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Discussion

Authors' reply ☆

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1. The goal and methodology of the paper

Farassat [2] commented that "the literature of acoustics on the subject of this paper is very extensive. The current state of the theoretical aeroacoustics is considerably more advanced than what is presented by these authors".

Whereas we agree that the literature on the subject is extensive, no references available to us provided answers to our question: why does this methodology not give correct answers in some simple situations? Therefore, the focus of Zinoviev and Bies [1] was on analysing the fundamental properties of the FW-H equation on the basis of the historically first paper on the subject by Curle [3]. In our analysis, we followed Curle's argument as closely as possible and stated this on p. 538 of Zinoviev and Bies. This can answer the questions by Farassat [2] which are concerned with justification of some of the techniques used in Zinoviev and Bies [1]. We answer these questions further in our reply.

2. The FW-H equation and the meaning of its terms

Let us first formulate the FW-H equation, which will serve as the basis of our response. To demonstrate its fundamental properties, we will stay within the linear approximation, so that the

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velocity of fluid particles and solid surfaces is much smaller than the sound speed, c_0, in the undisturbed fluid. Then, after neglecting Doppler factors in Eq. (5.1) of Ffowcs Williams and Hawkings [4], one can obtain the following equation:

$$4\pi c_0^2 \rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_{V_{\text{tot}}} \left[\frac{T_{ij}}{r}\right]_{\text{ret}} \mathrm{d}\mathbf{y} - \frac{\partial}{\partial x_i} \iint_S \left[\frac{n_j p_{ij}}{r}\right]_{\text{ret}} \mathrm{d}S(\mathbf{y}) + \frac{\partial}{\partial t} \iint_S \rho_0 \left[\frac{n_j v_j}{r}\right]_{\text{ret}} \mathrm{d}S(\mathbf{y}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the observation point, $\mathbf{y} = (y_1, y_2, y_3)$ is the source point, $r = |\mathbf{x} - \mathbf{y}|$, t is time, ρ_0 is the fluid density at equilibrium, $\rho'(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho_0$ is the density fluctuation, V_{tot} is the total volume of the fluid, S is the control surface, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal vector to S. Whereas in Curle [3] as well as in Zinoviev and Bies [1] the unit normal vector points from the fluid, we will follow Farassat [2] and assume that the vector $\mathbf{n}$ points into the fluid. Square brackets indicate the dependence on the retarded time, $\tau = t - |\mathbf{x} - \mathbf{y}|/c_0$.

Considering the importance of the source terms, p_{ij} and, especially, $\mathbf{v}$, for further analysis, let us justify our understanding of their meaning. Note that, by neglecting the nonlinear terms, Eq. (1) can be obtained also from Eqs. (2.15) and (2.16) of Curle [3], Eq. (2.2.3) of Howe [5], and Eq. (2–68) of Blake [6]. In all these publications, the meaning of the source terms in Eq. (1) is the same. Specifically, $T_{ij} = \rho v_i v_j + p_{ij} - c_0^2 \rho \delta_{ij}$ is the Lighthill's stress tensor [7, Eq. (5)], p_{ij} is the *total* compressive stress tensor (Eq. (6) of Lighthill [7]), which includes the viscous stresses, and $\mathbf{v}$ is the velocity of the surface S *with respect to a stationary observer* (Ffowcs Williams and Hawkings [4, p. 324]). This understanding of the source terms can be confirmed by theoretical considerations by Farassat [8, p. 795], as well as by applications of the FW-H theory to sound scattering by a sphere [2] and to sound radiation by a vibrating sphere [6, p. 81–83]. Note that Curle's formula for a stationary surface S can be obtained from Eq. (1) by neglecting the last term in the right-hand part of this equation.

3. Examples of application of the FW-H equation

In Section 2.0, Farassat [2] claims that, in direct contrast to the conclusion of Zinoviev and Bies [1], the FW-H and Curle equations give correct predictions for the two simple examples considered in the paper. While accepting some of Farassat's criticism, below we prove that, according to our original conclusion, the FW-H and Curle equations fail to produce the correct prediction for the radiated sound in some simple situations.

3.1. Scattering of a plane wave by a rigid sphere

Farassat [2] considered in detail the application of the FW-H equation to the example of scattering of a plane wave by a rigid sphere, which is considered in Zinoviev and Bies as Example 1. He commented that "...the Curle formula gives the correct result for this example".

We accept that our calculations related to this example contain an error. This error is related to an unjustified assumption that the second and the third terms in the divergence ($\partial/\partial x_2$ and $\partial/\partial x_3$)

in Eq. (37) of Zinoviev and Bies could be neglected due to the axial symmetry of the problem. Therefore, we confirm that Eq. (43) of Zinoviev and Bies is incorrect and the FW-H equation leads to the correct prediction in this case.

