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A reduced eigenvalue method for broadband analysis of a structure with vibration absorbers possessing rotatory inertia

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Abstract

Multi-tuned vibration absorbers are attractive in reducing vibration and noise over a broadband. Here, an efficient and robust dynamic reanalysis algorithm is presented for predicting the dynamic response of a base structure to which are attached multiple absorbers. The 'reduced eigenvalue method' uses the modal analysis results of the base structure without absorbers, computed just once, to obtain the response of the modified structure. The method presented has two salient features. One is that the resonances of the modified structure are provided, and can be summed directly to estimate broadband measures of the dynamic response. The other is that rotatory inertia of the absorbers is properly captured, which has been found from experiment to be significant when an absorber is attached to a point on the base structure undergoing rotation. The method is contrasted with the impedance-based approaches. The method is used to model a three-dimensional absorber on a structure, which has been built, and is shown to correlate with experiment and full-scale finite element analysis.

1. Introduction

Attachment of vibration absorbers to a vibrating plate or shell structure is an effective passive approach for reducing vibration and/or radiated sound power. The use of a large number of

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distributed absorbers and the desire to evaluate narrow or broadband responses requires an efficient method for dynamic analysis. Efficiency is even more important if, as is usually necessary, the absorber parameters must be optimized in an iterative process. An efficient analysis method is presented here, for harmonically excited structures. A single computation based on the modal response of the base structure without any absorbers, is used repeatedly to compute the response of the structure with absorbers. The eigenanalysis of the base structure is independent of frequency and absorbers. Further efficiency is achieved by using a reduced order model for the absorbers through assumed deformation modes. A key aspect of the method is that rotatory inertia of an absorber is taken into account. Motivation for this came from experiments, where it was noticed that theory and experiment did not match well when an absorber was attached to a point on the structure that was undergoing significant rotation.

Frahm in 1911 [1] and Ormondroyd and Den Hartog in 1927 [2] presented the original theory on a single tuned absorber attached to a single-degree-of-freedom system to control response within a narrow frequency band. Since then, researchers have studied the use of multiple tuned absorbers attached to multidegree-of-freedom systems [3,4]. Numerical methods for analysis with multiple absorbers, the focus of this work, may be broadly categorized as 'impedance' methods or 'eigenvalue methods'. In impedance methods, the effect of an absorber is treated as an impedance (frequency-dependent force) and equations of motion are solved for the velocities at the absorber attachment points as a function of frequency. In impedance methods, peaks in the response, which are summed to estimate broadband response, are not directly available. The other category of methods may be referred to as 'eigenvalue methods' where eigenvalues and eigenvectors (modes) are recalculated for the structure with absorbers. Broadband response is more easily estimated in this class of methods where peaks occur at the natural frequencies of the modified structure with absorbers.

In both impedance methods and eigenvalue methods, efficiency is achieved by assuming that the modes of the base structure without absorbers continue to provide basis functions that represent the response with absorbers attached. Thus, an eigenvalue analysis of the base structure is done only once, and is independent of absorber geometry. This is a valid assumption for light-weight absorbers attached to the structure at a point. In this paper and certain previous publications, the assumption has been verified by comparison to a full-scale finite element model as well as by experiment.

Among impedance methods, Neubert's book [5] provides a good summary of the different formulations. Level et al. [6] present an efficient technique for inverting the impedance matrix at harmonic excitation frequencies. In Klasztorny's approach [7], evaluation of coupling between absorbers requires a complex eigenvalue analysis of the full structure, thus limiting its usefulness for repeated reanalysis of large models. A more practical method of analysis of structures with several tuned absorbers is given by Hamill and Andrew [8] and by Kitis et al. [9]. A modified version of their approach is used by Constans et al. [10–12]. Kitis et al. [9] assemble the absorber stiffness and mass matrices directly into the existing (base structure) mass and stiffness matrices, but truncate the higher modes of the combined structure. The number of degrees of freedom is then fixed, and from there, the method treats modifications in the same manner as the impedance method. Huang et al. [13] developed a method for extending the basis shapes of the undamped, unmodified structure to include additional degrees of freedom. A static analysis is performed to obtain Ritz vectors that are used to find approximate eigenvalues and eigenvectors for the modified structure. Rice [14] adds the structural modifications as additional degrees of freedom to the dynamic stiffness after truncating the higher-frequency modes, creating a reduced dynamic stiffness matrix. This frequency-dependent matrix is then inverted at each frequency to find the harmonic receptance of the modified structure.

The method proposed here falls in the category of the ‘reduced eigenvalue approach’, where additional stiffness and mass matrices $\Delta\mathbf{k}$, $\Delta\mathbf{m}$, representing the absorbers are coupled to the already available modal response of the base structure. Modal characteristics of the modified structure are found by solving a reduced-size eigenvalue problem. This approach has been discussed in the literature with certain limitations. Discussion on static structural analysis is a simpler case as no additional degrees of freedom are involved (see Ref. [15] for a review). In this paper, a general technique is given for deriving $\Delta\mathbf{k}$, $\Delta\mathbf{m}$ matrices, and their subsequent integration with the modal quantities of the base structure. Importantly, rotatory inertia effects of the attachments are treated in a general manner and a technique requiring smaller dimensional $\Delta\mathbf{k}$, $\Delta\mathbf{m}$ matrices is presented. Here too, it is assumed that the modes of the base structure continue to represent response with absorbers attached—however, unlike the impedance approach, additional degrees of freedom are introduced to properly describe rotatory inertias and couple these to the base structure. Higher modes of the structure are not truncated in this approach. It is also shown as to why the reduced eigenvalue approach is more attractive than the impedance approach for evaluation of broadband frequency response.

A few other techniques can be used to analyze a structure with attachments. The component mode synthesis approach is often used for large components which have been independently analyzed for their modal response before combining them. However, this approach is unwieldy and inefficient for treatment of small low-impedance absorbers attached to the base structure at one point. Moreover, large numbers of modes are needed in the component mode synthesis method to obtain accuracy beyond the first few resonances. While several papers exist in the literature, we refer to a recent paper on this [16].

The various sections in this paper are arranged as follows. Basic analysis using mode superposition of the base structure without absorbers is first presented. Following this, the impedance method and difficulty in evaluating broadband response by this class of methods is discussed. Then, the reduced eigenvalue method of this paper is presented in Sections 5 and 6. This method is applied to a simple example, and then generalized to treat a three-dimensional absorber which is then built for experimental purposes. Applications are given in Sections 7–9. In Section 9, correlation with a full-scale NASTRAN model of the structure with absorbers and experiment is given.

2. Analysis of base structure (without absorbers)

The first step is to determine the eigenvalues and eigenvectors of the unmodified structure. We denote $\mathbf{\Phi}_0$ = matrix whose columns are eigenvectors and λ_0 = diagonal matrix whose elements are eigenvalues. A natural frequency in rad/s is obtained from the eigenvalue as $\omega = \sqrt{\lambda}$. Dimension of the matrix $\mathbf{\Phi}_0$ is (number of degrees of freedom, number of modes in the basis).

