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Journal of Sound and Vibration 281 (2005) 835–847
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

# Axisymmetric oscillation of a viscous liquid covered by an elastic structure

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Received 7 February 2003; received in revised form 23 June 2003; accepted 4 February 2004
Available online 25 September 2004

## Abstract

If the free liquid surface of a viscous liquid is completely covered by an elastic structure, the damped natural frequencies of such a hydroelastic system are shifted to values larger than those obtained with a free liquid surface. In addition, the sloshing of the liquid is considerably reduced and exhibits, depending on the tension or stiffness of the covering elastic structure, smaller deflection amplitudes and drastically reduced liquid forces and moments participating in the dynamic behavior of the system. The hydroelastic frequencies of a viscous liquid in a circular cylindrical tank of moderate to large liquid height aspect ratio may be covered completely by a flexible membrane or an elastic plate, and exhibit an increase with increasing liquid height ratio $h/a$. In addition, the damping becomes stronger with the increasing $h/a$ and remains—as the frequency does—nearly constant from a certain height ratio onwards. This indicates that from a certain moderate liquid height ratio neither frequency nor damping does change considerably, since in the lower portion of the container the velocity distribution and its change remain quite small.

With the increase in the membrane tension parameter $T^* \equiv Ta/\rho v^2$, the magnitude of both natural frequency and the decay of the oscillations increase. The same happens for the increase of the stiffness parameter $D^* \equiv D/\rho v^2 a$. In addition, higher modes show larger damping.


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0022-460X/$ - see front matter 
doi:10.1016/j.jsv.2004.02.002

**Nomenclature**

| | |
|---|---|
| $A$, $A_0$, $D$ | coefficients |
| $A_n$ | coefficients in Eqs. (13)–(15) |
| $A_n^*$ | $\equiv A_n a^2/\nu$ |
| $a$ | radius of tank |
| $\bar{D}$ | flexural rigidity of the plate |
| $E$ | Young's modulus of the plate |
| $g$ | gravitational acceleration |
| $g^*$ | $\equiv g a^3/\nu^2$ |
| $h$ | liquid height |
| $I_0$ | modified Bessel function of order zero |
| i | imaginary unit |
| $J_m$ | Bessel function of the first kind of order $m$ |
| $k$ | constant in Eqs. (6)–(8) |
| $K$ | $\equiv ka$ |
| $K_n(S)$ | roots of transcendental equation (12) |
| $\bar{K}$ | distributed stiffness of plate |
| $M$ | number of division |
| $M_r$ | moment |
| $P_0^*$ | constant |
| $p$ | liquid pressure |
| $p_0$ | static liquid pressure |
| $r$, $\varphi$, $z$ | coordinate system |
| $S$ | $\equiv s a^2/\nu = \delta \pm \mathrm{i}\omega^*$ |
| $s$ | characteristic index |
| $T$ | tension of membrane |
| $T^*$ | $\equiv Ta/\rho\nu^2$ |
| $t$ | time |
| $u$, $w$ | velocity components of liquid |
| $V_r$ | shear force |
| $\beta^2$ | $\equiv a^2(\mu s^2 + \rho g)/T = (\mu^* S^2 + g^*)/T^*$ |
| $\bar{\delta}$ | plate thickness |
| $\eta$ | dynamic viscosity of liquid |
| $\mu$ | mass/unit area of membrane or plate |
| $\mu^*$ | $\equiv \mu/\rho a$ |
| $\nu$ | $\equiv \eta/\rho$ |
| $\bar{\nu}$ | Poisson's ratio of the plate |
| $\bar{\xi}, \bar{\varsigma}$ | displacement components of membrane or plate |
| $\rho$ | density of liquid |
| $\omega^*$ | $\mathrm{Im}(S) = \omega a^2/\nu$ |
| $\delta$ | $\mathrm{Re}(S) = \sigma a^2/\nu$ |

