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A perturbation approach for evaluating natural frequencies of moderately thick elliptic plates

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Abstract

Natural frequencies for moderately thick elliptic plates are calculated by perturbing initial values corresponding to the Kirchhoff classical theory of plate bending. The proposed approach utilizes a universal algebraic equation for perturbed eigenvalues, previously derived by the authors. For elliptic plates of a small eccentricity, a dual procedure is developed to evaluate the sought for natural frequencies starting from those for a circular thin plate. A comparison with finite-element computations is presented. An importance of the perturbation techniques in question for interpreting of finite-element data is emphasized.

1. Introduction

General asymptotic analysis of the 3D equations in elasticity for plates and shells leads to numerous conclusions of practical importance. In particular, it results in universal algebraic equations for natural frequencies corresponding to bending and extension vibration of moderately thick plates (see Refs. [1,2]). These are applicable for various plate shapes and boundary conditions on plate edges. In the case of bending vibration such an equation is based on

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perturbation of the eigenvalues corresponding to the classical Kirchhoff theory of plates. High efficiency of the proposed approach has been demonstrated in Refs. [1,2] by comparing with the 3D "exact" solutions for circular and rectangular plates with free and simply supported edges. Further applications dealing with plates of more complicated shape will apparently require numerical evaluation of natural frequencies according both to the Kirchhoff theory (for initial setting) and 3D elasticity (for estimating accuracy of computed frequencies).

In this paper we apply the aforementioned algebraic equation to moderately thick elliptic plates. To our best knowledge, all the known publications on the subject are devoted only to thin plates governed by the 2D classical theory (see, e.g., Refs. [3–10]). Below we evaluate natural frequencies of moderately thick elliptic plates by perturbing those of thin plates; in doing so, our main concern is a clamped plate.

Initial settings for natural frequencies corresponding to the classical theory can be established, for example, starting from considerations in Refs. [3–7]. Explicit variational formulae proposed in Ref. [9] appear to be accurate enough over a representative parameter range. We also utilize numerical results in Ref. [10], based on a direct computational procedure.

For an elliptic plate of small eccentricity perturbed eigenvalues of a circular plate may also provide a reasonable zero-order approximation. In the latter case we proceed to a dual perturbation procedure; its two stages correspond to perturbations with respect to eccentricity (or a similar geometrical parameter expressing ellipticity) and thickness.

To test accuracy of the developed methodology we utilize a 3D finite-element code; in doing so we separate the sought for bending finite-element values from those associated with plate thickness vibrations discussed in brief in the appendix.

2. Theoretical background

Consider an elastic elliptic plate of uniform thickness $2h$ with semi-major and minor radii a and b, respectively, clamped along the boundary C. In terms of the classical Kirchhoff theory natural vibrations of a clamped plate are described by equation of motion

$$\nabla^4 w - k^4 w = 0 \tag{1}$$

with

$$k^4 = \frac{2\rho h}{D}\omega^2,$$

and boundary conditions

$$w|_C = \left.\frac{\partial w}{\partial \mathbf{n}}\right|_C = 0 \quad \text{with } C = \left\{x, y : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}. \tag{2}$$

Here w=$w(x,y)$ is the transverse displacement of the middle plane of the plate, x and y are the Cartesian coordinates, ∇^4 is the bi-harmonic operator, k is the wave number, $\mathbf{n}$ is the unit normal to C, ω is the circular frequency, ρ is the density and $D = 2Eh^3/[3(1-\nu^2)]$, where E is the Young modulus and ν is the Poisson ratio.

Higher-order asymptotic theories of plate bending also result in a bi-harmonic governing equation. It is [1,2]

$$\nabla^4 w - \frac{3(1-\nu)}{2}\frac{\Omega^2}{\eta^2 a^4}\sum_{j=0}^{N} A_j(\eta\Omega)^j w = 0 \tag{3}$$

with

$$\Omega = \frac{\omega a}{c_2}, \quad \eta = \frac{h}{a}, \tag{4}$$

and

$$A_0 = 1,\ A_1 = \sqrt{\frac{3(1-\nu)}{2}}\frac{17-7\nu}{15(1-\nu)}, \quad A_2 = \frac{1179-818\nu+409\nu^2}{2100(1-\nu)},$$
$$A_3 = \sqrt{\frac{3(1-\nu)}{2}}\,\frac{5951-2603\nu+9953\nu^2-4901\nu^3}{126000(1-\nu)^2}, \ldots, \tag{5}$$

where

$$c_2 = \sqrt{\frac{E}{2(1+\nu)\rho}}.$$

The integer N in Eq. (3) defines the order of approximation; zero-order approximation ($N=0$) corresponds to the Kirchhoff theory, cf. Eq. (1). Below we start from the third-order theory ($N=3$ in Eq. (3)). As it has been shown in Ref. [11], this theory provides an excellent approximation for the ‘‘exact’’ zero-order anti-symmetric Lamb mode (i.e. bending mode) over a wide frequency range up to the first thickness resonance. Apparently, the latter is optimal for treating moderately thick plates.