Modal information can be found from a finite element model or from experiment. Here we use finite elements to determine modal response. The basic equations for this are given below.

Equations of motion of the forced vibration for a finite element representation of a base strucure with structural damping are

$$-\omega^2\mathbf{m}_0\mathbf{X} + [\mathbf{k}_0 + \mathrm{i}\eta\mathbf{k}_0]\mathbf{X} = \mathbf{F}_0, \tag{1}$$

where $\mathbf{m}_0, \mathbf{k}_0, \eta, \mathbf{F}_0$, and $\mathbf{X}$ are the mass and stiffness matrices, the material loss factor, and the complex forcing and response amplitude vectors, respectively, assuming harmonic excitation. If the forcing vector and damping are set to zero, the normal modes can be found by solving the eigenvalue problem

$$\mathbf{k}_0\mathbf{\Phi}_0 = \lambda_0\mathbf{m}_0\mathbf{\Phi}_0. \tag{2}$$

The eigenvectors satisfy

$$\mathbf{\Phi}_0^{\mathrm{T}}\mathbf{k}_0\mathbf{\Phi}_0 = \lambda_0, \quad \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{m}_0\mathbf{\Phi}_0 = \mathbf{I}. \tag{3}$$

Using mode superposition, the forced response of the structure can be given as

$$\mathbf{X} = \sum_{j=1}^{m} q_j \Phi_0^j = \mathbf{\Phi}_0\mathbf{q}, \tag{4}$$

where $\mathbf{q}$ is a vector of modal 'participation factors' or modal 'coordinates' given by

$$q_j = \frac{\Phi_0^{j\mathrm{T}}\mathrm{F}}{\left[-\omega^2 + (1+\mathrm{i}\eta)\lambda_0^j\right]}, \quad j = 1, \ldots, m. \tag{5}$$

At the kth resonance, whence $\omega = \omega_k$, and $\lambda^k = {\omega_k}^2$, we have $q_j = \Phi_0^{j\mathrm{T}}\mathbf{F}/(\mathrm{i}\eta\lambda_0^j)$. Other quantities such as kinetic energy and radiated sound power can now be computed.

3. Analysis of modified structure by the impedance method

As discussed in the introduction, a few different methods exist for dynamic analysis of the structure with vibration absorbers attached to it. Of these, the impedance method and the reduced eigenvalue method are most attractive, since in each of these Eq. (2) is solved only once. We first discuss the impedance method (see Ref. [12] for further details). The reanalysis problem is formulated in terms of added impedances as

$$-\omega^2\mathbf{m}_0\mathbf{X} + [\mathbf{k}_0 + \mathrm{i}\eta\mathbf{k}_0]\mathbf{X} = \mathbf{F}_0 = \mathbf{F}_{\mathrm{in}} - \mathrm{i}\omega\mathbf{z}\mathbf{X}, \tag{6}$$

where $\mathbf{F}_{\mathrm{in}}$ is the forcing vector, and $\mathbf{z}$ is the local impedance matrix of the modification. The impedance matrix is diagonal if each modification is independent and discrete as is the case with simple spring–mass absorbers. For example, the impedance for a simple mass m takes the expression $z = \mathrm{i}\omega m$, and for a spring–mass system with parameters k, m it takes the form $z = \mathrm{i}\omega mk/(k - m\omega^2)$. Replacing $\mathbf{F}$ by $\mathbf{F}_{\mathrm{in}} - \mathrm{i}\omega\mathbf{z}\mathbf{X}$, Eqs. (4) and (5) yield

$$\mathbf{X} = \mathbf{\Phi}_0\left[-\omega^2 + (1+\mathrm{i}\eta)\lambda_0\right]^{-1}\mathbf{\Phi}_0^{\mathrm{T}}(\mathbf{F}_{\mathrm{in}} - \mathrm{i}\omega\mathbf{z}\mathbf{X}). \tag{7}$$

Defining a diagonal matrix $\mathbf{A} = \left[-\omega^2 + \lambda_0(1 + \mathrm{i}\eta)\right]^{-1}$, we can write the local solution

$$\mathbf{X}_z = \left[\mathbf{I} + \mathrm{i}\omega\mathbf{\Phi}_z\mathbf{A}\mathbf{\Phi}_z^{\mathrm{T}}\mathbf{z}_z\right]^{-1}\mathbf{\Phi}_z\mathbf{A}\mathbf{\Phi}_0^{\mathrm{T}}\mathbf{F}_{\mathrm{in}}, \tag{8}$$

where $\mathbf{z}_z$ is the matrix of impedances and $\mathbf{\Phi}_z$ is the matrix of eigenvectors corresponding only to (non-zero) impedance locations. Solution to Eq. (8) gives the response only at the impedance locations, $\mathbf{X}_z$. In Eq. (8), only a small $p \times p$ matrix, were p is the number of impedance (or absorber) locations, is inverted for each desired frequency. Response of the modified structure at a general degree of freedom (as opposed to where an absorber is attached) is obtained by

$$\mathbf{X} = \Phi_0\mathbf{q}_z = \Phi_0\mathbf{A}\Phi_0^{\mathrm{T}}(\mathbf{F}_0 - \mathrm{i}\omega\mathbf{z}\mathbf{X}_m), \tag{9}$$

where $\mathbf{X}_m$ is the vector of $\mathbf{X}_z$ found through the equation augmented with zero values at the zero impedance locations, and $\mathbf{q}_z$ is the vector of modal coordinates of the modified structure.

The computational procedure may be summarized as follows. Given a set of absorbers with known locations and parameters, $\mathbf{z}$ is first defined. Then, for the specified frequency ω, Eqs. (8) and (9) are solved to obtain the displacement amplitude of the base structure $\mathbf{X}(\omega)$. Velocities are obtained from $\dot{\mathbf{X}} = \mathrm{i}\omega\mathbf{X}$. Other quantities such as kinetic energy of the base structure are readily determined from the velocities.

4. A disadvantage of the impedance method for estimating broadband response

Two main difficulties exist with the impedance method. One is the derivation of expressions for impedance, $\mathbf{z}_z$, that incorporate rotatory inertia of the absorbers (as a result of base rotation). The other difficulty is as follows. The impedance method yields the response at a specified frequency ω. Peak values of kinetic energy or other performance metric which occur at resonance frequencies not known a priori are not easily determined. The kinetic energy must be calculated at enough discrete frequencies that the peaks (or sum of peaks or an integral measure) over the broadband are accurately captured. The following figure illustrates the difficulty just mentioned. The kinetic energy of the base structure (shown in Fig. 1) is computed and plotted at various frequencies for the given resolution. Only a single peak is included in the broadband for illustration. Evaluation of kinetic energy at equal increments in frequency misses this peak value. For low structural damping as is generally the case, extremely small steps must be used, and even this may not be accurate. Noting that small increments means more computation, the problem of determining broadband response now becomes evident. This problem has not received much attention in the past as absorbers were used to target only a fixed frequency. Use of distributed tuned absorbers for broadband energy/sound reduction has exacerbated the problem of determining multiple response peaks.