## 1. Introduction

The trend for thin and light structures, predominant in modern technology, leads to high flexibility of such systems. This is true for large-capacity liquid containers, such as propellant storage tanks or containers in airplanes, missiles, space vehicles, satellites or space stations. There are often strong interactions of propellants with the control system and the elastic structure, which usually endangers the integrity of the system and finally, of course, the success of a mission. Thus, the shifting of these—to instability leading—frequencies is usually the only way to remedy such disturbing and dangerous problems. This may in some cases be achieved by covering the free liquid surface with a flexible structural member, such as a membrane or a thin elastic plate of various possible boundary conditions. In this way, a hydroelastic system as treated here is showing frequencies away from the dangerous ones of the original system. It is, of course, mandatory to know the magnitude of these hydroelastic frequencies and assure the designer that they are located to a great extent above the control frequency. In recent years, some of the first research activities in this hydroelastic area, where a lot of experiments and analyses [1–23] have been performed for cylindrical tanks partially filled with frictionless liquid, emphasized the importance of such hydroelastic investigations. Some of them treated the interaction and influence of an elastic sidewall [2–5,9–15,17] with the free liquid surface behavior, while others [6–8] investigated the effect of an elastic container bottom on the sloshing of non-viscous liquid with a free surface. Similar investigations have been performed for rectangular tanks with an elastic liquid surface cover [24]. If the free liquid surface is completely covered by an elastic structure, the natural frequencies of the liquid–structure system shall yield the required increase of the frequencies [23], whose magnitude depends mainly on the tension or stiffness of the elastic coverage. There are, however, also hydrodynamic systems for which the free liquid surface is only partially covered with an elastic member. Results for the coupled frequencies of such systems are presented for a cylinder [25] and for a rectangular container by Bauer and Eidel [26]. In all these investigations, the liquid has been assumed to be incompressible and frictionless. Recently, the coupled frequencies of the liquid in a circular cylindrical tank with an elastic cover and filled with frictionless liquid has been determined by Bauer [23]. No results have been known for viscous liquids until recently. For a cylindrical container covered by an elastic structure such as a membrane or elastic thin plate and filled with viscous liquid, the coupled frequencies were determined for small liquid height ratios $h/a$ by Bauer and Chiba [27]. In this treatment, adhesive conditions at the container bottom were satisfied, while at the small sidewall area only the normal velocity condition has been observed. This seems to be justified for shallow containers. This analytical treatment yields approximate complex frequencies, of which the real parts describe the damping decay, while the imaginary parts represent the oscillation frequencies. The procedure is based on the fact that in a container of small aspect ratio $h/a$, say at least $h/a<1/2$, the friction at the sidewall can be neglected in comparison with that on the container's bottom and top. This assumption is based on the experimental and theoretical observation that the liquid motion takes place in the upper portion, thus creating a large velocity at the bottom and top of the tank, which represents an area of $2\pi a^2$, while the sidewall area is only $2\pi ah$. The smaller the value of $h$, the less the damping created by the sidewall. It was found that viscosity decreases the oscillation frequencies in comparison to the coupled hydroelastic frequencies of frictionless liquid and that a new phenomenon appears exhibiting for certain small liquid height ranges $h/a$ only aperiodic motion. With increasing angular and radial mode numbers, these aperiodic ranges of $h/a$ decrease. In addition, it was found that higher modes show larger damping. An increase in the membrane tension decreases the aperiodic region, while an increase in the mass of the membrane increases it.

In what follows, an investigation is presented of the coupled frequencies of a hydroelastic system consisting of a circular cylindrical tank filled with incompressible viscous liquid, the liquid surface of which is covered by an elastic element, such as a flexible membrane or an elastic thin plate with various boundary conditions. In this analysis, the liquid height ratio is considered large, containing the complete region of liquid motion. The lower portion of the liquid does not participate much in the motion of the liquid and behaves nearly as a rigid body. This indicates that the velocity distribution at the bottom is quite small and exhibits only very small and negligible values in the radial direction. It may therefore be considered to vanish in this approximation theory. The large magnitude of the sidewall requires, however, to satisfy all boundary conditions.

In the case where the surface is covered by an elastic plate, we may consider various boundary conditions. First of all, the plate may be clamped, or it may be simply supported or guided such that the rim of the plate may only be able to move up and down the wall of the cylinder. Another case may be just the freely floating plate on the liquid surface. The case of an elastically supported boundary, where the edge rotation would be opposed by spiral springs showing distributed stiffness ($\bar{K}$ moment per unit length) and its longitudinal motion up and down would be opposed by springs ($\bar{k}$ distributed force per unit length), could also be mentioned.

The case, treated here valid for large liquid height ratios $h/a$, will satisfy at the cylindrical wall all adhesive conditions, while at the container bottom only the axial velocity conditions are satisfied.