Another important feature of dynamic asymptotic analysis in Refs. [1,2] is that refinement of boundary conditions appears to be secondary, in a sense, compared with that of the equations of motion. In particular, the refined equation of motion (3) may be utilized together with simplest classical boundary conditions, e.g. for a clamped edge these take form (2).

Let a natural frequency in problem (1) and (2) be known and denoted as ω_*. Then the sought for perturbed value ω in problem (3) and (2) for the natural frequency of the associated moderately thick plate may be easily found by equating relevant inertial terms in Eqs. (1) and (3). Finally, we have (see Refs. [1,2] for details)

$$\Omega^2(1 + A_1\eta\Omega + A_2\eta^2\Omega^2 + A_3\eta^3\Omega^3) = \Omega_*^2, \tag{6}$$

where

$$\Omega_* = \frac{\omega_* a}{c_2}. \tag{7}$$

The fifth-order algebraic equation (6) allows elementary numerical treatment including that based on modern standard routines. It is not restricted to any specific bending mode but assumes evaluation of the initial approximations for natural frequencies corresponding to the classical plate theory. These values ω_* may be determined, for example, from the exact solution of problem

(1) and (2) expressed in terms of Mathieu functions and modified Mathieu functions (see, e.g., Ref. [3]) or using various numerical methods. Simple analytical estimations are also of a great importance. Among the latter, we mention those following from Rayleigh [3] and Galerkin [4] procedures as well as from a refined Rayleigh–Ritz-type qualitative approach recently developed in Ref. [9].

The estimation for the fundamental frequency, derived in the papers [3,4,9], may be written in a unified form as

$$\Omega_* = \mu_0 \eta \sqrt{\frac{2}{3(1-\nu)}} \sqrt{\frac{3}{8}\left[1 + \frac{2}{3}\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4\right]}, \tag{8}$$

where the value of the coefficient μ_0 depends on the method exploited. In particular, the Rayleigh quotient gives $\mu_0 \approx 10.328$, see Ref. [3]. Comparison with the exact solution demonstrates that the optimal value is given by Ref. [9], $\mu_0 = \lambda_0^2$, where λ_0 is the first root of the equation

$$J_n(\lambda) I_n'(\lambda) - I_n(\lambda) J_n'(\lambda) = 0 \tag{9}$$

with $n=0$, i.e. $\mu_0 \approx 10.216$. As usual, J_n and I_n denote the Bessel and modified Bessel functions, respectively.

Similar formulae for a few next natural frequencies may be also found in Ref. [9]. In particular, the upper estimation for eigenvalue corresponding to the mode with nodal line along the minor ellipse axis (the first mode which is symmetric about the x-axis and anti-symmetric about the y-axis) can be written as

$$\Omega_* = \mu_1 \eta \sqrt{\frac{2}{3(1-\nu)}} \sqrt{\frac{1}{8}\left[5 + 2\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4\right]}, \tag{10}$$

whereas for the mode with the nodal line along the major axis (the first mode which is anti-symmetric about the x-axis and symmetric about the y-axis) the sought for formula becomes

$$\Omega_* = \mu_1 \eta \sqrt{\frac{2}{3(1-\nu)}} \sqrt{\frac{1}{8}\left[1 + 2\left(\frac{a}{b}\right)^2 + 5\left(\frac{a}{b}\right)^4\right]}. \tag{11}$$

with $\mu_1 = \lambda_1^2$, where λ_1 is the first root of Eq. (9) for $n = 1$, i.e. $\mu_1 \approx 21.260$.

Natural forms of a thin clamped elliptic plate are schematically shown in Fig. 1 along with their analogues for a circular plate. In the notations SS, SA, AS, AA the first (second) letter corresponds to symmetry (S) or anti-symmetry (A) about the x-axis (the y-axis). Formulae (8), (10) and (11) describe the first natural forms of SS, SA and AS types, respectively.

It should be emphasized that the algebraic Eq. (6) is applicable, in principle, to a plate of arbitrary shape and treats general boundary conditions on a plate edge. In this case the initial value ω_* follows from the solution of Eq. (1) subject to proper boundary conditions, whereas the parameter a denotes a typical linear size.