We have attempted, with moderate success, a variable step frequency sweep with a response-dependent resolution, in conjunction with peak refinement based on a Golden section search strategy. However, there is no way of avoiding repeated evaluations of Eqs. (8) and (9). A more robust method is to involve the direct calculation of the modified structure's resonance frequencies, as discussed next.

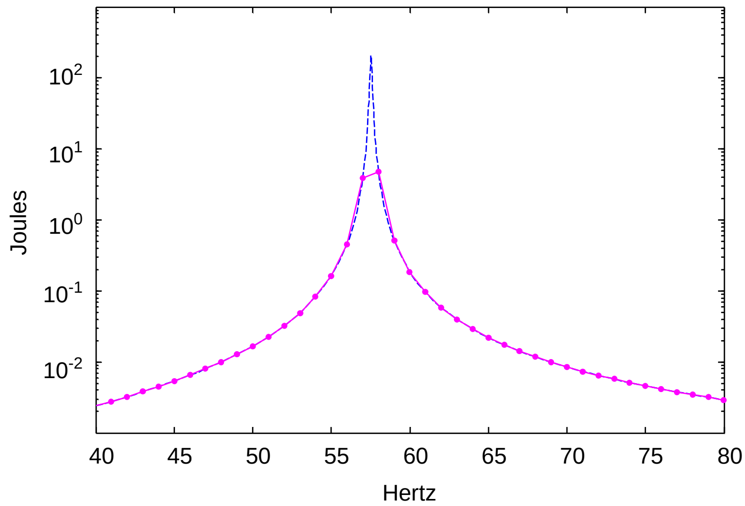

Fig. 1. A constant resolution sweep missing a peak: - - - , Peak with low damping; —●—, constant resolution sweep.

5. Reduced eigenvalue reanalysis method

As before, let $\mathbf{M}_0$ and $\mathbf{K}_0$ refer to the mass and stiffness matrices of the base structure without absorbers. An absorber is described by its own mass and stiffness matrices $\mathbf{M}_{\text{abs}}$ and $\mathbf{K}_{\text{abs}}$. Some degrees of freedom of these matrices coincide (are shared) with those of base structure where they are attached, while other degrees of freedom are independent. Thus, the modification mass and stiffnesses of the absorbers may be partitioned as

$$\mathbf{M}_{\text{abs}} = \begin{bmatrix} \Delta\mathbf{m} & \mathbf{m}_a \\ \mathbf{m}_a^{\mathrm{T}} & \mathbf{m}_z \end{bmatrix}, \quad \mathbf{K}_{\text{abs}} = \begin{bmatrix} \Delta\mathbf{k} & \mathbf{k}_a \\ \mathbf{k}_a^{\mathrm{T}} & \mathbf{k}_z \end{bmatrix}, \tag{10}$$

where $\Delta\mathbf{m}$ is the added mass matrix at the shared degrees of freedom, $\mathbf{m}_z$ is the added mass matrix at the new degrees of freedom, $\mathbf{m}_a$ is the coupling mass matrix, with similar descriptions for the stiffness submatrices. The modification element matrices are assembled into the base structure's mass and stiffness matrices as

$$-\omega^2 \begin{bmatrix} \mathbf{m}_0 + \Delta\mathbf{m} & \mathbf{m}_a \\ \mathbf{m}_a^{\mathrm{T}} & \mathbf{m}_z \end{bmatrix} \begin{Bmatrix} \mathbf{X}_0 \\ \mathbf{X}_z \end{Bmatrix} + \begin{bmatrix} (1+i\eta)\mathbf{k}_0 + \Delta\mathbf{k} & \mathbf{k}_a \\ \mathbf{k}_a^{\mathrm{T}} & \mathbf{k}_z \end{bmatrix} \begin{Bmatrix} \mathbf{X}_0 \\ \mathbf{X}_z \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_0 \\ 0 \end{Bmatrix}. \tag{11}$$

Harmonic excitation, response, and modal superposition as defined for the unmodified structure is assumed. As stated earlier, attachment of small vibration absorbers allow us to assume that response of the base structure with additions can be represented in the original modes. Thus, modal superposition parallels Eq. (4) for the unmodified structure, but with added terms from the modifications:

$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{X}_z \end{Bmatrix} = \begin{bmatrix} \Phi_0 & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{X}_z \end{Bmatrix}. \tag{12}$$

Eqs. (11) and (12) is combined to give

$$\left[-\omega^2\begin{bmatrix}\mathbf{m}_0+\Delta\mathbf{m} & \mathbf{m}_a\\ \mathbf{m}_a^{\mathrm{T}} & \mathbf{m}_z\end{bmatrix}+\begin{bmatrix}(1+\mathrm{i}\eta)\mathbf{k}_0+\Delta\mathbf{k} & \mathbf{k}_a\\ \mathbf{k}_a^{\mathrm{T}} & \mathbf{k}_z\end{bmatrix}\right]\begin{bmatrix}\Phi_0 & 0\\ 0 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{q}\\ \mathbf{X}_z\end{Bmatrix}=\begin{Bmatrix}\mathbf{F}_0\\ 0\end{Bmatrix}. \tag{13}$$

Both sides of the equation are pre-multiplied by the modal matrix in Eq. (12), and the result is simplified by taking advantage of the orthogonality conditions:

$$\left[-\omega^2\begin{bmatrix}\mathbf{I}+\Phi_0^{\mathrm{T}}\Delta\mathbf{m}\Phi_0 & \Phi_0^{\mathrm{T}}\mathbf{m}_a\\ \mathbf{m}_a^{\mathrm{T}}\Phi_0 & \mathbf{m}_z\end{bmatrix}+\begin{bmatrix}(1+\mathrm{i}\eta)\lambda_0+\Phi_0^{\mathrm{T}}\Delta\mathbf{k}\Phi_0 & \Phi_0^{\mathrm{T}}\mathbf{k}_a\\ \mathbf{k}_a^{\mathrm{T}}\Phi_0 & \mathbf{k}_z\end{bmatrix}\right]\begin{Bmatrix}\mathbf{q}\\ \mathbf{X}_z\end{Bmatrix}=\begin{Bmatrix}\Phi_0^{\mathrm{T}}\mathbf{F}_0\\ 0\end{Bmatrix}. \tag{14}$$