## 2. Basic equations and method of solution

A circular cylindrical container of radius $a$ (Fig. 1) is filled to a height $h>a$ with an incompressible and viscous liquid of density $\rho$ and kinematic viscosity $\nu = \eta/\rho$. The sidewall of the tank $r = a$ and the bottom of the tank at $z = -h$ are considered rigid, while the free surface at $z = 0$ is totally covered with a flexible membrane or an elastic plate. The plate may exhibit various attachments to the cylindrical wall. It may be clamped, simply supported, free, guided or it may be elastically supported. Assuming small velocities and displacements, the motion of the liquid–structure system satisfies for axisymmetric oscillations ($\partial/\partial\varphi \equiv 0$) the Stokes equations

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right], \tag{1}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] - g \tag{2}$$

and the continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{3}$$

These equations have to be solved with the sidewall boundary condition

$$u = w = 0 \quad \text{at } r = a \tag{4}$$

and the bottom boundary condition

$$u = w = 0 \quad \text{at } z = -h, \tag{5}$$

of which $u = 0$ has to be abandoned at $z = -h$. This is justified for $h>a$ since the motion of the liquid is mainly present in the immediate vicinity of the elastic surface cover (near $z = 0$). With $s = \sigma + \mathrm{i}\omega$

$$u(r, z, t) = U(r)\cosh[k(z+h)]\mathrm{e}^{st}, \tag{6}$$

$$w(r, z, t) = W(r)\sinh[k(z+h)]\mathrm{e}^{st}, \tag{7}$$

$$\bar{p}(r, z, t) = \bar{P}(r)\cosh[k(z+h)]\mathrm{e}^{st}, \tag{8}$$

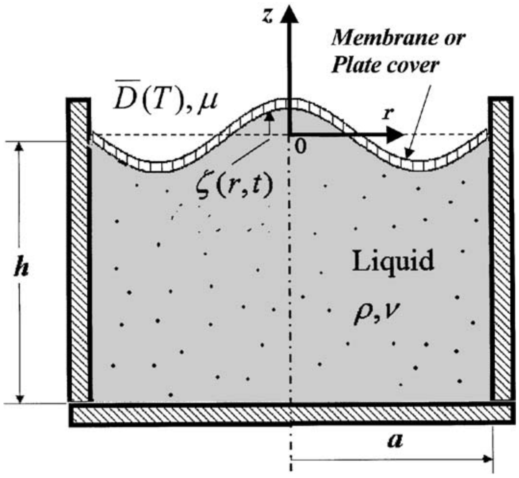


Fig. 1. Geometry and coordinate system.

where the pressure distribution

$$p(r,\ z,\ t) = p_0 - \rho g z + \overline{p}(r,\ z,\ t) \tag{9}$$

has been used. After the application of the vector operation "divergence" to the Stokes equations, we obtain with the use of the above continuity equation div $\vec{v} = 0$ for the pressure, the Laplace equation

$$\Delta \overline{p} = 0,$$

which renders the solution

$$\bar{P}\,(r) = \frac{\eta}{a} D J_0(kr). \tag{10}$$

Introducing these results into the Stokes equations and the continuity equation, we obtain

$$\frac{\mathrm{d}^2 U}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r} + \left[k^2 - \frac{1}{r^2} - \frac{s}{v}\right] U = -\frac{k}{a} D J_1(kr),$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}W}{\mathrm{d}r} + \left[k^2 - \frac{s}{v}\right] W = \frac{k}{a} D J_0(kr)$$

and

$$\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{U}{r} + kW = 0,$$

which exhibits the solutions ($K \equiv ka, S \equiv sa^2/v = \delta + \mathrm{i}\omega^*$)

$$U(r) = A J_1\left(\sqrt{K^2 - S}\frac{r}{a}\right) + D\frac{K}{S} J_1\left(K\frac{r}{a}\right),$$

$$W(r) = -A\frac{\sqrt{K^2 - S}}{K} J_0\left(\sqrt{K^2 - S}\frac{r}{a}\right) - D\frac{K}{S} J_0\left(K\frac{r}{a}\right).$$

The side wall conditions (4) yield

$$D = -A\frac{SJ_1(\sqrt{K^2 - S})}{KJ_1(K)} \tag{11}$$

and the transcendental equation for the determination of $K_n(S)$:

$$KJ_0(K)J_1\left(\sqrt{K^2 - S}\right) - \sqrt{K^2 - S}J_1(K)J_0\left(\sqrt{K^2 - S}\right) = 0. \tag{12}$$