At the same time, we insist that this result is only fortuitous and that the conclusion of Zinoviev and Bies that Curle's formula does not describe properly the scattering of sound by a rigid object is correct. Below we demonstrate this conclusion on another example.

3.2. A spherical wave converging on a rigid sphere

Consider an incident spherical wave *converging* on an acoustically small rigid immovable sphere or radius, R_0, with the centre at the point $x = 0$. The negative temporal dependence, $e^{-i\omega t}$, will be used here as in Farassat [2]. The incident wave pressure, P_{inc}, can be written as follows:

$$P_{inc} = A\,\frac{e^{-ikx}}{x}, \tag{2}$$

where x is a spherical coordinate, so that $\mathbf{x} = (x, \Phi, \Theta)$, and A is a constant. This situation can be described by the FW-H theory, if the volume sources determined by the first term in the right-hand part of Eq. (1) are symmetrical with respect to the centre of the sphere at $x = 0$. As the spatial layout in this case is the same as in the case of sound scattering by a plane wave, calculation of the amplitude of the reflected (scattered) wave, P_{sc}, can be carried out by means of Eq. (11) of Farassat [2], which we rewrite here as

$$4\pi P_{sc} = ik\iint_S P_{tot}\cos\alpha\,\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,dS. \tag{3}$$

In Eq. (3), $\cos\alpha$ is determined by Eq. (13) of Farassat [2].

To determine the total pressure P_{tot} on the surface in our case, the scattered (reflected) wave will be represented as a diverging spherical wave with unknown complex amplitude, B:

$$P_{sc} = B\,\frac{e^{ikx}}{x}. \tag{4}$$

The incident and reflected waves must satisfy the following condition on the surface of an immoveable sphere [9, p. 425]:

$$\left.\frac{\partial P_{inc}}{\partial x}\right|_{x=R_0} = -\left.\frac{\partial P_{sc}}{\partial x}\right|_{x=R_0}. \tag{5}$$

Substituting Eqs. (2) and (4) into Eq. (5), one obtains

$$B = Ae^{-2ikR_0}\,\frac{ikR_0+1}{ikR_0-1}. \tag{6}$$

With the use of Eqs. (2), (4) and (6), the total pressure field, $P_{tot} = P_{inc} + P_{sc}$, on the surface of the sphere, can be written as

$$P_{tot}|_{x=R_0} = \frac{A}{R_0}\,e^{-ikR_0}\left(1 + \frac{ikR_0+1}{ikR_0-1}\right). \tag{7}$$

By extending the right-hand part of Eq. (7) into Taylor series, it can be shown that, for the acoustically small sphere ($kR_0 \ll 1$), Eq. (7) takes the following form:

$$P_{\mathrm{tot}}|_{x=R_0} = -2A\mathrm{i}k + O(kR_0)^3. \tag{8}$$

Let us now find the scattered far field based on the FW-H theory. With the use of Eqs. (14) and (16) of Farassat [2] and our Eq. (8), Eq. (3) becomes

$$P_{\mathrm{sc}}^{\text{FW-H}}|_{x\to\infty} = \frac{A}{2\pi}(kR_0)^2 \frac{\mathrm{e}^{\mathrm{i}kx}}{x}\int_0^{\pi}\int_0^{2\pi}(1-\mathrm{i}kR_0\cos\alpha)\cos\alpha\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta. \tag{9}$$

The evaluation of the integral in Eq. (9) produces the following result:

$$\int_0^{\pi}\int_0^{2\pi}(1-\mathrm{i}kR_0\cos\alpha)\cos\alpha\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = -\tfrac{4}{3}\mathrm{i}\pi kR_0, \tag{10}$$

which, after substitution into Eq. (9), gives the following expression for the spherical wave reflected from the rigid sphere:

$$P_{\mathrm{sc}}^{\text{FW-H}} = -\tfrac{2}{3}\mathrm{i}A(kR_0)^3\frac{\mathrm{e}^{\mathrm{i}kx}}{x}. \tag{11}$$

The correct expression for the scattered wave can be easily derived from Eqs. (4) and (6). It takes the form of

$$P_{\mathrm{sc}} = -A\frac{\mathrm{e}^{\mathrm{i}kx}}{x} + O(kR_0)^3, \tag{12}$$

meaning the physically obvious result that the amplitude of the reflected wave is equal to the amplitude of the incident wave. One can observe that Eq. (11) is a different order of magnitude than Eq. (12) with respect to the small parameter kR_0. Therefore, the FW-H theory does not produce the correct result for this example.

3.3. A rigid sphere in a variable velocity field

3.3.1. Origin of the problem

This problem (Example 2 from Zinoviev and Bies) deals with a rigid sphere submerged in a fluid. The fluid moves back and forth with negligible pressure fluctuations. Referring to this example, Farassat [2] commented that "The origin of this problem as discussed by authors is somewhat confusing" and that "...this example is on the determination of the radiation field of a rigid sphere oscillating ... along the x-axis". We do not agree with both statements and provide below two situations that can be described by such a model.