Eq. (14) can be denoted as

$$\left[-\omega^2\hat{\mathbf{M}}+\hat{\mathbf{K}}\right]\hat{\mathbf{X}}=\hat{\mathbf{F}}, \tag{15}$$

where

$$\hat{\mathbf{X}}=\begin{bmatrix}\mathbf{q}\\ \mathbf{X}_z\end{bmatrix}. \tag{16}$$

Eq. (15) involves inverting a smaller matrix. Dimension of $\hat{\mathbf{X}}$ equals m number of modes used in Eq. (4) plus the number of independent degrees of freedom associated with the absorbers. Solution of $\hat{\mathbf{X}}$ from the above equation can again be obtained using modal superposition. We set $\hat{\mathbf{F}}=0$ and solve for the modes from

$$\hat{\mathbf{K}}\hat{\phi}^j=\lambda_i\hat{\mathbf{M}}\hat{\phi}^j,\quad j=1,\ldots,\hat{m}. \tag{17}$$

We then have

$$\hat{\mathbf{X}}=\sum_{j=1}^{\hat{m}}\psi_j\hat{\phi}^j, \tag{18}$$

where

$$\hat{\phi}=\begin{bmatrix}\mathbf{\Phi}_q\\ \mathbf{\Phi}_z\end{bmatrix}. \tag{19}$$

As in Section 2, we may use orthogonality properties to write the modal response as

$$\psi_j=\frac{\hat{\phi}^{j^{\mathrm{T}}}\hat{\mathbf{F}}}{(-\omega^2+\lambda_m)}, \tag{20}$$

where λ_m is the (complex) eigenvalue of the modified structure with absorbers. From Eq. (12), we have $\mathbf{X}_m=\mathbf{\Phi}_0\mathbf{q}$, which together with Eqs. (16), (18)–(20) yields the response of the base structure degrees of freedom (i.e., excluding the absorber degrees of freedom) as

$$\mathbf{X}_m=\sum\psi_j\mathbf{\Phi}_m^j, \tag{21}$$

where the modes of the modified system are given by

$$\mathbf{\Phi}_m=\mathbf{\Phi}_0\mathbf{\Phi}_q. \tag{22}$$

Eq. (21) can be written as

$$\mathbf{X}_m = \Phi_m\left[-\omega^2 + \lambda_m\right]^{-1}\Phi_m^{\mathrm{T}}\mathbf{F}_0 \tag{23}$$

which represents the forced response of the modified base structure.

While the impedance approach discussed earlier only provides $\mathbf{X}(\omega)$, and a search technique is needed to determine the peak responses, in the reduced eigenvalue approach each peak response is immediately obtained by setting the real part of $\lambda_m = \omega_m^2$ in Eq. (23).

6. An efficient technique for generating absorber matrices, $\mathbf{M}_{\mathrm{abs}}$ and $\mathbf{K}_{\mathrm{abs}}$

Matrices $\mathbf{M}_{\mathrm{abs}}$ and $\mathbf{K}_{\mathrm{abs}}$ represent stiffness and mass properties of the absorbers attached to the base structure. As evident from Eq. (10), some of the degrees of freedom are shared with the attachment point while others are additional. While a full scale finite element model of the absorbers is possible, this will introduce several additional dof. Here, a method is recommended which involves very few additional dof and in addition includes rotatory inertia effects.

Let $\mathbf{q}_a = [q_1, q_2, \ldots, q_6]^{\mathrm{T}}$ represent the dof of a node on the base structure at an attachment point. Typically, q_1, q_2 , q_3 represent translations about x, y, z and q_4, q_5, q_6 represent rotations about x, y, z. Let q_7, q_8, ... represent additional dof that adequately describe the deformed states of the absorber. Note that actual dof numbers will not be 7, 8, ... but will be $m+1$, $m+2$, ..., where m is the number of modes used to describe response of the base structure in Eq. (4). Then, we define $\phi_1(\mathbf{x})$ to be the motion induced in the absorber material due to prescribed displacements $q_1 = 1$, $q_2 = q_3 = \cdots = q_8 = 0$. Similarly, $\phi_2(\mathbf{x})$ is defined to be the motion induced in the absorber material due to $q_1 = 0$, $q_2 = 1$, $q_3 = q_4 = \cdots = q_8 = 0$, and finally $\phi_8(\mathbf{x})$ is defined to be the motion induced in the absorber material due to $q_1 = q_2 = \cdots = q_6 = q_7 = 0$ and $q_8 = 1$. Then, the displacement field within the absorber may be written as $\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{8} q_i\phi_i(\mathbf{x})$, from which we can write the kinetic and strain energy in the absorber, respectively, as

$$T_{\mathrm{abs}} = \tfrac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\mathrm{abs}}\dot{\mathbf{q}} \;\text{ and }\; U_{\mathrm{abs}} = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}_{\mathrm{abs}}\mathbf{q}. \tag{24}$$

Rotatory inertia is automatically included in this formulation. This is because $[q_4, q_5, q_6]$ *represent rotation of the base structure at the attachment point, and the corresponding* ϕ*'s represent the displacements of the absorber induced by these rotations*. Recent experiments have showed us that rotations are even more significant than translations when absorbers are attached to points on the base structure that undergo significant rotation (the tip of a cantilever beam is an example). The expressions in Eq. (24) are derived using familiar finite element shape functions, as illustrated in the examples below. The examples below also show how the ϕ's may be constructed for a specific geometry.

7. A simple example

This problem has been solved in Thomson's book [17] using the 'component mode synthesis' method. The greater simplicity of the method in this paper is noticeable. The base structure is simply a horizontal beam, and the 'absorber' attached to this is a vertical beam (Fig. 2). The base structure is two-dimensional, and is modeled using two dof.

Construction of ϕ_i in Fig. 3 follows the discussion in Section 6, and results in only one extra degree of freedom to the base structure. From elementary beam equations, the following expressions follow from applying appropriate boundary conditions:

$$\phi_2 = 1, \quad \phi_2 = -\frac{y^3}{2L^2} + \frac{3y^2}{2L} - y, \quad \phi_3 = \frac{-3}{L_b^3}\left(\frac{y^3}{6} - \frac{L_b y^2}{2}\right).$$

The displacement of the modification beam is given by

$$u(y) = q_1\phi_1 + q_2\phi_2 + q_3\phi_3.$$

The strain energy in the modification beam is

$$U = \tfrac{1}{2}EI\int_{L_b}\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2}\right)^2 \mathrm{d}y \ \text{ or } \ U = \tfrac{1}{2}\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^{\mathrm{T}} EI \begin{bmatrix} \int_{L_b}(\phi_2'')^2\,\mathrm{d}y & \int_{L_b}\phi_2''\phi_3''\,\mathrm{d}y \\ \int_{L_b}\phi_2''\phi_3''\,\mathrm{d}y & \int_{L_b}(\phi_3'')^2\,\mathrm{d}y \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

which defines the stiffness contributions relating to the dof 2, 3. The mass matrix is evaluated by considering the kinetic energy of the modification beam

$$T = \tfrac{1}{2}\int_m \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 \mathrm{d}m \ \text{ or } \ T = \tfrac{1}{2}\rho A\omega^2 \int_{L_b} u^2\,\mathrm{d}y$$