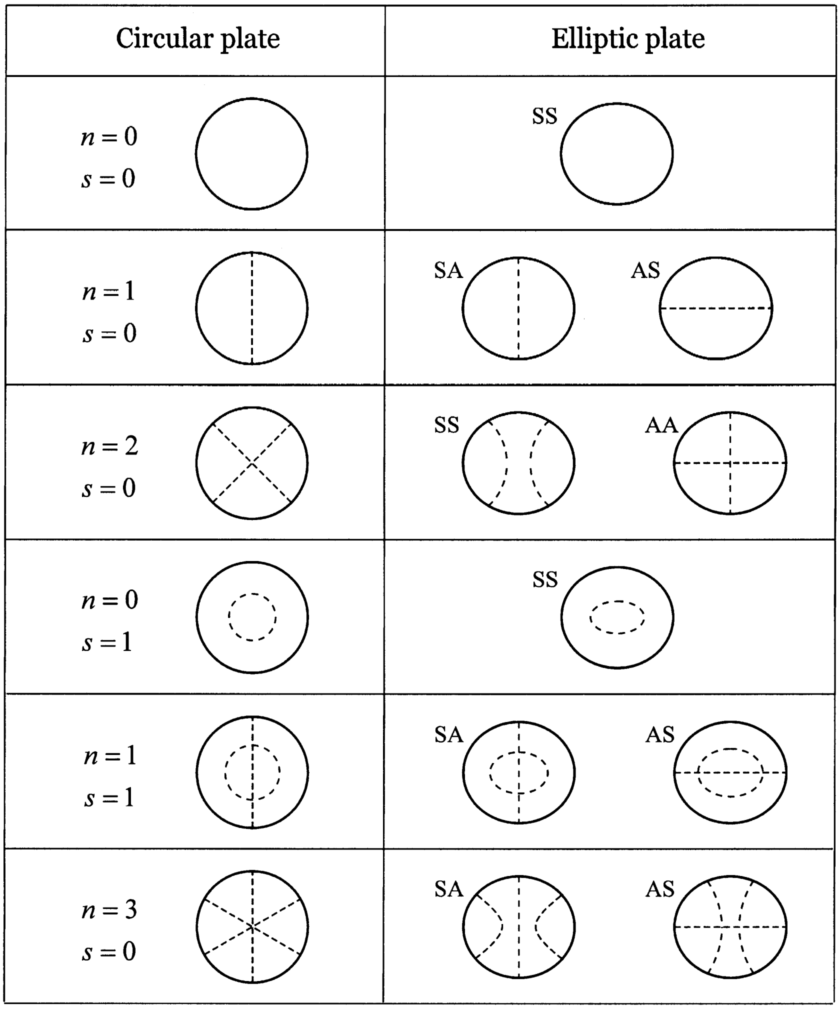


Fig. 1. Natural forms for circular and elliptic plates. n and s denote numbers of nodal diameters and nodal circles for circular plate.

3. Dual perturbation procedure

For nearly circular plates we develop a dual perturbation procedure, schematically sketched in Fig. 2; the stages 1 and 2 deal with perturbations with respect to small eccentricity and thickness, respectively. Thus we evaluate first the natural frequencies for a thin elliptic plate by perturbing those for a circular plate. For simplicity sake, below we restrict ourselves only to unperturbed axisymmetric motions. This problem was tackled by the boundary perturbation method in Ref. [8]. Since the cited paper contains mistakes in final formulae, we are forced to revise in brief the derivation in Ref. [8].

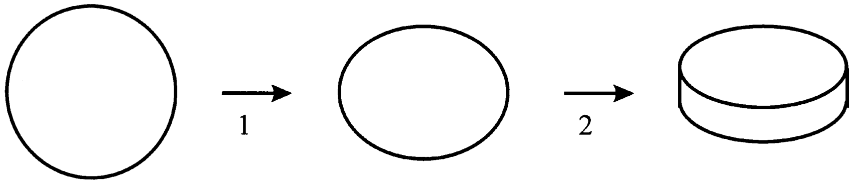


Fig. 2. Dual perturbation scheme.

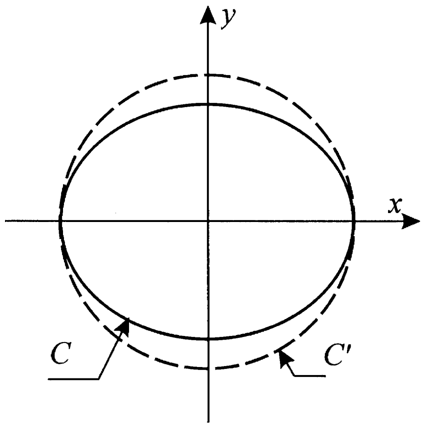


Fig. 3. Boundary perturbation.