First, this model is applicable if a typical vortex size in a turbulent flow is much larger than the diameter of the sphere submerged in the flow. If a trail of vortices moves with a constant velocity, the variable velocity component of the flow near the sphere can be approximated as spatially uniform and oscillating with harmonic temporal dependence. The pressure fluctuations in such a flow can be neglected in the linear approximation, as the relative changes of density are proportional to the Mach number squared [7, p. 571]. This justification (without going into details) has been provided in Zinoviev and Bies (p. 545).

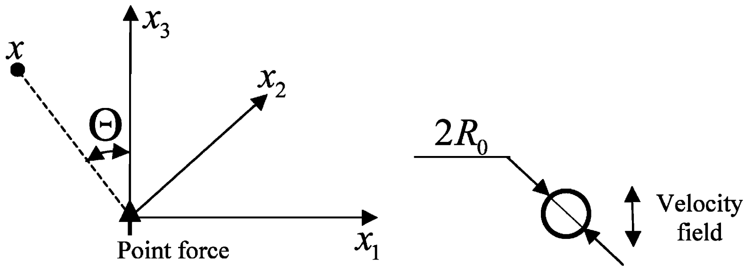

Fig. 1. A small rigid sphere in the near field of a point force.

In addition, this situation occurs in acoustics. Consider a point force, **F**, located at the origin and directed along the x_3-axis (Fig. 1). The pressure field produced by such a source can be determined as follows:

$$P_{\mathrm{pf}}(x,\Theta)=\frac{F}{4\pi}\frac{e^{ikx}}{x^2}(1-ikx)\cos\Theta, \tag{13}$$

where x is the length of the radius vector of the observation point and Θ is the angle between the radius vector and the x_3-axis. Eq. (13) can be easily obtained from a more general Eq. (4-4.5) of Pierce [9] by substituting the harmonic temporal dependence of the force.

Assume that a sphere of radius, R_0, is located along the x_1-axis, so that its centre is at the point $(x_1,x_2,x_3)=(X,0,0)$ (Fig. 1). The sphere is small and located in the near field of the point force, so that the following conditions are satisfied:

$$kX \ll 1, \tag{14}$$

$$R_0/X = O(kX)^2. \tag{15}$$

3.3.2. Velocity and pressure fields generated by the point force near the sphere

Let us consider in detail the spatial distribution of the amplitude of the pressure and velocity fields generated by the point force in the vicinity of the sphere. Values of the velocity and the pressure will be found at a point $(x_1,x_2,x_3)=(X+R_0,R_0,-R_0)$ located near the sphere.

From Euler's equation, $\partial\mathbf{u}/\partial t=-\nabla P/\rho_0$, one can derive the following equation for the fluid velocity:

$$\mathbf{u}=-\frac{i}{\rho_0\omega}\nabla P, \tag{16}$$

from which the components of the velocity can be easily obtained.

Let us first find the x_3-component. The corresponding derivative of the pressure can be expressed through the spherical coordinates as follows [10, p. 718]:

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3}=\cos\Theta\,\frac{\partial P_{\mathrm{pf}}}{\partial x}-\frac{\sin\Theta}{x}\frac{\partial P_{\mathrm{pf}}}{\partial\Theta}. \tag{17}$$

The substitution of Eq. (13) to Eq. (17) leads to the following expression for the x_3-derivative of the pressure field of the point force:

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3} = \frac{F}{4\pi x^3}\,\mathrm{e}^{\mathrm{i}kx}[(3\cos^2\Theta - 1)(\mathrm{i}kx - 1) + (kx)^2\cos^2\Theta], \tag{18}$$

which at $kx \ll 1$ reduces to

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3} = \frac{F}{4\pi x^3}\,(1 - 3\cos^2\Theta) + O(kx)^2. \tag{19}$$

To obtain the value of the above derivative in the vicinity of the sphere, first substitute $\Theta = \pi/2 + \alpha,\ \alpha = R_0/X = O(kX)^2$ to Eq. (19). The substitution leads to

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3} = \frac{F}{4\pi x^3} + O(kx)^2 + O(\alpha^2). \tag{20}$$

With the use of the following series with respect to α:

$$\frac{1}{(X+R_0)^3} = \frac{1}{X^3}\,\frac{1}{(1+\alpha)^3} = \frac{1}{X^3} + O(\alpha). \tag{21}$$

Eq. (20) is reduced to

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3} = \frac{F}{4\pi X^3} + O(kX)^2. \tag{22}$$

