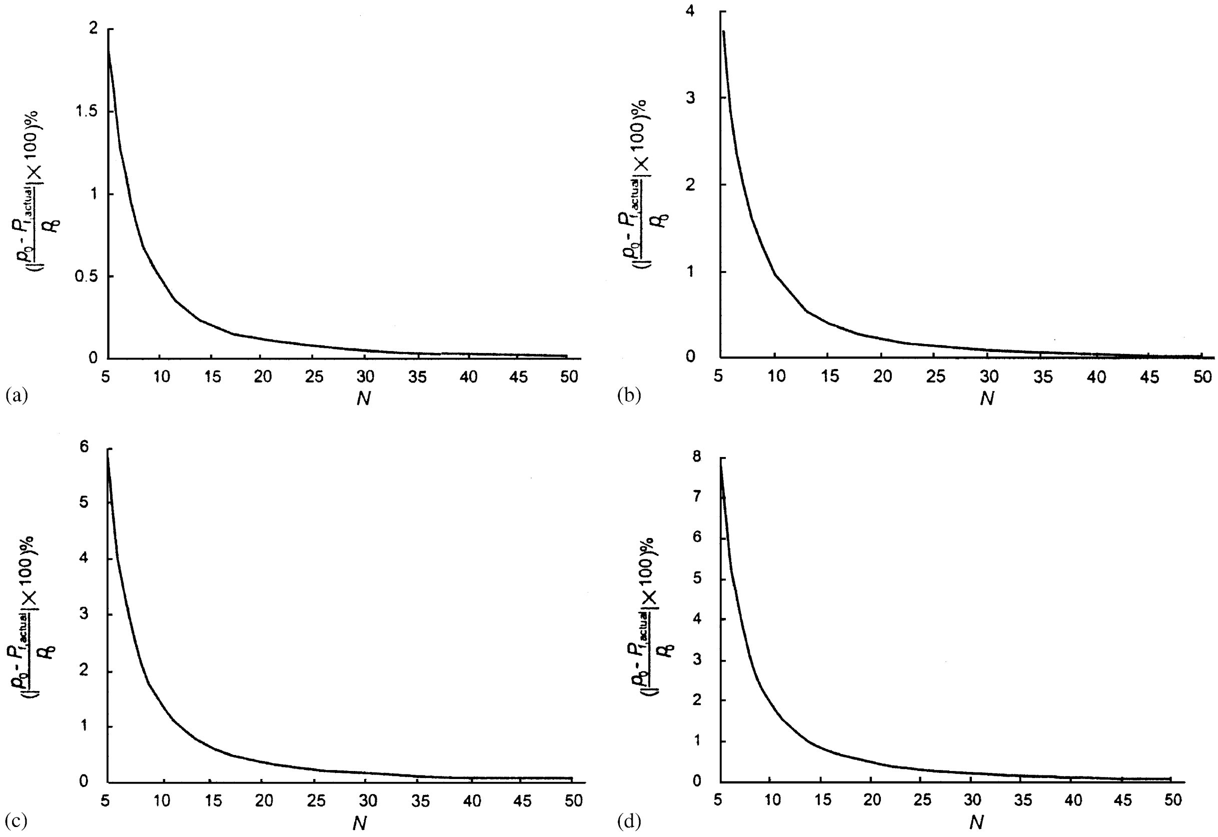


Fig. 2. Error in probability of failure as a function of $N$, for various allowable probabilities of failure $p_0$: (a) $p_0 = 0.1$, (b) $p_0 = 0.01$, (c) $p_0 = 0.001$, (d) $p_0 = 0.0001$.

than unity when $N$ increases and because $p_0$ is, naturally, smaller than unity, the actual probability of failure always exceeds the allowed value. For a fixed value for $N$, say $N = 10$ ($\delta = 0.998951$), we obtain the following results: $P_{f,\text{actual}} = 1.00474p_0$ for $p_0 = 0.1$; $P_{f,\text{actual}} = 1.00951p_0$ for $p_0 = 0.01$; $P_{f,\text{actual}} = 1.01429p_0$ for $p_0 = 0.001$; $P_{f,\text{actual}} = 1.0191p_0$ for $p_0 = 0.0001$ (Fig. 3).

When increasing the number of $N$, for a fixed codified value of $p$, the value of $P_{f,\text{actual}}$ decreases, maintaining its value greater than $p_0$. When $N$ tends to infinity, $P_{f,\text{actual}}$ reaches the allowable value $p_0$.