Below the boundary C is treated as a perturbation of a circumscribing circle C' of radius a (see Fig. 3); and a typical ellipticity parameter ε is defined as

$$\varepsilon = \frac{a}{b} - 1. \tag{12}$$

The latter is assumed to be small, i.e. $\varepsilon \to 0$. In this case the transverse displacement w and wavenumber k may be expanded as

$$w(r, \psi) = \sum_{j=0}^{m} w^{(j)} \varepsilon^j, \tag{13}$$

$$k^4 = \sum_{j=0}^{m} k_j^4 \varepsilon^j, \tag{14}$$

where r and ψ denote polar coordinates ($x = r\cos\psi$, $y = r\sin\psi$).

The functions $w^{(j)}$ ($j = 0, 1, \ldots, m$) must satisfy equivalent boundary conditions on the boundary $C'(r = a)$; explicit expressions for these boundary conditions are derived in Ref. [12]. In addition, a set of equations for $w^{(j)}$ is also presented in Ref. [8]. Here we write out only the zero-order equation

$$(\nabla^4 - k_0^4)\, w^{(0)} = 0 \tag{15}$$

with

$$w^{(0)}\big|_{C'} = \frac{\partial w^{(0)}}{\partial r}\bigg|_{C'} = 0. \tag{16}$$

In the axisymmetric case the problem possesses the well-known solution

$$w^{(0)}(r) = A\left[J_0(k_0 r) - \frac{J_0(\lambda_0)}{I_0(\lambda_0)} I_0(k_0 r)\right] \tag{17}$$

with $\lambda_0 = k_0 a$. Here the eigenvalue λ_0 represents the $(s+1)$th root ($s = 0,1,2,\ldots$) of the frequency Eq. (9) with $n=0$.

By solving the set of the problems for $w^{(i)}$ $(i = 0, 1, 2)$, we finally arrive at the second-order expansion for the eigenvalues $\lambda = ka$. It is

$$\lambda^4 = \lambda_0^4\left[1 + 2\varepsilon + \beta(s)\varepsilon^2\right] \tag{18}$$

with

$$\beta(s) = \tfrac{1}{2}\lambda_0 \,\frac{J_1(\lambda_0)}{J_0(\lambda_0)} - \tfrac{1}{4}\lambda_0^2 \,\frac{I_0(\lambda_0)J_0(\lambda_0)}{I_1(\lambda_0)J_1(\lambda_0)}, \tag{19}$$

where s is the root number. The values of $\beta(s)$ corresponding to the modes $s = 0,1,2,3,4$ are listed in Table 1.

It should be noted that formula (19) for $\beta(s)$ as well as numerical data differ from that in Ref. [8]. At the same time for the fundamental mode the proposed formula agrees with that obtained in Ref. [3] for a plate of small eccentricity.

Now we proceed to a moderately thick plate. Assuming then that the value ω_* is known we can rewrite Eq. (6) using formula (18). It becomes

$$\lambda_0^4\left[1 + 2\varepsilon + \beta(s)\varepsilon^2\right] = \frac{3(1-\nu)}{2\eta^2}\Omega^2\left(1 + A_1\eta\Omega + A_2\eta^2\Omega^2 + A_3\eta^3\Omega^3\right). \tag{20}$$

Note that for circular plate Eq. (8) for the fundamental frequency transforms to Eq. (20) provided that the parameter μ_0 is defined as in Ref. [9], i.e. $\mu_0 = \lambda_0^2$ for $s = 0$.

Table 1
Numerical values of coefficient $\beta(s)$

s	0	1	2	3	4
$\beta(s)$	2.4499	8.9363	20.459	36.932	58.346

4. Numerical results

Computations below deal with clamped plates; only Tables 2 and 3 also refer to a simply supported plate. Moderately thick elliptic plates are considered including the limiting case of a circular plate. The estimations based on the developed perturbation approach are compared with 3D finite-element results. Below the numerical data are presented for the dimensionless frequency parameter $\Omega = \omega a/c_2$ (or $\Omega/\eta = \omega a^2/(hc_2)$ in Tables 4 and 5). At the same time some of the cited publications on the subject operate with another dimensionless frequency parameters, e.g., in the notation of this paper, $\omega a^2\sqrt{2\rho h/D}$ in Refs. [4,5,9] or $\omega ab\sqrt{2\rho h/D}$ in Refs. [7,10]. It should be