With these results the velocity- and pressure distribution are given by

$$u(r,z,t) = \sum_{n=1}^{\infty} A_n\left\{J_1\left(\sqrt{K_n^2 - S}\frac{r}{a}\right) - \frac{J_1\sqrt{K_n^2 - S}}{J_1(K_n)}J_1\left(K_n\frac{r}{a}\right)\right\}\cosh\left[K_n\frac{(z+h)}{a}\right]e^{st}, \tag{13}$$

$$w(r,z,t) = -\sum_{n=1}^{\infty} A_n\left\{\frac{\sqrt{K_n^2 - S}}{K_n}J_0\left(\sqrt{K_n^2 - S}\frac{r}{a}\right) - \frac{J_1\left(\sqrt{K_n^2 - S}\right)}{J_1(K_n)}J_0\left(K_n\frac{r}{a}\right)\right\}\sinh\left[K_n\frac{(z+h)}{a}\right]e^{st} \tag{14}$$

and

$$\bar{p}(r,z,t) = -\frac{\eta}{a}\sum_{n=1}^{\infty} A_n\frac{SJ_1\left(\sqrt{K_n^2 - S}\right)}{K_nJ_1(K_n)}J_0\left(K_n\frac{r}{a}\right)\cosh\left[K_n\frac{(z+h)}{a}\right]e^{st}. \tag{15}$$

If the liquid free surface at $z = 0$ is covered by a flexible membrane, we have to solve the membrane equation

$$T\left[\frac{\partial^2\bar{\varsigma}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\varsigma}}{\partial r}\right] = \mu\frac{\partial^2\bar{\varsigma}}{\partial t^2} - \left(p - 2\eta\frac{\partial w}{\partial z}\right)_{z=0}, \tag{16}$$

while the boundary condition

$$\bar{\varsigma} = 0 \quad \text{at } r = a \tag{17}$$

and the compatibility condition

$$\frac{\partial\bar{\zeta}}{\partial t} = w \quad \text{at } z = 0. \tag{18}$$

If the surface cover is described as an elastic plate, we have to employ the plate equation

$$\bar{D}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]^2\bar{\varsigma} + \mu\frac{\partial^2\bar{\varsigma}}{\partial t^2} = \left(p - 2\eta\frac{\partial w}{\partial z}\right)_{z=0}, \tag{19}$$

which has to be solved [31] with either

$$\text{(a) clamped boundary}: \quad \bar{\varsigma} = 0 \ \text{ and } \ \frac{\partial \bar{\varsigma}}{\partial r} = 0 \quad \text{at } r = a, \tag{20}$$

$$\text{(b) simply supported boundary}: \quad \bar{\varsigma} = 0 \ \text{ and } \ M_r = 0 \quad \text{at } r = a, \tag{21}$$

$$\text{(c) free boundary}: \quad M_r = 0 \ \text{ and } \ V_r = 0 \quad \text{at } r = a, \tag{22}$$

$$\text{(d) guided boundary}: \quad \frac{\partial \bar{\varsigma}}{\partial r} = 0 \ \text{ and } \ V_r = 0 \quad \text{at } r = a, \tag{23}$$

$$\text{(e) elastically supported boundary}: \quad M_r = \overline{K}\frac{\partial \bar{\varsigma}}{\partial r} \ \text{ and } \ V_r = -\bar{k}\bar{\zeta} \quad \text{at } r = a. \tag{24}$$

In these equations, $T$ is the membrane tension, $\mu$ is the mass per unit area, $\bar{\varsigma}(r,t)$ is the elastic deflection in axial direction, while $\bar{D} = E\bar{\delta}^3/12(1-\bar{v}^2)$ is the flexural rigidity with $\bar{\delta}$ as the thickness of the plate, $\bar{v}$ its Poisson's ratio and $E$ the modulus of elasticity. The values of $V_r$ and $M_r$ are given by

$$M_r = -\overline{D}\left[\frac{\partial^2\bar{\zeta}}{\partial r^2} + \frac{\bar{v}}{r}\frac{\partial\bar{\zeta}}{\partial r}\right] \ \text{ and } \ V_r = -\overline{D}\frac{\partial}{\partial r}\left[\frac{\partial^2\bar{\zeta}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\zeta}}{\partial r}\right].$$

The edge rotation in the boundary condition (e) is opposed by torsional springs having a distributed stiffness $\overline{K}$ (moment per unit length), while the translation in the direction of $\bar{\zeta}$ is opposed by spring having a distributed stiffness $\bar{k}$ (force/unit length).