The actual probability of failure should not be more than the allowed one, but result demonstrate that the error made because of discretization is a "bad error"; this is because the actual value obtained is not on the safe side for the design of the structure.

It should be noted that analogous qualitative results are obtained in the cases in which the elasticity modulus follows some other distributions (Rayleigh distribution or uniform distribution).

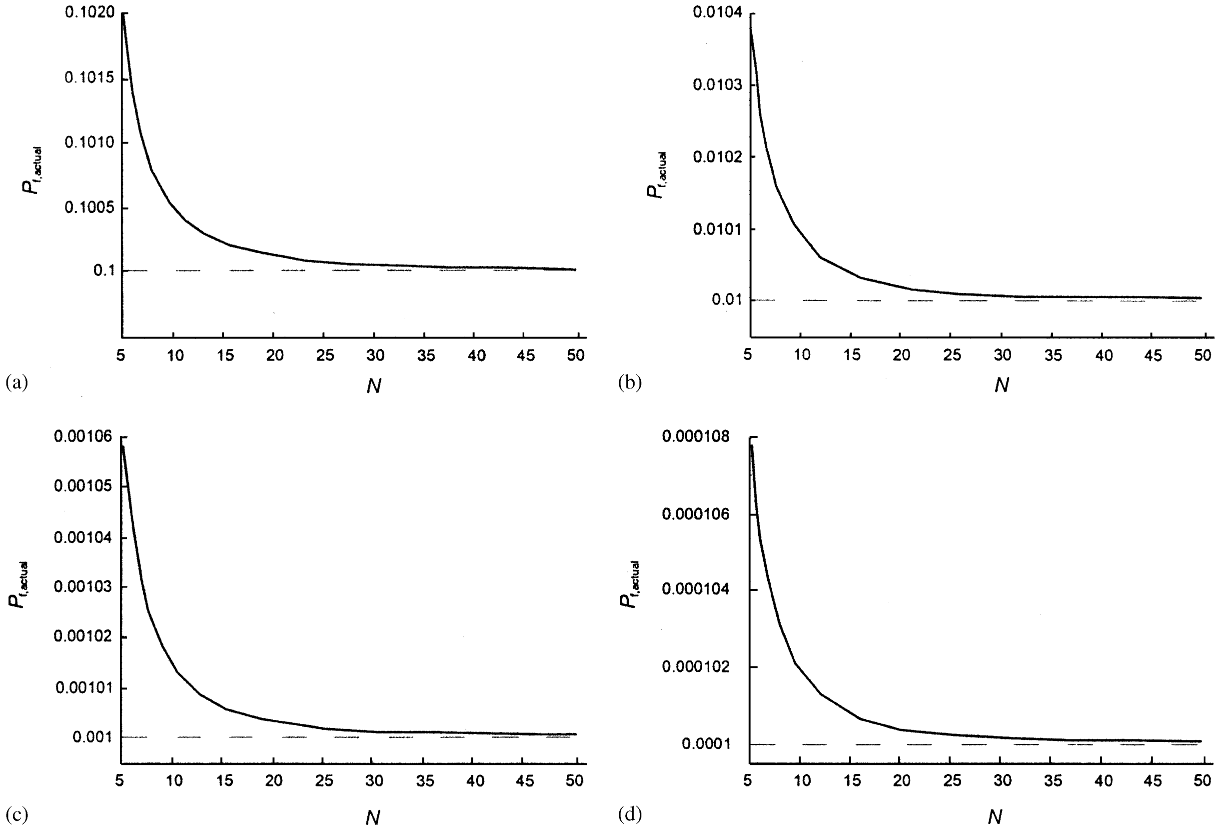


Fig. 3. Variation of the *actual* probability of failure versus $N$, for various levels of $p_0$: (a) $p_0 = 0.1$, (b) $p_0 = 0.01$, (c) $p_0 = 0.001$, (d) $p_0 = 0.0001$.

## 5. Analysis using second-order central differences

Using the second-order central difference method the expressions for the first and second derivatives of the displacement are [14,15]

$$\dot{U}_i = (1/12h)(U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}),$$
$$\ddot{U}_i = (1/12h^2)(-U_{i-2} + 16U_{i-1} - 30U_i + 16U_{i+1} - U_{i+2}). \tag{29}$$

The equation for the longitudinal vibrations can be rewritten as

$$-U_{i-2} + 16U_{i-1} + \left(\frac{12\rho h^2\omega^2}{E}\right)U_i + 16U_{i+1} - U_{i+2} = 0, \tag{30}$$

where $U$, $i$, $\rho$, $h$, $\omega$ and $E$ have the same meaning of the ones in Eqs. (1) and (3).