Taking account of Eq. (16), one can obtain the following equation for the x_3-component of the fluid velocity near the sphere at the point $(x_1, x_2, x_3) = (X + R_0, R_0, -R_0)$:

$$u_3 = -\frac{\mathrm{i}}{4\pi\rho k c_0}\,\frac{F}{X^3} + O(kX)^2. \tag{23}$$

Consider now the x_1-component of the velocity field. The corresponding derivative of the pressure takes the form [10, p. 718]:

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_1} = \sin\Theta\,\cos\Phi\,\frac{\partial P_{\mathrm{pf}}}{\partial x} + \frac{1}{x}\cos\Theta\,\cos\Phi\,\frac{\partial P_{\mathrm{pf}}}{\partial\Theta} - \frac{\sin\Phi}{x\,\sin\Theta}\,\frac{\partial P_{\mathrm{pf}}}{\partial\Phi}. \tag{24}$$

The substitution of Eq. (13) to Eq. (24) leads to the following equation:

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_3} = \frac{F}{4\pi x^3}\,\mathrm{e}^{\mathrm{i}kx}[\sin\Theta\,\cos\Theta\,\cos\Phi(3\mathrm{i}kx - 3 + (kx)^2)], \tag{25}$$

which, for $kx \ll 1$, is reduced to

$$\frac{\partial P_{\mathrm{pf}}}{\partial x_1} = -3\,\frac{F}{4\pi x^3}\sin\Theta\,\cos\Theta\,\cos\Phi + O(kx)^2. \tag{26}$$

Using Eqs. (15) and (21) as well as the following expansions with respect to $\alpha = R_0/X$:

$$\cos(\alpha) = 1 + O(\alpha^2), \tag{27}$$

$$\sin(\pi/2+\alpha)\cos(\pi/2+\alpha) = -\alpha + O(\alpha^3), \tag{28}$$

the x_1-component of the velocity field generated by the point force at the point $(x_1, x_2, x_3) = (X + R_0, R_0, -R_0)$ can be written as

$$u_1 = -\frac{3\mathrm{i}}{4\pi\rho_0 k c_0}\frac{F}{X^3}\frac{R_0}{X} = 3u_3\frac{R_0}{X} = O(kX)^2. \tag{29}$$

To find the pressure of the point force near the sphere, we expand Eq. (13) into the Taylor series and obtain:

$$P_{\mathrm{pf}} = -\frac{F}{4\pi X^2}\frac{R_0}{X} = -\frac{F}{4\pi k X^3}\frac{R_0}{X}(kX) = u_3\frac{\rho_0 c_0}{\mathrm{i}}\frac{R_0}{X}(kX) = O(kX)^3. \tag{30}$$

Eqs. (22), (29) and (30) show that the amplitudes of the x_1-component of the velocity and the pressure are proportional to a small parameter and thus can be neglected. Therefore, the sphere shown in Fig. 1 can be considered as being immersed in a variable spatially uniform velocity field with negligible pressure fluctuations and Example 2 of Zinoviev and Bies [1] has a clear justification in acoustics.

3.3.3. *Sound wave radiated by the sphere in the velocity field*

3.3.3.1. Equivalence of a motion of the fluid and the sphere. Farassat [2] noted that "...the surface of the sphere cannot be stationary because the momentum equation dictates that $\partial p'/\partial n' = 0$ which is not what the authors assume (see Eq. (57), Zinoviev and Bies)". In response to this comment, we point out that Eq. (57) of our paper [1] describes only the scattered (radiated) component of pressure fluctuations, which satisfies the boundary conditions, determined by Eq. (5) above. The total velocity field, which is the sum of the incident and scattered components, is zero on the surface meaning that the sphere is stationary.

The following point is important for our analysis. As stated in Landau and Lifshitz [11, p. 297], a motion of the fluid relative to the sphere is equivalent to a motion of the sphere in the fluid. As this point can be confusing, below we provide the proof by solving the corresponding boundary value problems.

3.3.3.2. A vibrating rigid sphere in a stationary fluid. Consider a rigid sphere with the centre coinciding with the origin. The sphere vibrates harmonically along the x_3-axis with the velocity amplitude, U. The sound radiated by the sphere is described by the Helmholtz equation:

$$(\nabla^2 + k^2)P = 0. \tag{31}$$

The condition on the surface of the sphere is the equality of the normal (radial) components of the velocities of the surface, $v_x = U\cos\Theta$, and the fluid, u_x:

$$u_x = v_x = U\cos\Theta. \tag{32}$$

Eqs. (31) and (32), together with the radiation condition at infinity, determine a boundary value problem, particular solutions of which can be determined as [12, p. 318]:

$$P_n(x, \Phi, \Theta) = \begin{cases} \mathrm{e}^{\pm\mathrm{i}kx}/x, & n = 0, \\ h_n^{(1)}(kx)Y_n(\Theta, \Phi), & n = 1, 2, .., \end{cases} \tag{33}$$

where $h_n^{(1)}(kx)$ is a spherical Bessel function of the third kind and $Y_n(\Theta, \Phi)$ is a spherical surface harmonic. Note that the choice of $h_n^{(1)}(kx)$ is determined by the radiation condition at infinity.