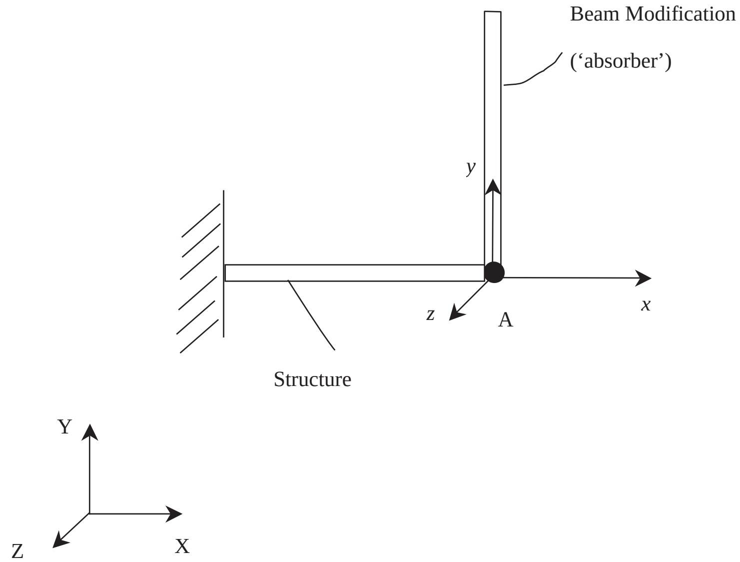

Fig. 2. Two-beam structure: $EI = 10$, $A = 1$, $\rho = 1$, $L_b = 1.7783$.

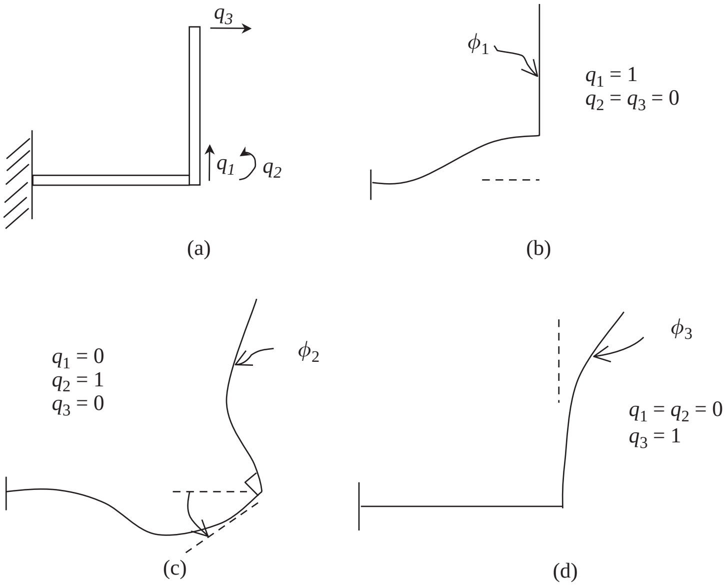

Fig. 3. Construction of ϕ_i for example problem: (a) horizontal base with vertical beam absorber; (b) ϕ_1 with $q_1 = 1$, $q_2 = q_3 = 0$; (c) ϕ_2 with $q_1 = q_3 = 0$, $q_2 = 1$; (d) ϕ_3 with $q_1 = q_2 = 0$, $q_3 = 1$.

which can be reduced to

$$T = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \rho A \begin{bmatrix} L_b & 0 & 0 \\ 0 & \int_{L_b} \phi_2^2 \,\mathrm{d}y & \int_{L_b} \phi_2\phi_3 \,\mathrm{d}y \\ 0 & \int_{L_b} \phi_2\phi_3 \,\mathrm{d}y & \int_{L_b} \phi_3^2 \,\mathrm{d}y \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

which defines the mass contributions. The terms of the element mass and stiffness matrices are either integrated using (six-point) Gaussian quadrature or in closed form. The additional stiffness and mass matrices corresponding to the modification/absorber beam are

$$K_{\mathrm{abs}} = \frac{3EI}{L^3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & L^2 & L \\ 0 & L & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 16.8702 & 9.4868 \\ 0 & 9.4868 & 5.3348 \end{bmatrix},$$

$$M_{\mathrm{abs}} = \frac{\rho AL}{840}\begin{bmatrix} 840 & 0 & 0 \\ 0 & 16L^2 & -33L \\ 0 & -33L & 198 \end{bmatrix} = \begin{bmatrix} 1.7783 & 0 & 0 \\ 0 & 0.1071 & -0.1242 \\ 0 & -0.1242 & 0.4192 \end{bmatrix}.$$

Comparison of the first two natural frequencies of the two-beam system are given in Table 1.

Table 1
Comparison of the first two natural frequencies of the two-beam system

	Reduced eigenvalue	CMS [17]	ANSYS
1st frequency	1.173	1.172	1.170
2nd frequency	3.213	3.198	3.200

8. $\mathbf{M}_{\mathrm{abs}}$ and $\mathbf{K}_{\mathrm{abs}}$ matrices for a three-dimensional single-beam absorber

The broadband vibration absorber or 'BBVA' used in the next section (shown in Fig. 4) is constructed by assembling multiple, single-beam absorbers of the type discussed now (Fig. 5). The single-beam absorber is comprised of a rigid base, of height L_h, a flexible beam of length L, and a mass at the end of the beam. The absorber is rigidly attached to the base structure at point 'A', and the six 'local' dof of the absorber at that point are shared with the base structure. The system is defined in terms of the six shared degrees of freedom, and only two additional dof per beam. Thus, the reduced eigenvalue reanalysis method is very effective.

The motion $u(\xi)$ of any location ξ on the modification is expressed in terms of basis shapes and the degrees of freedom shown in Fig. 5, as

$$u(\xi) = \sum_{i=1}^{8} q_i \phi_i(\xi). \tag{25}$$

As in the previous simple example, rotatory inertia of the absorbers are automatically included since base structure rotations q_4, q_5, and q_6 couple with the other dof. The eight basis shapes are shown in Figs. 6 and 7.