Actually, the displacement of the plate $\bar{\xi}$ in radial direction is in comparison to that in axial direction $\bar{\zeta}$ negligibly small. For this reason, the compatibility conditions are given by

$$u = 0 \quad \text{and} \quad \frac{\partial \bar{\zeta}}{\partial t} = w \quad \text{at} \quad z = 0, \tag{25}$$

of which the first condition yields the expression

$$\sum_{n=1}^{\infty} A_n \left\{ J_1\left(\sqrt{K_n^2 - S}\,\frac{r}{a}\right) - \frac{J_1\left(\sqrt{K_n^2 - S}\right)}{J_1(K_n)} J_1\left(K_n \frac{r}{a}\right) \right\} \cosh\left(K_n \frac{h}{a}\right) \mathrm{e}^{st} = 0. \tag{26}$$

If the liquid surface at $z = 0$ is totally covered by a flexible membrane, we obtain from the compatibility condition with Eq. (14) and the deflection $\bar{\varsigma}$ of the membrane

$$\bar{\zeta}(r,t) = \bar{\zeta}_0(r)\mathrm{e}^{st} = \sum_{n=1}^{\infty} \left\{ \bar{\zeta}_{n1} J_0\left(\sqrt{K_n^2 - S}\,\frac{r}{a}\right) + \bar{\zeta}_{n2} J_0\left(K_n \frac{r}{a}\right) \right\} \mathrm{e}^{st}, \tag{27}$$

$$\bar{\varsigma}_{n1} = -\frac{\sqrt{K_n^2 - S}}{SK_n} A_n^* \sinh\left(K_n \frac{h}{a}\right), \quad \bar{\varsigma}_{n2} = \frac{J_1\sqrt{K_n^2 - S}}{SJ_1(K_n)} A_n^* \sinh\left(K_n \frac{h}{a}\right). \tag{28}$$

The equation of the membrane is given by ($A_n^* \equiv A_n a^2/v$)

$$\frac{d^2\bar{\zeta}_0}{dr^2} + \frac{1}{r}\frac{d\bar{\zeta}_0}{dr} - \frac{\beta^2}{a^2}\bar{\zeta}_0 = \frac{\eta}{Ta}\sum_{n=1}^{\infty} A_n \frac{SJ_1\left(\sqrt{K_n^2 - S}\right)}{K_n J_1(K_n)} J_0\left(K_n \frac{r}{a}\right)\cosh\left(K_n \frac{h}{a}\right)$$
$$- \frac{2\eta}{Ta}\sum_{n=1}^{\infty} A_n \left\{\sqrt{K_n^2 - S}J_0\left(\sqrt{K_n^2 - S}\frac{r}{a}\right) - \frac{K_n J_1\left(\sqrt{K_n^2 - S}\right)}{J_1(K_n)} J_0\left(K_n \frac{r}{a}\right)\right\}\cosh\left(K_n \frac{h}{a}\right) - \frac{p_0}{T}, \tag{29}$$

and exhibits the solution $\left(\beta^2/a^2 \equiv (\mu s^2 + \rho g)/T, T^* \equiv Ta/\rho v^2\right)$

$$\bar{\zeta}_0(r) = A_0 I_0\left(\beta \tfrac{r}{a}\right) - \tfrac{1}{T^*}\sum_{n=1}^{\infty} A_n^* \frac{SJ_1(\sqrt{K_n^2 - S})(1 + 2k_n^2/S)}{K_n J_1(K_n)\left(K_n^2 + \beta^2\right)}\cosh\left(K_n \frac{h}{a}\right) J_0\left(K_n \frac{r}{a}\right)$$
$$+ \tfrac{2}{T^*}\sum_{n=1}^{\infty} A_n^* \frac{\sqrt{K_n^2 - S}\cosh(K_n h/a)}{\left(K_n^2 - S + \beta^2\right)} J_0\left(\sqrt{K_n^2 - S}\frac{r}{a}\right) + P_0^*. \tag{30}$$

The boundary condition (17) gives, i.e.

$$A_o I_0(\beta) - \tfrac{1}{T^*}\sum_{n=1}^{\infty} A_n^* \frac{SJ_1(\sqrt{K_n^2 - S})(1 + 2K_n^2/S)}{K_n J_1(K_n)\left(K_n^2 + \beta^2\right)}\cosh(K_n \frac{h}{a}) J_0(K_n)$$
$$+ \tfrac{2}{T^*}\sum_{n=1}^{\infty} A_n^* \frac{\sqrt{K_n^2 - S}\cosh(K_n h/a)}{\left(K_n^2 - S + \beta^2\right)} J_0\left(\sqrt{K_n^2 - S}\right) + P_0^* = 0. \tag{31}$$