As the boundary condition (Eq. (32)) contains only the dipole term, only the term with $n = 1$ will be different from zero in Eq. (33). Substitution of the known expressions for $h_1^{(1)}(kx)$ and $Y_1(\Theta, \Phi)$ in Eq. (33) leads to the following expression for the pressure field radiated by the sphere:

$$P(x, \Theta) = -A\,\frac{\mathrm{e}^{\mathrm{i}kx}}{kx}\left(1 + \frac{\mathrm{i}}{kx}\right)\cos\,\Theta, \quad A = \mathrm{const}. \tag{34}$$

The radial velocity component can be determined by means of Eq. (16) and can be written as

$$v_x = -\,\frac{\mathrm{i}A\mathrm{e}^{\mathrm{i}kx}}{\rho_0 c_0 (kx)^3}\,(2\mathrm{i} + 2kx - \mathrm{i}(kx)^2)\cos\,\Theta. \tag{35}$$

Substitution of the above equation into the boundary condition allows one to find the following expression for the complex amplitude A:

$$A = \frac{\rho_0 c_0 U \mathrm{e}^{-\mathrm{i}kR_0}(kR_0)^3}{2 - 2\mathrm{i}kR_0 - k^2R_0^2}, \tag{36}$$

which, after substitution into Eq. (34), gives an equation determining the pressure field radiated by the vibrating sphere:

$$P(x, \Theta) = -\,\frac{\rho_0 c_0 U (kR_0)\mathrm{e}^{\mathrm{i}k(x-R_0)}}{2 - 2\mathrm{i}kR_0 - (kR_0)^2}\left(\frac{R_0}{x}\right)^2 (kx + \mathrm{i})\cos\,\Theta. \tag{37}$$

Note that the above equation can be easily transformed to the expression for the velocity potential of a vibrating sphere [11, p. 286].

For the far field ($kx \to \infty$) of an acoustically small sphere ($kR_0 \ll 1$), Eq. (37) is reduced to

$$P(x, \Theta) = -\tfrac{1}{2}\rho_0 c_0 U (kR_0)^2\,\frac{R_0}{x}\,\mathrm{e}^{\mathrm{i}kx}\,\cos\,\Theta. \tag{38}$$

3.3.3.3. A stationary rigid sphere in a moving fluid. Consider a stationary sphere immersed in a variable velocity field as described in Section 3.3.2. The fluid moves back and forth with harmonic temporal dependence and a complex velocity amplitude, $-U$. (We use the amplitude with the negative sign to keep the same *relative* motion of the sphere and the fluid as in the previous example.) In the linear problem, all effects of the motion of the fluid on sound propagation are neglected. Therefore, this case is described by the same Helmholtz Eq. (31) and the only difference between this case and the previous one is in the boundary condition.

As the sphere is stationary, the total fluid velocity on the surface is zero. However, apart from the radiated sound field, there is an external velocity field in the fluid; thus, the condition of zero velocity on the surface must be written as follows:

$$v_x|_{x=R_0} = u_x^{\mathrm{rad}}|_{x=R_0} + u_x^{\mathrm{inc}}|_{x=R_0} = 0, \tag{39}$$

where u^{rad} is the velocity field of the sound wave radiated by the sphere and u^{inc} is the incident velocity field in the fluid. Note that such boundary conditions for the incident and radiated fields are used widely in solving linear problems of sound radiation and scattering. For example, a similar condition can be found in [9, p. 425].

Substitution of $u_x^{\mathrm{inc}} = -U \cos \Theta$ into this equation gives the condition for the radiated sound field on the surface of the sphere:

$$u_x^{\mathrm{rad}}|_{x=R_0} = U \cos \Theta, \tag{40}$$

which coincides with the boundary condition for the case of a vibrating sphere (Eq. (32)). As the boundary value problems in both cases are identical, the cases of a vibrating sphere in a stationary fluid and of a stationary sphere in a moving fluid will be clearly equivalent in terms of the radiated sound.