Numerically, three different shapes are used to represent the flexible displacement of the beam portion of the absorber. The basis shapes must satisfy the boundary conditions for the absorber, and the beam is clamped at the rigid link. The clamped deflection due to a unit displacement at q_2 and zero slope at the rigid link, ψ_1, is used in ϕ_2, ϕ_3, and ϕ_4:

$$\psi_1(x) = -\frac{3x^2}{2L^2} + \frac{x^3}{2L^3} + 1.$$

The deflection due to a unit rotation at q_5, ψ_2, is used in ϕ_5 and ϕ_6:

$$\psi_2(x) = \frac{-x^3}{2L^2} + \frac{3x^2}{2L} - x.$$

The deflection due to a unit displacement at q_7, ψ_3, is used in ϕ_7 and ϕ_8:

$$\psi_3(x) = -\frac{x^3}{2L^3} + \frac{3x^2}{2L^2}.$$

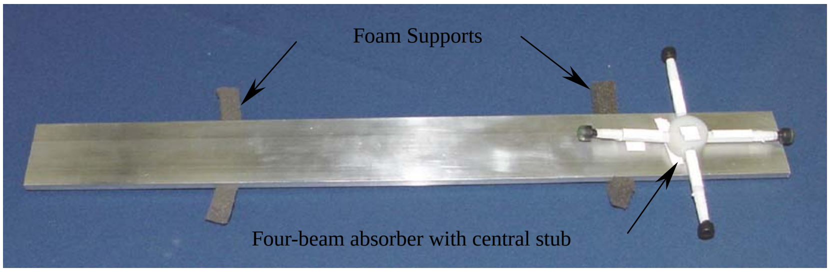

Fig. 4. Four-beam absorber with end masses attached to a foam supported aluminum beam.

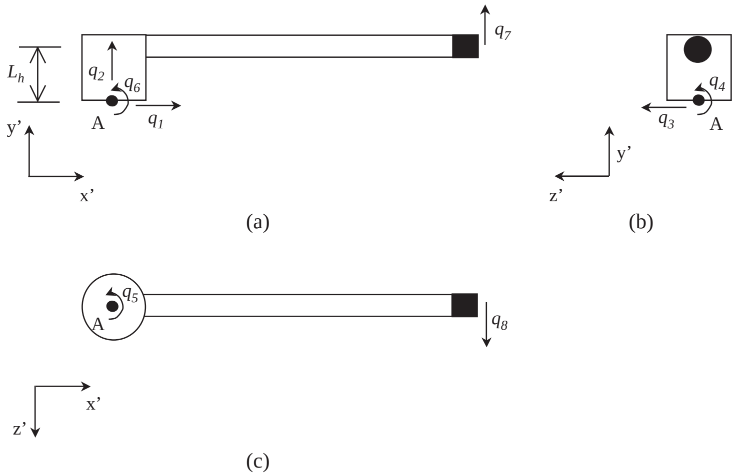

Fig. 5. Single-beam absorber with local degrees of freedom indicated; (a) $x'y'$ view; (b) $y'z'$ view; (c) $x'z'$ view.

It is noted that rotation q_4 leads to a motion $L_h\, q_4$ at the top of the stub. This is considered while defining ϕ_4. Also, the rigid stub does not contribute any stiffness, but does contribute to the mass matrix.

The kinetic energy can be expressed in matrix form in terms of the participation degrees of freedom and the basis shapes, as $T = \frac{1}{2}\omega^2 \sum_i \rho_i A_i \int_i u(\xi)^2\, \mathrm{d}\xi$, which gives an expression for the mass matrix, $M'_{j,k} = \sum_i \rho_i A_i \int_i \phi_j \phi_k\, \mathrm{d}\xi$. Likewise, the element stiffness matrix is determined by considering the strain energy of the single beam absorber as $U = \frac{1}{2} EI \int_\xi \left(\frac{\mathrm{d}^2 u(\xi)}{\mathrm{d}\xi^2}\right)^2 \mathrm{d}\xi$ which yields $K'_{j,k} = EI \int_\xi \left(\frac{\mathrm{d}^2\phi_j}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2\phi_k}{\mathrm{d}\xi^2}\right) \mathrm{d}\xi$.

The element stiffness and mass matrices are defined by simple polynomials, and the integrals are evaluated analytically. Guyan reduction with some care can be used to verify the resulting matrices.

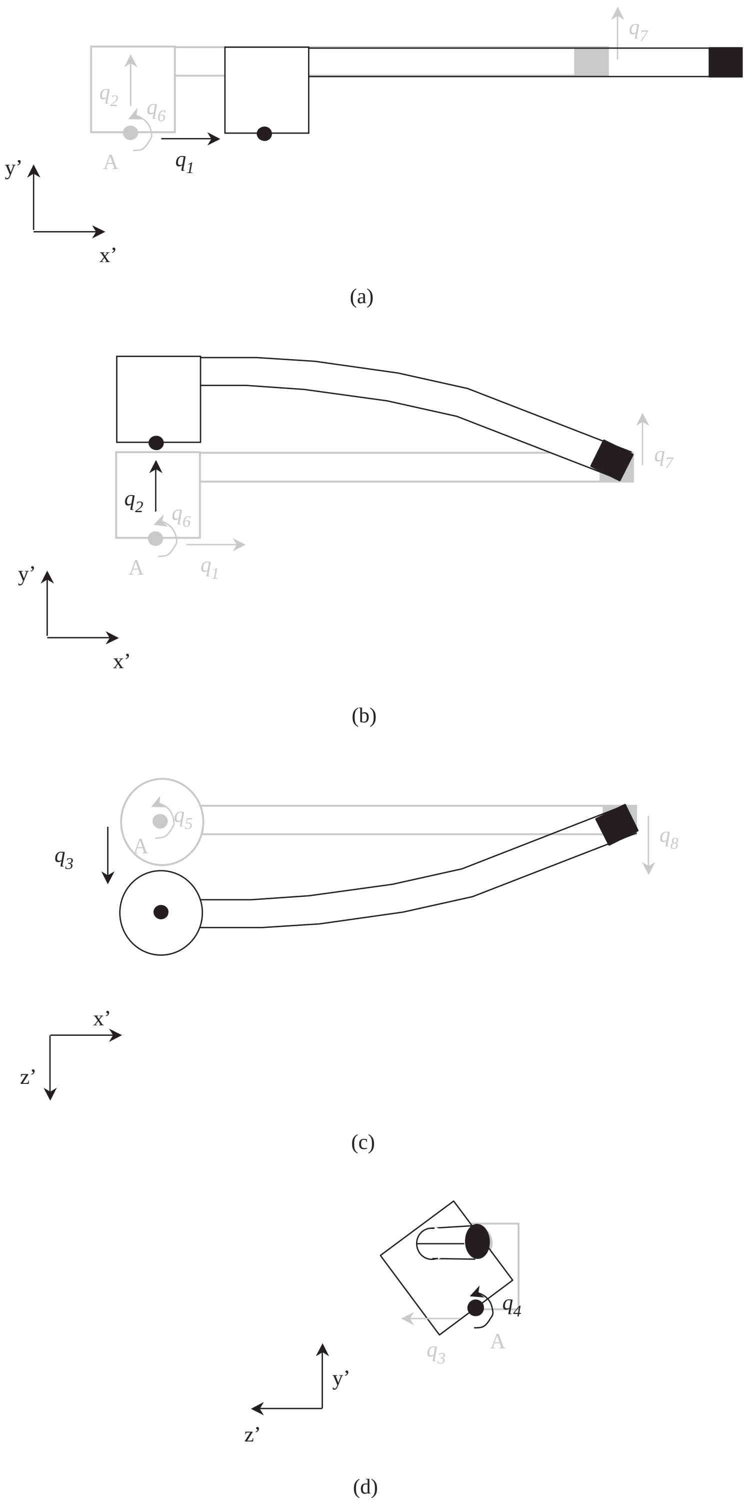

Fig. 6. Basis shapes 1–4 for the single-beam absorber geometry: (a) ϕ_1; (b) ϕ_2; (c) ϕ_3; (d) ϕ_4.