To obtain the solution of the difference equation (30) we can again set $U_i = A\lambda^i$ getting the following equation in $\lambda$:

$$\left(\frac{1}{\lambda}+\lambda\right)^2 - 16\left(\frac{1}{\lambda}+\lambda\right) + 4\left(7 - 3\frac{\rho h^2\omega^2}{E}\right) = 0. \tag{31}$$

Solutions in $\lambda$ are given by

$$\lambda_{1,2} = 4 + 3\sqrt{1+\frac{\rho h^2\omega^2}{3E}} \pm \sqrt{\left(4+3\sqrt{1+\frac{\rho h^2\omega^2}{3E}}\right)^2 - 1},$$
$$\lambda_{3,4} = 4 - 3\sqrt{1+\frac{\rho h^2\omega^2}{3E}} \pm \sqrt{\left(4-3\sqrt{1+\frac{\rho h^2\omega^2}{3E}}\right)^2 - 1}. \tag{32}$$

The general solution has the following expression:

$$U_j = C_1\lambda_1^j + C_2\lambda_2^j + C_3\lambda_3^j + C_4\lambda_4^j. \tag{33}$$

The boundary conditions needed to find the constants of integration are expressed by

$$U_1 = U_{-1}, \quad U_2 = U_{-2}, \quad U_{N+1} = U_{N-1}, \quad U_{N+2} = U_{N-2}. \tag{34}$$

Under the condition that the determinant should vanish we arrive at

$$\left[\left(4-\sqrt{(\gamma-5)(\gamma-3)}-\gamma\right)^N + \left(4+\sqrt{(\gamma-5)(\gamma-3)}-\gamma\right)^N\right]\sqrt{(\gamma-5)(\gamma-3)}\sqrt{(\gamma+5)(\gamma+3)}$$
$$\times\gamma^2(2\gamma^2-33)\left[\left(4-\sqrt{(\gamma+5)(\gamma+3)}+\gamma\right)^N + \left(4+\sqrt{(\gamma+5)(\gamma+3)}+\gamma\right)^N\right] = 0, \tag{35}$$

where

$$\gamma = 3\sqrt{1+\frac{\rho h^2\omega^2}{3E}}. \tag{36}$$

Fixing $N$, respectively, to be equal to 5, 10 and 15 we obtain the following expressions for the fundamental natural frequencies:

$$\omega_1^2|_{N=5} = \frac{25\left(61+\sqrt{5}-16\sqrt{2(5+\sqrt{5})}\right)}{24}\frac{E}{\rho L^2}, \tag{37}$$

$$\omega_1^2|_{N=10} = \frac{25\left(30+\sqrt{\frac{5+\sqrt{5}}{2}}-16\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}\right)}{3}\frac{E}{\rho L^2}, \tag{38}$$

$$\omega_1^2|_{N=15} = \frac{75\left(119+\sqrt{5}+2\sqrt{3\frac{5+\sqrt{5}}{2}}\right)-32\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}}}{16}\frac{E}{\rho L^2}. \tag{39}$$

We fix $N$ at five and obtain the expressions of approximate reliability and actual one given by

$$R_{\text{approx}}^{(5)} = 1-\exp\left[-\frac{24}{25\left(61+\sqrt{5}-16\sqrt{2(2+\sqrt{5})}\right)}\frac{\rho L^2\omega_0^2}{M[E]}\right], \tag{40}$$

$$R_{\text{actual}}^{(5)} = 1-\exp\left[\frac{25\left(61+\sqrt{5}-16\sqrt{2(2+\sqrt{5})}\right)}{6\pi^2}\ln(1-r_0)\right]. \tag{41}$$

For $N=10$:

$$R_{\text{approx}}^{(10)} = 1-\exp\left[-\frac{3}{25\left(30+\sqrt{\frac{5+\sqrt{5}}{2}}-16\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}\right)}\frac{\rho L^2\omega_0^2}{M[E]}\right], \tag{42}$$

$$R_{\text{actual}}^{(10)} = 1-\exp\left[\frac{100\left(30+\sqrt{\frac{5+\sqrt{5}}{2}}-16\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}\right)}{3\pi^2}\ln(1-r_0)\right]. \tag{43}$$