3.3.3.4. Prediction of the FW-H theory for the sound radiated by a vibrating sphere in a stationary fluid. Blake [6] used the FW-H theory to calculate the sound generated by a rigid sphere vibrating in a fluid with the velocity amplitude, U. He showed that the contribution of the monopole and dipole sources in the far field, $P_{\mathrm{mon}}(\mathbf{x})$ and $P_{\mathrm{dip}}(\mathbf{x})$, determined by the third and second terms in the right-hand part of Eq. (1), respectively, can be written as follows:

$$P_{\mathrm{mon}}(\mathbf{x}) = -\tfrac{1}{3}\rho_0 c_0 U (kR_0)^2 \frac{R_0}{x} \mathrm{e}^{\mathrm{i}kx} \cos \Theta, \tag{41}$$

$$P_{\mathrm{dip}}(\mathbf{x}) = -\tfrac{1}{6}\rho_0 c_0 U (kR_0)^2 \frac{R_0}{x} \mathrm{e}^{\mathrm{i}kx} \cos \Theta, \tag{42}$$

and the total field, $P_{\mathrm{sph}}(\mathbf{x})$, radiated by the sphere is

$$P_{\mathrm{sph}}^{\mathrm{FW\text{-}H}}(\mathbf{x}) = P_{\mathrm{mon}}(\mathbf{x}) + P_{\mathrm{dip}}(\mathbf{x}) = -\tfrac{1}{2}\rho_0 c_0 U (kR_0)^2 \frac{R_0}{x} \mathrm{e}^{\mathrm{i}kx} \cos \Theta. \tag{43}$$

Note that this formula coincides with Eq. (38) above and, therefore, the FW-H theory produces correct predictions for the sound wave radiated by the vibrating sphere.

3.3.3.5. Prediction of the FW-H theory for the sound radiated by a stationary sphere in a moving fluid. For the FW-H theory, however, the equivalence of the two situations does not stand. As described in Section 2, the velocity component v_j in the third term of Eq. (1) is measured with respect to a stationary observer. As a result, for the immovable sphere $P_{\mathrm{mon}}(\mathbf{x})$ vanishes. At the same time, the force on the fluid from the sphere must still be described by Eq. (42), since it is determined by *relative* motion of the sphere and the fluid around it. Therefore, the prediction of the FW-H theory for the pressure amplitude of the acoustic wave radiated by the stationary sphere in the variable velocity field is

$$P^{\mathrm{FW\text{-}H}}(\mathbf{x}) = P_{\mathrm{dip}}(\mathbf{x}) = -\tfrac{1}{6}\rho_0 c_0 U (kR_0)^2 \frac{R_0}{x} \mathrm{e}^{\mathrm{i}kx} \cos \Theta. \tag{44}$$

One can note that Eq. (44) differs from the correct prediction determined by Eq. (43) by a factor of 3. Therefore, the conclusion of Zinoviev and Bies [1] that the FW-H equation gives a wrong prediction for the acoustic wave radiated by a rigid sphere in a variable velocity field is correct.

3.4. A rigid sphere embedded in a flow

Fundamental problems of the FW-H theory can be demonstrated even more easily using another example. Let the sphere considered in Section 3.3 be *embedded* in the velocity field in such a way that *it is stationary with respect to the fluid but moving with the fluid with respect to a stationary observer*.

As there is no force acting upon the fluid from the sphere in this case, the second term in Eq. (1) vanishes. At the same time, the sphere moves with respect to a stationary observer and, therefore, the contribution of the third term in Eq. (1) is exactly the same as for the vibrating sphere (Eq. (41)). On the other hand, it can be easily proven by conventional methods of solution of boundary value problems that there will be no radiation from such a sphere.

Thus, the result of the FW-H equation contradicts the obvious prediction of the absence of radiation from a sphere which does not move with respect to the surrounding fluid flow.

4. Some aspects of the theoretical argument

As we mentioned in Section 1, in Zinoviev and Bies [1] we followed the logic of Curle [3] as close as possible. On this ground, many of the comments in Farassat [2] can be answered. Below we provide our response to these comments.

4.1. Justification of Eq. (5) of Zinoviev and Bies

Farassat [2] noted that "The authors start with a Kirchhoff-like formula for the density perturbation $\rho'(\mathbf{x}, t)$ Eq. (4), which they present without derivation of reference". This equation is, indeed, the sum of the solution of Lighthill's equation for a flow without boundaries and the Kirchhoff formula for the density perturbations. This is also the starting point of the analysis by Curle [3], who made reference to Stratton [13, p. 427] as the source of this equation, which provides the most general solution of the inhomogeneous Lighthill's equation. As Lighthill's equation is definitely applicable to a general flow field, we agree with Curle that Eq. (2.4) of his paper [3], which is also our Eq. (4), has a solid foundation in application to a general flow.

4.2. Method of integration

Farassat [2] commented that "Most of the algebraic manipulations involving integration by parts in the paper are confusing and not mathematically precise because the authors do not clearly indicate evaluation of the integrand at the retarded time". In response to this comment, we emphasize that our choice of the method of integration is also determined by Curle's logic. For example, Eqs. (5) and (6) of Zinoviev and Bies are also utilised by Curle as Eqs. (2.9) and (2.11).

Taking this into consideration, the explicit evaluation of the integrands as shown in Eq. (22) of Farassat [2] cannot affect our analysis and conclusions.