Comparing the deflection $\bar{\zeta}_0(r)$, i.e. Eq. (30) with that obtained from the compatibility conditions (27) and (28) yields

$$A_0 I_0\left(\beta \tfrac{r}{a}\right) + \sum_{n=1}^{\infty} A_n^* \left\{\frac{2\cosh(K_n h/a)}{T^*\left(K_n^2 - S + \beta^2\right)} + \frac{\sinh(K_n h/a)}{SK_n}\right\}\sqrt{K_n^2 - S}J_0\left(\sqrt{K_n^2 - S}\tfrac{r}{a}\right)$$
$$- \sum_{n=1}^{\infty} A_n^* \frac{J_1(\sqrt{K_n^2 - S})}{K_n J_1(K_n)}\left\{\frac{2K_n^2 + S}{T^*(K_n^2 + \beta^2)}\cosh\left(K_n \frac{h}{a}\right) + \frac{K_n}{S}\sinh\left(K_n \frac{h}{a}\right)\right\} J_0\left(K_n \frac{r}{a}\right) + P_0^* = 0. \tag{32}$$

The Eqs. (26) and (32) represent two conditions as functions of the radial variable $r/a$ and Eq. (31) one condition at $r = a$, and may be satisfied for a finite number of $r/a = \lambda/M$ in the range $0 \leqslant r/a \leqslant 1$.

We satisfy these equations at $r/a = \lambda/M$ with λ = 0, 1, 2, …, (M−1), i.e. at a finite number of points $r/a$. This yields for Eqs. (26), (31) and (32) the expressions

$$\sum_{n=1}^{2M-2} A_n^* \left\{ J_1\left(\sqrt{K_n^2 - S}\frac{\lambda}{M}\right) - \frac{J_1\left(\sqrt{K_n^2 - S}\right)}{J_1(K_n)} J_1\left(K_n \frac{\lambda}{M}\right) \right\} \cosh\left(K_n \frac{h}{a}\right) = 0 \qquad (26')$$

for $\lambda = 1, 2, \ldots, (M-1)$.

It may be noticed that $\lambda = 0$ and $\lambda = M$ are satisfied identically. These are $(M-1)$ algebraic equations in $A_n^*$. Eq. (31) is a single equation, while Eq. (32) is represented as

$$A_o I_0\left(\beta \frac{\lambda}{M}\right) + \sum_{n=1}^{2M-2} A_n^* \left\{ \frac{2\cosh(K_n h/a)}{T^*(K_n^2 - S + \beta^2)} + \frac{\sinh(K_n h/a)}{S K_n} \right\} \sqrt{K_n^2 - S} J_0\left(\sqrt{K_n^2 - S}\frac{\lambda}{M}\right)$$

$$- \sum_{n=1}^{2M-2} A_n^* \frac{J_1(\sqrt{K_n^2 - S})}{K_n J_1(K_n)} \left\{ \frac{2K_n^2 + S}{T^*(K_n^2 + \beta^2)} \cosh\left(K_n \frac{h}{a}\right) + \frac{K_n}{S} \sinh\left(K_n \frac{h}{a}\right) \right\} J_0\left(K_n \frac{\lambda}{M}\right) + P_0^* = 0$$

$$(32')$$

for $\lambda = 0, 1, 2, \ldots, M$.

These are $M+1$ algebraic equations in the constants $A_0$, $A_n^*$ and $P_0^*$. The Eqs. (26) and (32), together with Eq. (31), are thus reduced to a homogeneous system of $2M$ algebraic equations. The solution of these $2M$ equations requires the sum of the above series to run from $n = 1$ to $n = 2M - 2$ to be able to describe the system of equations of $2M$ unknowns for $A_0$, $A_n^*$ and $P_0^*$ $(n = 1, 2, \ldots, 2M - 2)$.

The velocity distribution and pressure distribution may be obtained from Eqs. (13)–(15). From this, we could easily obtain the liquid forces and liquid moments by integration over the container surfaces, as has been performed in similar other cases [29]. Such computations have been omitted here.