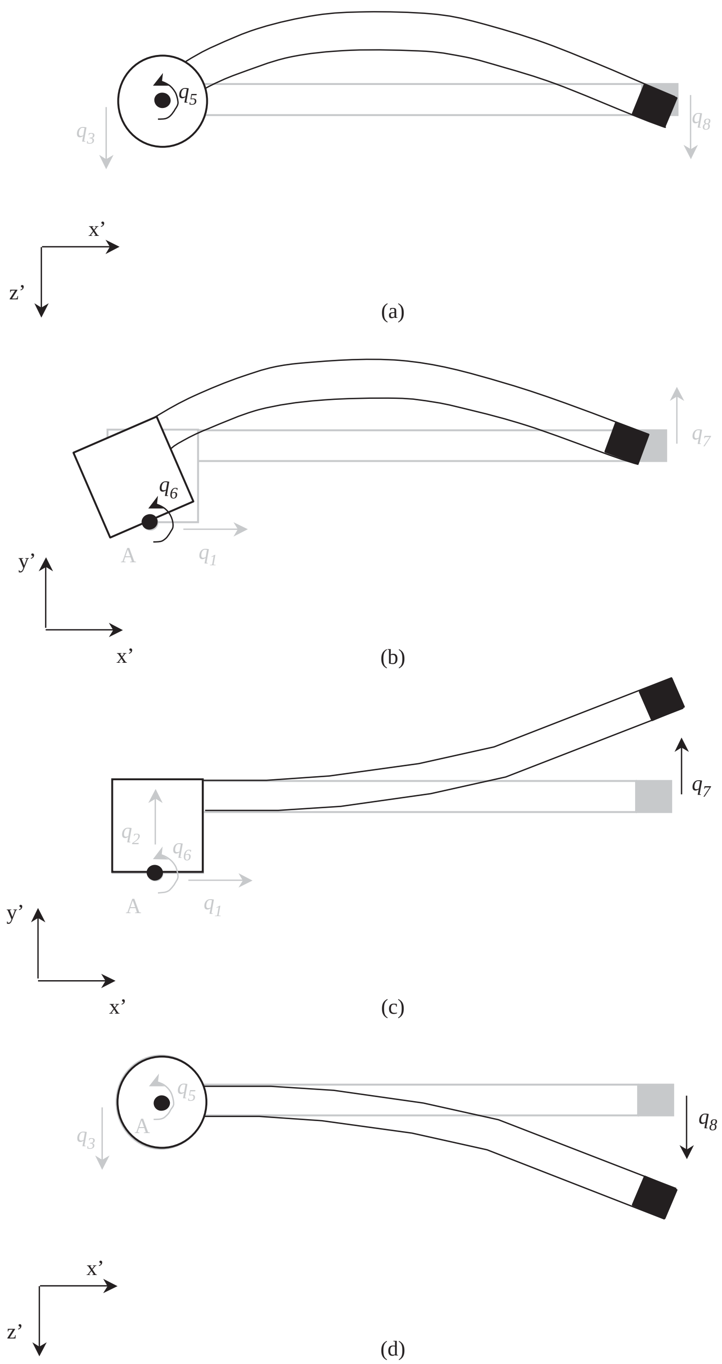

Fig. 7. Basis shapes 5–8 for the single-beam absorber geometry; (a) ϕ_5; (b) ϕ_6; (c) ϕ_7; (d) ϕ_8.

To model the absorber in Fig. 4, each single-beam absorber stiffness and mass must be transformed into global coordinates, prior to being assembled. This is done readily as

$$\mathbf{k} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}, \quad \mathbf{m} = \mathbf{L}^{\mathrm{T}}\mathbf{m}'\mathbf{L}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{R} & & \\ & \mathbf{R} & \\ & & \mathbf{I} \end{bmatrix};$$

R is the well-known direction cosines matrix, and **I** is a 2×2 identity matrix.

9. Structural application

The beam-type absorber in Fig. 4 is now applied to a structure. The structure is also a beam and is modeled using shell finite elements. A simple structure is chosen in order to verify the approach in this paper with experiments.

The first step is to determine modes of the structure without absorbers, viz. $\mathbf{\Phi}_0$. This is done using NASTRAN. The beam (structure) dimensions are 50.80 cm × 5.08 cm × 0.65 cm. To obtain nearly free–free boundary conditions, the beam is supported with a very compliant foam at two locations indicated in Fig. 8. Each foam support is modeled as a set of five 856 N/m springs applied across the width of the beam. The spring stiffness was chosen to match the first rigid body mode of the beam with the foam supports. Young's modulus of the beam is $6.70 \times 10^{10}\,\mathrm{N/m^2}$, and the density is $2713\,\mathrm{kg/m^3}$. The finite element mesh of the unmodified beam consists of 640 quadrilateral plate elements, and 10 spring elements to model the foam supports.

The beam structure is subject to a unit force, across the frequency band, located at 0.64 cm from one end of the beam. The response is calculated/measured at a point 6.35 cm from the other end of the beam.

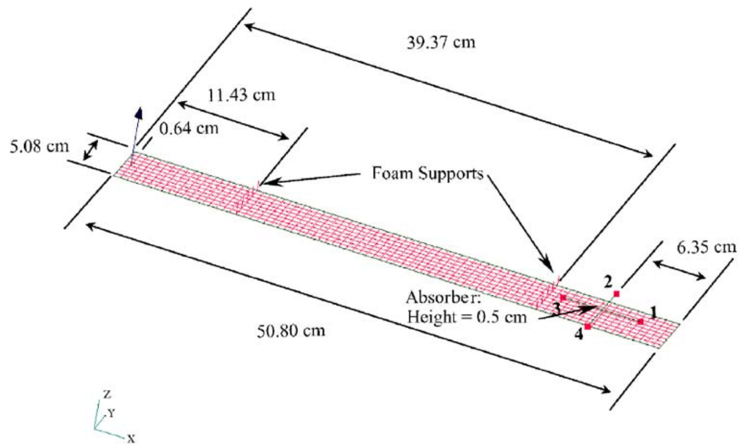

Fig. 8. Finite element model of the beam (absorber and foam supports are shown).