4.3. The surface divergence

Farassat [2] observed that "...it is not at all clear what the authors mean by the discontinuity of the functions F_i on the boundary S_v. The proper mathematical tool to study this problem is generalised functions... Also in differential geometry, the surface divergence has a clearly defined meaning. The authors' definition does not correspond to this definition".

Answering this comment, we point out that the formulation of the surface divergence of a vector field, which we used in Zinoviev and Bies [1], can be found in Korn [12, Eq. (5.6–4)]. It has the clear meaning of the scalar product of the normal unit vector to a surface S and the difference between the values of the vector field on both sides of S, which is represented by Eq. (9) of Zinoviev and Bies. Since inside the solid object, Lighthill's stress tensor is obviously zero, whereas it is not zero in general on the exterior side of the surface of the object, the vector fields determined by Eqs. (15) and (16) of Zinoviev and Bies are discontinuous on the surface. Note that we did not use the theory of generalised functions in our analysis, as Curle [3] did not base his argument on this mathematical formulation.

However, it is also possible to obtain our result (Eq. (34) of Zinoviev and Bies) without using the notion of the surface divergence, which we demonstrate below.

The theoretical argument in our article [1] differs from that of Curle [3] in only one aspect. According to Curle, integrals over the total volume, V_{tot}, of the fluid, can be reduced to integrals over the rigid surface, S, as follows:

$$\iiint_{V_{\text{tot}}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \iint_S n_i \frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\,\mathrm{d}S(\mathbf{y}), \tag{45}$$

$$\iiint_{V_{\text{tot}}} \frac{\partial}{\partial y_i}\left(T_{ij}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \iint_S n_i \frac{T_{ij}}{r}\,\mathrm{d}S(\mathbf{y}). \tag{46}$$

The notion of the surface divergence was used in Zinoviev and Bies to prove that, in fact, the right-hand parts of Eqs. (45) and (46) vanish, so that the equations take the form

$$\iiint_{V_{\text{tot}}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = 0, \tag{47}$$

$$\iiint_{V_{\text{tot}}} \frac{\partial}{\partial y_i}\left(T_{ij}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = 0. \tag{48}$$

In can be shown quite easily that utilising Eqs. (47) and (48) in place of Eqs. (45) and (46) in Curle's analysis leads to Eq. (34) of Zinoviev and Bies rather than to Eqs. (2.15) and (2.18) of Curle [3] and to the FW-H Eq. (1).

At the same time, we would like to draw readers' attention to a simpler way to prove the correctness of Eqs. (47) and (48), which we specified in Subsection 3.6 of Zinoviev and Bies. As the regions in the fluid where $T_{ij} \neq 0$ are finite [2], it is always possible to specify a closed surface, S_1, which will enclose all such regions. Due to the divergence theorem, the volume integrals can be represented as surface integrals over S_1:

$$\iiint_{V_{\mathrm{tot}}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \iint_{S_1} n_i \frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\,\mathrm{d}S(\mathbf{y}), \tag{49}$$

$$\iiint_{V_{\mathrm{tot}}} \frac{\partial}{\partial y_i}\left(T_{ij}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \iint_{S_1} n_i \frac{T_{ij}}{r}\,\mathrm{d}S(\mathbf{y}). \tag{50}$$

As $T_{ij} \equiv 0$ on S_1, the right-hand parts of Eqs. (49) and (50) vanish and the volume integrals take the form of Eqs. (47) and (48). Therefore, utilising the notion of the surface divergence is not necessary at all for reaching our conclusions. The surface divergence was used in Zinoviev and Bies to demonstrate that, if carried out correctly, the methodology of Curle [3] leads to a conclusion different from that stated in his article.

4.4. Converting the derivatives

It is stated in Section 3.0 of Farassat [1] that "The manipulations for converting the derivatives with respect to the source variable (of) the Lighthill stress tensor to derivatives with respect to the observer (variable) are not necessary". We would like to point out that this statement provides another proof for the correctness of our result.

In fact, a major part of Curle's analysis [3] is devoted to the transformation of the derivatives of the Lighthill's stress tensor T_{ij} with respect to $\mathbf{x}$ to the derivatives with respect to $\mathbf{y}$. Curle obtained his Eq. (2.15) (which can be reduced to Eq. (1) above) by using the following transformation:

$$\begin{aligned}\iiint_{V_{\mathrm{tot}}}\left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\right]\frac{\mathrm{d}\mathbf{y}}{r} = \frac{\partial^2}{\partial x_i \partial x_j}\iiint_{V_{\mathrm{tot}}}\left[\frac{T_{ij}}{r}\right]\mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\iint_S\left[\frac{n_j T_{ij}}{r}\right]\mathrm{d}S(\mathbf{y}) \\ + \iint_S\left[\frac{n_j}{r}\frac{\partial T_{ij}}{\partial y_j}\right]\mathrm{d}S(\mathbf{y}).\end{aligned} \tag{51}$$

If such a transformation were not necessary, it would, apparently, mean that the second and the third terms in Eq. (51) vanish. Substitution of the above equation without the surface integrals to Eq. (4) of Zinoviev and Bies (which is also the starting point of Curle's analysis) would immediately lead to Eq. (34) of Zinoviev and Bies, which is the main result of the article.