They must be satisfied together with Eq. (12). The existence of a non-trivial solution requires the determinant of the coefficients matrix to be zero. In the process of finding the zero determinant, we can get parameters for a damping δ and for frequency $\omega^*$ as complex values of $S = sa^2/\nu = \delta \pm i\omega^*$, which is contained in the matrix components. To find $S$ which makes the determinant zero, $S_{sol}$, we represent diagrams of real and imaginary parts of the determinant with contour lines in a predicted $S$ space where $S_{sol}$ supposed to exist. From these two diagrams, we can find a solution $S_{sol}$ as a cross point of zero contour lines of the real and imaginary parts of the determinant.

Plate cases may be treated in a similar way, but are omitted here.

## 3. Numerical evaluations and conclusions

Some of the above-obtained analytical results have been evaluated numerically and are presented here. The numerical evaluations have been restricted to a membrane cover (Fig. 1) in axisymmetric motion. In this case, vibration characteristics of the present viscous liquid–membrane coupled system are governed by the parameters: tension parameter $T^* \equiv Ta/\rho v^2$, gravitational parameter $g^* \equiv ga^3/v^2$, density ratio parameter $\mu^* \equiv \mu/\rho a$, liquid height ratio $h/a$, and the vibration mode ($m = 0$, $n$). In the numerical calculations, the number of unknown parameters $A_n$ was taken to be 20 to obtain reliable engineering data. It is to be noted here that there are two kinds of roots $K_n$ for transcendental equation (12), one is $\mathrm{Re}(K_n) > 0$ and $\mathrm{Im}(K_n) > 0$ type, the other is $\mathrm{Re}(K_n) > 0$ and $\mathrm{Im}(K_n) < 0$ type, both of which have to be used. The given results are especially of importance and of approximate validity for large liquid height ratios $h/a$, for which the liquid in the lower part of the container behaves like a rigid body and its motion takes place immediately below the membrane, penetrating the liquid only to a depth of about one wavelength. Therefore, the contribution of the lower part of the liquid (near the bottom) to the frequency and damping is of a minor and negligible effect.

In previous investigations [28,29] for a viscous liquid in a rigid container with a free liquid surface, the results show the important fact that, for small liquid height $h/a$, the liquid motion exhibits only an aperiodic motion, if disturbed. The decrease of the liquid surface tension

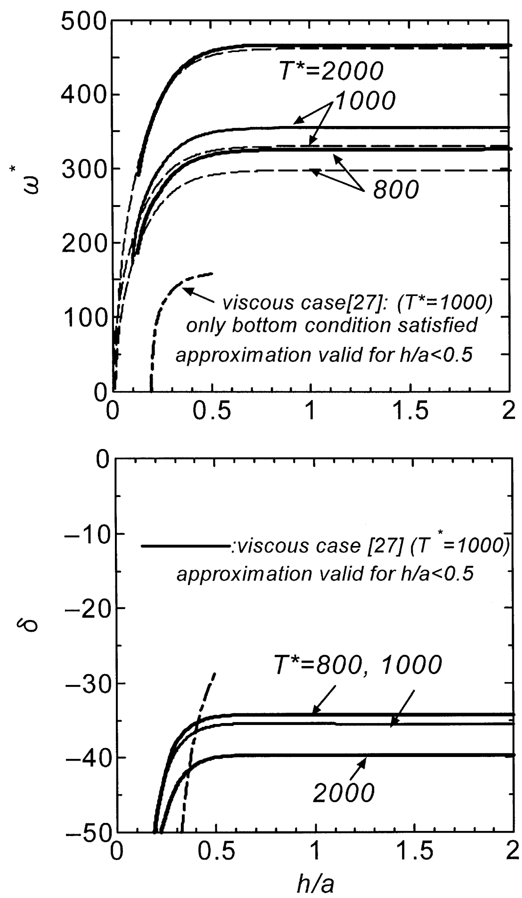


Fig. 2. Coupled complex frequencies for axisymmetric motion $m = 0$ with liquid height $h/a$; $T^* = 800, 1000, 2000$, $g^* = 10^4$ and $\mu = 10^{-2}$; —— viscous; $1 \ll h/a$, - - - -, non viscous —·—, viscous [27] $h/a \ll 1$ ($T^* = 1000$).

$\sigma^* \equiv \sigma a/\rho v^2$ increases this aperiodic region and decreases the decay magnitude and oscillation frequency for $h/a$ values above the aperiodic range. In addition, the aperiodic region decreases with increasing gravity parameter $g^* \equiv ga^3/v^2$, while in the oscillation region the decay magnitude as well as the oscillation frequency increase. Higher modes have been shown to exhibit significantly stronger damping.