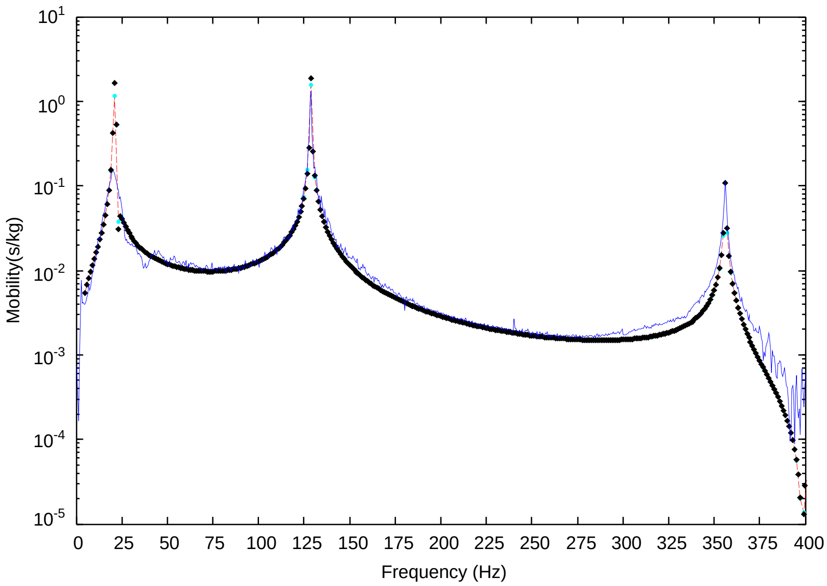

Fig. 9. Numerical and experimental mobility response comparison of the unmodified beam (without absorber): –●–, NASTRAN, ◆, reduced eigenvalue method; —, experimental.

The mobility response between the force location and the response location, for the structure without absorbers, is shown in Fig. 9. Experimentally, this excitation is realized with an instrumented impact hammer, and the response is measured with a laser velocimeter.

The four-beam absorber modification that was shown in Fig. 4 is then attached to the beam structure (Fig. 8). The absorber consists of a 0.5 cm rigid link ('stub') normal to the surface, a beam ('spoke') modeled with five beam elements, and an 8.5 g mass at the free end. The Young's modulus of the spoke material is $1.00 \times 10^9\ \mathrm{N/m^2}$, and the density is $1100\ \mathrm{kg/m^3}$. The loss factor of the material was found experimentally by the half-power method on a test sample. The spokes are assumed to be uniform tubes with circular cross-sections, with each cross-sectional area equal to $1.52 \times 10^{-5}\ \mathrm{m^2}$, and a moment of inertia of $8.01 \times 10^{-11}\ \mathrm{m^4}$. The absorber characteristics including the first natural frequency of each beam are given in Table 2. The relative orientation of the absorbers to each other and the beam is shown in Fig. 8.

The reduced eigenvalue approach in this paper is now verified, by comparing the mobility frequency response with both NASTRAN and experiment. The mobility response is calculated/measured at the four-beam absorber attachment location. NASTRAN calculations are based on a full finite element model where the beam structure and absorber (with stub, spokes, end-mass) are all modeled with finite elements. Fig. 10 shows agreement between the three methods. Capturing the rotatory inertia of the absorbers is necessary for this correlation. The deviation between the experimental and numerical results can be attributed to small variations in the absorber material properties especially the loss factors. Fig. 10 also shows the ill-effect of ignoring rotatory inertia of the absorbers simulated by ignoring coupling between base rotational dof and the absorber. It is noted that the reduced eigenvalue method is considerably faster than the NASTRAN analysis. Both methods solve the eigenvalue problem. The size of the NASTRAN inversion is equal to the

Table 2
Four-beam absorber characteristics

	Mass (g)	Damping (η)	Frequency (Hz)	Length (cm)
1	8.5	0.03	125.0	3.56
2	8.5	0.03	133.0	3.43
3	8.5	0.03	122.0	3.64
4	8.5	0.03	128.0	3.52

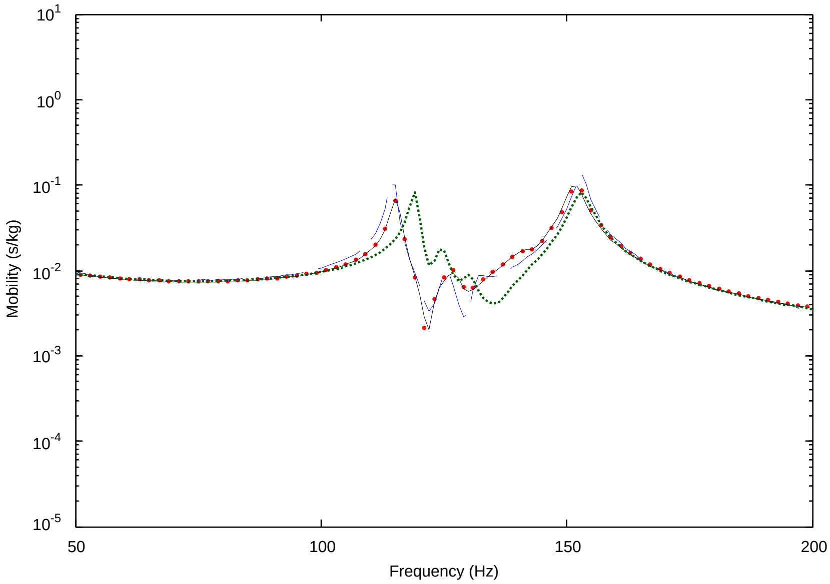

Fig. 10. Numerical frequency response comparison with a four-beam absorber: ● (red), NASTRAN, - - - (blue), experimental; — (black), reduced eigenvalue method; · · · · (green), without rotational components.

total number of degrees of freedom. The size of the reduced eigenvalue problem is the number of mode shapes plus two times the number of absorbers which is on the order of 100 times smaller than NASTRAN for the example above.

10. Conclusions

An efficient and robust dynamic reanalysis algorithm based on a reduced eigenvalue approach is presented for predicting the dynamic response of a base structure to which are attached multiple absorbers. In this approach, resonances of the modified structure are provided, and can be summed directly to estimate broadband measures of the dynamic response. Further, rotatory inertia of the absorbers is properly captured, which has been found from experiment to be significant when an absorber is attached to a point on the base structure undergoing rotation. The method is contrasted with the impedance-based approaches. The method is used to model a three-dimensional absorber on a structure, which has been built, and is shown to correlate with experiment and full-scale finite element analysis. This approach is efficient within an iterative optimization process or for Monte Carlo simulations.

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