Table 2
Axisymmetric natural frequencies $\Omega = \omega a/c_2$ for circular plate with $\eta = 0.05$ (s is number of nodal circles)

s	Simply supported circular plate			Clamped circular plate		
	Kirchhoff plate, Ω_*	Perturbed value, Eq. (6)	3D finite elements	Kirchhoff plate, Ω_*	Perturbed value, Eq. (6)	3D finite elements
0	0.2408	0.2387	0.239	0.4985	0.4897	0.487
1	1.450	1.380	1.38	1.941	1.819	1.79
2	3.618	3.233	3.23	4.348	3.811	3.72
3	6.749	5.586	5.59	7.719	6.256	6.09
4	10.84	8.266	8.27	12.05	8.993	8.73
5	15.90	11.15	11.2	17.35	11.91	11.6
6	21.92	14.16	14.2	23.61	14.94	14.5
7	28.90	17.25	17.2	30.83	18.04	17.5
8	36.85	20.38	20.4	39.02	21.17	20.6
9	45.76	23.52	23.5	48.17	24.32	23.7

Table 3
Natural frequencies $\Omega = \omega a/c_2$ for circular plate with $\eta = 0.2$ (n is number of nodal diameters, s is number of nodal circles)

n	s	Simply supported circular plate			Clamped circular plate		
		Kirchhoff plate, Ω_*	Perturbed value, Eq. (6)	3D finite elements	Kirchhoff plate, Ω_*	Perturbed value, Eq. (6)	3D finite elements
0	0	0.9632	0.8552	0.864	1.994	1.607	1.49
1	0	2.713	2.068	2.13	4.150	2.880	2.59
2	0	4.999	3.310	3.33	6.807	4.135	3.71
0	1	5.801	3.689	3.71	7.763	4.532	4.09
3	0	7.799	4.547	4.57	9.960	5.367	4.85
1	1	9.462	5.186	5.24	11.87	6.025	5.47
4	0	11.09	5.764	5.78	13.60	6.575	6.00
2	1	13.69	6.602	6.62	16.51	7.428	6.79
0	2	14.47	6.840	6.86	17.39	7.671	7.04
5	0	14.87	6.959	6.98	17.71	7.757	7.15

Table 4
Natural frequencies $\Omega/\eta = \omega a^2/(hc_2)$ for clamped elliptic Kirchhoff plate

	$a/b = 1.1$	$a/b = 1.2$	$a/b = 1.5$	$a/b = 2$	$a/b = 3$
Eq. (8)	11.041	12.261	16.789	27.074	57.271
Ref. [10]	11.039	12.251	16.717	26.717	55.432
Eq. (10)	21.864	23.143	27.993	39.503	74.808
Ref. [10]	21.856	23.111	27.786	38.546	69.869
Eq. (11)	24.041	27.689	40.719	69.203	151.05
Ref. [10]	24.032	27.655	40.487	68.175	146.48

Table 5
Axisymmetric natural frequencies $\Omega/\eta = \omega a^2/(hc_2)$ for clamped elliptic Kirchhoff plate (s is number of nodal circles)

s		$a/b = 1.1$	$a/b = 1.2$
0	Eq. (18)	11.032	12.202
	Ref. [10]	11.039	12.251
1	Eq. (18)	44.072	51.454
	Ref. [10]	43.999	50.737
2	Eq. (18)	103.06	129.51
	Ref. [10]	100.52	117.43

noted that the eigenvalues expressed in terms of the latter do not depend on the Poisson ratio in the framework of the Kirchhoff theory. However, this is not a feature of 3D theory, as it also follows from Eq. (6). Below we set $\nu = 0.3$ for the Poisson ratio.

4.1. Circular plate

Consider first a clamped circular plate, which appears to be a good starting point for investigating the efficiency of the algebraic Eq. (6) applied to this type of boundary conditions for simpler geometry. As we have already mentioned, the accuracy of Eq. (6) had been demonstrated only for free and simply supported plates (see Refs. [1,2]).

The graphs of four lowest axisymmetric natural frequencies versus the plate half-thickness are displayed in Fig. 4. The straight thin lines correspond to the classical Kirchhoff theory. The perturbed natural frequencies evaluated from Eq. (20) with $\varepsilon = 0$ (solid lines) are presented together with finite-element computations (white dots).