If the free liquid surface is covered by a flexible membrane, we have found that the range of aperiodicity is much larger than that for a liquid without such a cover. These results are presented for small liquid height ratios $h/a < 1/2$ in Ref. [27], where for $m = 0$ the fictive liquid with $g^* \equiv ga^3/v^2 = 10^4$and $T^* = 10^3$ has been chosen.

In the present numerical evaluation, the tension parameter was chosen to be $T^* = 800$, 1000 and 2000, while the gravity parameter $g^* \equiv ga^3/v^2 = 10^4$ and the density parameter $\mu^* = 0.01$. In Fig. 2, we represent for these cases the coupled frequency $\omega^* \equiv \text{Im}(S)$ and $\delta \equiv \text{Re}(S)$ both as functions of the liquid height ratio $h/a$ ($h/a \leqslant 2.0$). In addition, the coupled frequency for frictionless liquid is presented as a dashed line (– – –). The dash–dotted results ($T^* = 1000$) are those in which only the normal boundary condition at the sidewall is satisfied and all adhesive boundary conditions at the container bottom are observed [27]. This solution exhibits only, as already mentioned above, a validity for small $h/a$ values. We notice that with increasing liquid height ratio $h/a$ the coupled frequency increases, reaching soon, with increasing $h/a$, a magnitude with very little change. The same is true for the decay magnitude $\delta$. With the increase of $h/a$ the damping becomes slightly stronger in the range of validity (hardly visible in the figure). The increase in the tension parameter $T^*$ exhibits for the coupled frequency $\omega^*$ increased values, while the decay magnitude $\delta$ the results show also increased magnitude. This means that for increasing membrane tension $T^*$ the damping of the liquid is stronger. In addition, we may note that for lower $T^*$ the coupled frequency of frictionless liquid rapidly approaches the coupled frequency of viscous liquid.

In comparison with the results of a viscous liquid with a free liquid surface in axisymmetric motion [30], on the addition of an elastic membrane covering the total free surface area we find increased damped oscillation frequency $\omega^* \equiv \text{Im}(S)$ and a decay magnitude $|\delta| \equiv |\text{Re}(S)|$, which is considerably higher than that for the free liquid surface case. In addition, the increase of the

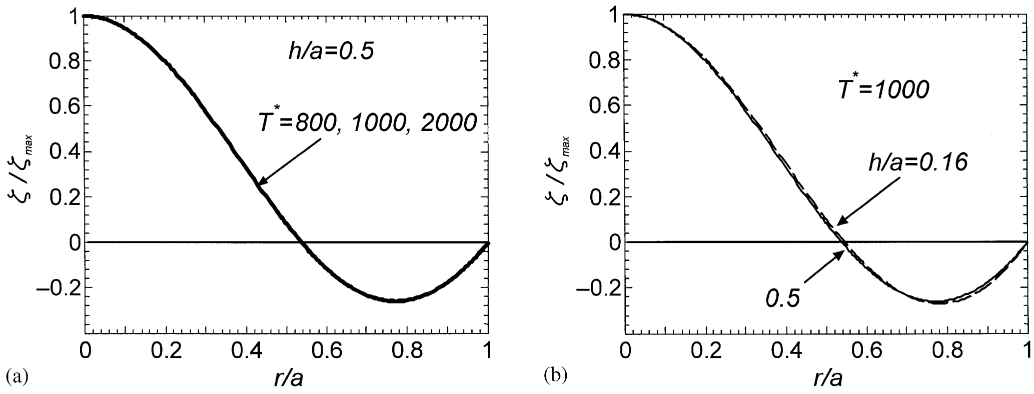


Fig. 3. Mode shape for $g^* = 10^4$, $\mu^* = 10^{-2}$, (a) $T^* = 800$, 1000, 2000, $h/a = 0.5$; (b) $T^* = 1000$, $h/a = 0.16, 0.5$.

damping with growing liquid height ratio $h/a$ is much reduced. It exhibits in the here presented range only a slight increase of $|\delta|$.

In Fig. 3, we observe the fundamental mode shape for $T^* = 10^3$, $g^* = 10^4$, $\mu^* = 10^{-2}$ and a liquid height ratio $h/a = 0.5$ (a) and the effect of liquid height ratio $h/a = 0.16, 0.5$ when $T^* = 1000$ (b).

## Acknowledgments

The authors would like to thank Dr. W. Eidel, Institut für Raumfahrttechnik, Universität der Bundeswehr München, for his helpful suggestions in the numerical method.

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