Table 1
Errors between actual reliability $R_{\text{actual}}$ and actual reliability $r_0$ using first-order and second-order finite difference schemes, with $N=5$

| $N=5$ | | $R_{\text{actual}}$ | | $\varepsilon=[(r_0-R)/r_0]\times 100$ (%) | | $P_{f,\text{actual}}$ | |
|---|---|---|---|---|---|---|---|
| | | 1st order | 2nd order | 1st order | 2nd order | 1st order | 2nd order |
| $r_0$ | 0.90 | 0.898094 | 0.899975 | 0.21 | 0.0027 | 0.101906 | 0.100025 |
| | 0.99 | 0.989615 | 0.98995 | 0.0389 | 0.000499 | 0.0104 | 0.0100049 |
| | 0.999 | 0.998942 | 0.998999 | 0.0058 | 0.000074 | 0.0011 | 0.0010007 |
| | 0.9999 | 0.999892 | 0.9999 | 0.00078 | $9.9\times10^{-6}$ | 0.0001078 | 0.0001001 |

Table 2
Errors between actual reliability $R_{\text{actual}}$ and actual reliability $r_0$ using first-order and second-order finite difference schemes, with $N = 10$

| $N = 10$ | | $R_{\text{actual}}$ | | $\varepsilon = [(r_0 - R)/r_0] \times 100$ (%) | | $P_{f,\text{actual}}$ | |
|---|---|---|---|---|---|---|---|
| | | 1st order | 2nd order | 1st order | 2nd order | 1st order | 2nd order |
| $r_0$ | 0.90 | 0.899526 | 0.899998 | 0.053 | 0.000219 | 0.100474 | 0.100002 |
| | 0.99 | 0.989905 | 0.99 | 0.0096 | 0.0000398 | 0.010095 | 0.01 |
| | 0.999 | 0.998986 | 0.999 | 0.0014 | $5.9 \times 10^{-6}$ | 0.001014 | 0.001 |
| | 0.9999 | 0.999898 | 0.9999 | 0.00019 | $7.9 \times 10^{-7}$ | 0.000102 | 0.0001 |

Table 3
Errors between actual reliability $R_{\text{actual}}$ and actual reliability $r_0$ using first-order and second-order finite difference schemes, with $N = 15$

| $N = 15$ | | $R_{\text{actual}}$ | | $\varepsilon = [(r_0 - R)/r_0] \times 100$ (%) | | $P_{f,\text{actual}}$ | |
|---|---|---|---|---|---|---|---|
| | | 1st order | 2nd order | 1st order | 2nd order | 1st order | 2nd order |
| $r_0$ | 0.90 | 0.899789 | 0.9 | 0.023 | 0.000011 | 0.100211 | 0.1 |
| | 0.99 | 0.989958 | 0.99 | 0.0043 | $2.1 \times 10^{-6}$ | 0.010042 | 0.01 |
| | 0.999 | 0.998994 | 0.999 | 0.000634 | $3.1 \times 10^{-7}$ | 0.001006 | 0.001 |
| | 0.9999 | 0.999899 | 0.9999 | 0.000084 | $4.1 \times 10^{-8}$ | 0.000101 | 0.0001 |

For $N = 15$:

$$R^{(15)}_{\text{approx}} = 1 - \exp\left[-\frac{16}{75\left(119 + \sqrt{5} + 2\sqrt{3\frac{5+\sqrt{5}}{2}}\right) - 32\sqrt{7 + \sqrt{5} + \sqrt{6(5+\sqrt{5})}}}\frac{\rho L^2 \omega_0^2}{M[E]}\right], \quad (44)$$

$$R^{(15)}_{\text{actual}} = 1 - \exp\left[\frac{75\left(119 + \sqrt{5} + 2\sqrt{3\frac{5+\sqrt{5}}{2}}\right) - 32\sqrt{7 + \sqrt{5} + \sqrt{6(5+\sqrt{5})}}}{4\pi^2}\ln(1 - r_0)\right]. \quad (45)$$

A comparison of the errors obtained between calculated actual reliability, $R_{\text{actual}}$, and the required reliability, $r_0$, is given in Tables 1–3. The results obtained by use of the first-order and second-order finite difference schemes are listed.

From the examination of the results we see that in the evaluation of reliability of bars in vibrations, second-order central differences analysis is preferable.

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