Analysis of finite-element computations assumes a delicate qualitative insight. The point is that the total eigenspectrum of a moderately thick plate is not obviously restricted to studied bending eigenvalues. Another eigenvalues may correspond, for example, to thickness vibrations demonstrating sinusoidal stress and strain distributions along the plate thickness. Figs. 5(a) and (b) show schematically the displacement variation along the thickness for bending and thickness vibration, respectively. An asymptotic theory for thickness vibrations is exposed in the

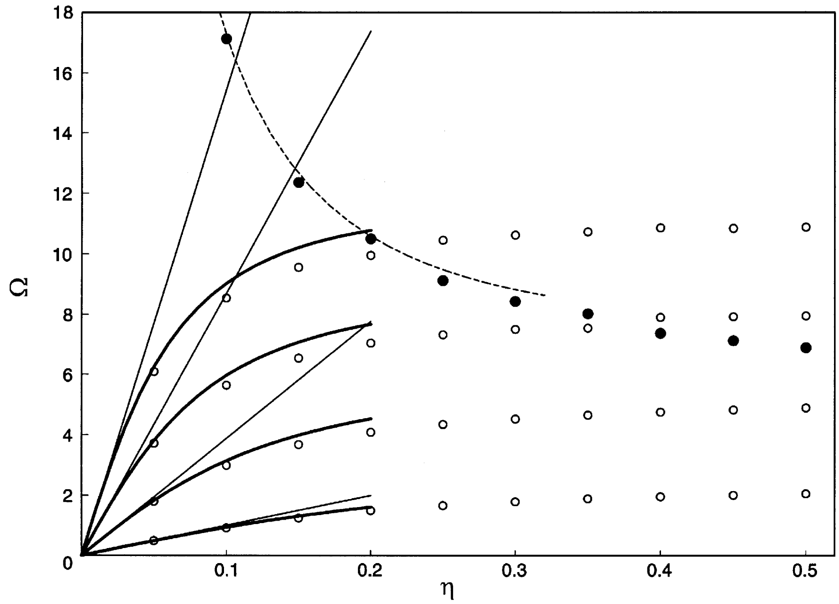
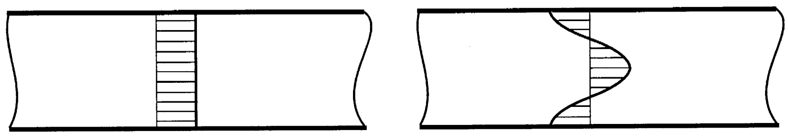


Fig. 4. Axisymmetric natural frequencies for circular plate. —, Kirchhoff theory; —, Eq. (20) with $\varepsilon = 0$; ---- Eq. (A.6); ◦, finite elements, bending; •, finite elements, thickness.



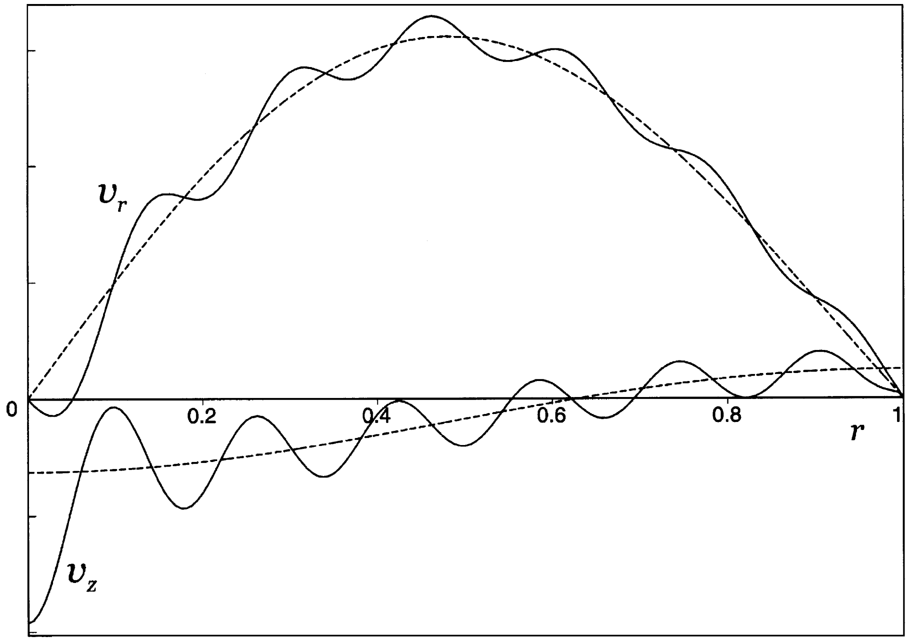
(a) (b)

Fig. 5. Displacement variation along plate thickness. (a) Bending vibration, (b) thickness vibration.