4.5. The linearised FW-H equation and its solution

In Section 1.0, Farassat [2] demonstrated the derivation of the linearised FW-H Eq. (5), which we re-write below for a stationary surface S determined by the condition $f = 0$:

$$\Box^2 p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \nabla\cdot[p'\mathbf{n}\delta(f)], \tag{52}$$

where p' is the acoustic pressure. We agree that this equation, indeed, correctly describes sound scattering by a rigid stationary surface in a linear approximation. The sound radiated by the volume distribution of Lighthill's sources, described by the first term in the right-hand part of the above equation, can be considered as the incident wave. At the same time, the second term determines the sound wave scattered by the surface.

However, the method of solving Eq. (52) used by Farassat [2] in application to the plane wave scattering by a rigid sphere does not appear to be mathematically justified. Consider the method of solution of this equation, which is known from textbooks on scattering (see, for example, [10, p. 16]). The solution of Eq. (52) can be represented as follows:

$$p'(\mathbf{x}) = p'_{\mathrm{inc}}(\mathbf{x}) + \frac{1}{4\pi}\iint_S p'(\mathbf{y})\,\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}S, \tag{53}$$

where the source term $p'(\mathbf{y})$ is not the total pressure measured on the surface S. Instead, the source term is determined by two following integral equations:

$$2p'_{\mathrm{inc}}(\mathbf{x}) = p'(\mathbf{x}) - \frac{1}{2\pi}\iint_S^{*} p'(\mathbf{y})\,\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}S, \tag{54}$$

$$4\pi\,\frac{\partial}{\partial n}\,p'_{\mathrm{inc}}(\mathbf{x}) = -\frac{\partial}{\partial n}\iint_S p'(\mathbf{y})\,\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}S, \tag{55}$$

where $(\mathbf{x}, \mathbf{y} \in S)$ and the asterisk denotes the Cauchy principal value. The above integral equations correspond to the boundary conditions on the surface for the acoustic pressure and its normal derivative, respectively.

Note that in Farassat [2] there is no indication of the necessity to formulate and solve the integral equations to determine the source term in the linearised FW-H equation. Moreover, in practical applications of the FW-H equation by other authors (see, for example, Howe [14, pp. 211 and 212]), the source term is also understood as the pressure measured at the surface S rather than as the solution of an integral equation. Therefore, the conventional method of solution of the linearised FW-H equation lacks mathematical foundation.

5. Conclusions

In our response, we acknowledge that the application of the FW-H theory to the case of a plane wave scattering by a rigid sphere in Zinoviev and Bies [1] contains an error caused by an unjustified assumption about the surface distribution of the dipole sources. We confirm that the predictions of the FW-H equation coincide with the correct result in this situation.

At the same time, we have shown that this coincidence is purely accidental. On the example of the reflection of a converging spherical wave from a rigid sphere, we have demonstrated that the FW-H equation leads to incorrect predictions for the reflected sound wave.

We have shown that the case of an immoveable sphere in a variable velocity field, considered in Zinoviev and Bies, has a clear origin, and that the FW-H equation does not lead to the correct prediction in this case. We have also shown that the FW-H equation fails to produce the correct result in a simple case of a rigid sphere embedded in a variable velocity field.

We have explained in more detail the statement of Zinoviev and Bies [1], that the logic of this paper follows that of Curle [3]. This answers a number of comments of Farassat [2] such as the justification of the starting point of our analysis (Eq. (4) of Zinoviev and Bies) and the choice of the method of integration.

We have shown that the surface divergence of a vector field discontinuous at a surface has a clear meaning related to the difference between the values of the vector field on both sides of the surface. We have also demonstrated that, as shown in Zinoviev and Bies, the main result of the paper can be obtained by direct evaluation of the volume integral without using the surface divergence.

In addition to the conclusions of Zinoviev and Bies, in this response we have shown that the conventionally used method of solution of the FW-H equation lacks mathematical foundation as it ignores the necessity to determine the source term by solving integral equations arising from the boundary conditions on the rigid surface.

In summary, in our response to Dr Farassat's comments, we have confirmed the main conclusion of Zinoviev and Bies, that the theory based on equations by Curle and Ffowcs Williams and Hawkings has significant theoretical problems which must be answered before any further use of this theory in engineering applications.

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