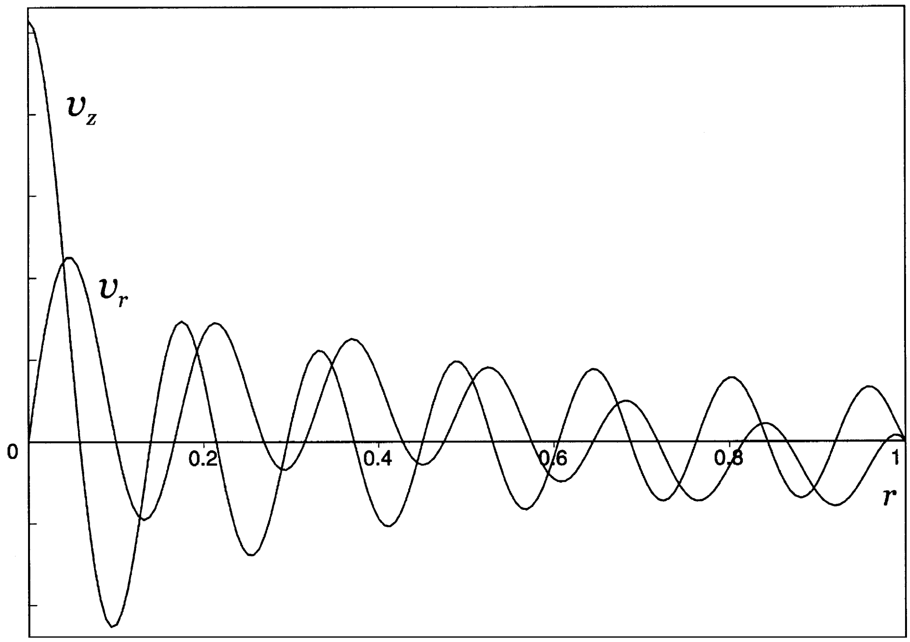
monograph [2] (see also references therein). It is important that for thicker plates, bending and thickness modes sometimes cannot be easily identified. In Fig. 4 the finite-element values of the sought for bending frequencies are given with white dots whereas those for the lowest thickness shear branch (see the appendix) are plotted with black dots. The asymptotic behavior of the latter is shown with a broken line, see Eq. (A.6).

We also mention that the axisymmetric problem in elasticity for a thick clamped circular plate has been treated analytically in Ref. [13]. In particular, this paper claims a decrease of the third axisymmetric bending natural frequency for $\eta > 0.3$. In reality this observation apparently relates to the transition from the analyzed third bending branch to the lowest thickness branch, marked in Fig. 4 by block dots.

As an illustration, the first thickness shear mode for the plate of half-thickness $\eta = 0.05$ is displayed in Fig. 6(a). Both face transverse and radial displacements are calculated. The solid line corresponds to finite-element computations whereas asymptotic results, see formulae (A.3) and (A.4) at $z = h$, are given with the dashed line. For comparison, its neighboring bending form



(a)



(b)

Fig. 6. Transverse and radial displacements at $z = h$ of circular plate. $\eta = 0.05$. (a) First thickness shear mode, $\omega a/c_2 = 32.1$, (b) neighboring bending mode, $\omega a/c_2 = 33.0$, —, finite elements; - - - -, Eqs. (A.3) and (A.4).

(13th bending mode) is presented in Fig. 6(b). It should be stressed again that as the parameter η increases, the difference between these two natural forms almost disappears.

Inspection of numerical data demonstrates that the perturbation approach based on formula (6) is useful for evaluating a few first natural frequencies of thicker plates as well as for refining natural frequencies of thinner plates in a wide frequency range. This approach appears to be even more efficient for boundary conditions other than those modeling a clamped edge. The reason is that in the Kirchhoff plate theory, 2D approximate boundary conditions on a clamped edge may possess lower asymptotic accuracy than those for free or simply supported edges, e.g., see Ref. [14] for more details.

Tables 2 and 3 list natural frequencies for plates with clamped and simply supported edges. In terms of 3D elasticity we assume for the latter zero circumferential and transverse displacements together with zero radial stress at $r = a$. The 2D analogue in the classical Kirchhoff theory results in zero mid-plane deflection and radial stress couple; in doing so, we assume that the value Ω_* corresponds to the solution of problem (1) subject to relevant 2D boundary conditions. The observation of the data in Tables 2 and 3 reveals higher accuracy of the perturbed natural frequencies for a simply supported edge. Further examples for plates with simply supported and free edges may be found in Refs. [1, 2]. At the same time it is clear that the developed methodology improves considerably the classical plate theory for a clamped edge as well.

4.2. Initial settings

Numerical analysis below for moderately thick clamped elliptic plates utilizes two types of approximate initial settings in the framework of the Kirchhoff theory. The first of them starts from the advanced variational formulae (8), (10) and (11), where for the fundamental mode the coefficient μ_0 in Eq. (8) is defined as in Ref. [9]. The second initial setting operates with the boundary perturbation of the associated circular plate resulting in Eq. (20).

The range of applicability of formulae (8), (10) and (11) is obviously dependent of the ellipticity parameter. Numerical results for various ratios a/b are presented in Table 4, illustrating the quality of this initial setting. Computations for a Kirchhoff elliptic plate in Ref. [10] are used as a benchmark. It may be verified that for $\varepsilon < 0.6$ the aforementioned formulae correspond to three lowest bending natural frequencies.

Table 5 illustrates the quality of the initial setting based on Eq. (18). It is clear that the perturbed values (18) are accurate enough for estimating axisymmetric natural frequencies. At the same time, the boundary perturbation method [8,12] is only applicable for lower vibration modes, when a typical wave length is much greater than ellipticity parameter, i.e. $s \ll \varepsilon^{-1}$.

4.3. Moderately thick elliptic plate

First ten natural frequencies (see Fig. 1) for an elliptic plate with ratio $a/b = 1.1$ are given in Table 6 for half-thicknesses $\eta = 0.05$ and 0.1. The initial approximation for Ω_* is based now on 2D calculations in Ref. [10]. Inspection of the presented data demonstrates that the perturbation approach is valuable for an elliptic plate as well. We also remark that for first three natural frequencies perturbed values based on the approximate formulae (8), (10) and (11) are almost identical to those in Table 6.

Table 6
Natural frequencies $\Omega = \omega a/c_2$ for clamped elliptic plate with $a/b = 1.1$

Model type	$\eta = 0.05$			$\eta = 0.1$		
	Kirchhoff plate, Ω_* Ref. [10]	Perturbed value, Eq. (6)	3D finite elements	Kirchhoff plate, Ω_* Ref. [10]	Perturbed value, Eq. (6)	3D finite elements
SS	0.5519	0.5413	0.538	1.104	1.027	0.998
SA	1.093	1.052	1.04	2.186	1.915	1.83
AS	1.202	1.153	1.14	2.403	2.082	1.99
SS	1.826	1.718	1.69	3.652	2.986	2.83
AA	1.881	1.767	1.74	3.763	3.062	2.90
SS	2.200	2.046	2.01	4.400	3.488	3.30
SA	2.707	2.481	2.43	5.414	4.128	3.89
AS	2.725	2.496	2.44	5.451	4.151	3.91
SA	3.169	2.866	2.81	6.338	4.679	4.42
AS	3.462	3.106	3.04	6.924	5.013	4.72

Table 7
Axisymmetric natural frequencies $\Omega = \omega a/c_2$ for clamped elliptic plate with $\eta = 0.05$ and $a/b = 1.05$

s	Kirchhoff plate, Ω_* Eq. (18)	Double perturbation, Eq. (20)	3D finite elements
0	0.5243	0.5146	0.512
1	2.056	1.921	1.89
2	4.665	4.056	3.95
3	8.428	6.731	6.47

As it has been already mentioned, the initial setting (20) appears to be useful for perturbation of a few first axisymmetric natural frequencies of the associated circular plate as $\varepsilon \ll 1$. Numerical data related to the dual perturbation procedure are given in Table 7.

Similarly to a circular plate we may expect that the perturbation approach should be even more efficient for other boundary conditions including those modeling a simply supported or a free edge. In addition, we mention that the occurrence of thickness natural frequencies is also an important feature of free vibrations of a moderately thick elliptic plate affecting 3D computations.

5. Concluding remarks

The consideration above demonstrates that the proposed approach is applicable over a wide range of problem parameters applied to a moderately thick elliptic plate. It does not involve 3D analysis and is especially useful if the initial approximation ω_* allows a simple estimation, e.g. when the advanced variational formulae (8), (10) and (11) are valid. The paper also clearly indicates further prospects for plates of more general shapes. Another important feature is that the perturbation methodology appears to be useful for interpreting numerical results for thicker plates since related eigenspectra are not ultimately restricted to bending vibration modes.

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Appendix

For the axisymmetric motion of a circular plate the lowest branch of thickness shear vibrations is described by the 1D asymptotic equation (e.g., see Ref. [2])

$$u_r'' + \frac{1}{r}u_r' + \left(\frac{\Lambda^2 - \Lambda_{\rm sh}^2}{h^2 P} - \frac{1}{r^2}\right)u_r = 0 \tag{A.1}$$

with

$$\Lambda = \frac{\omega h}{c_2}, \quad P = 1 + \frac{8\kappa\ \cot(\kappa\Lambda_{\rm sh})}{\Lambda_{\rm sh}}, \quad \kappa = \sqrt{\frac{1-2\nu}{2(1-\nu)}}, \tag{A.2}$$

where $\Lambda_{\rm sh} = \pi/2$ is the first thickness shear resonance frequency, $u_r = u_r(r)$ is the radial long-wave amplitude. This equation describes long wave natural forms with the frequencies Λ close to $\Lambda_{\rm sh}$, i.e. $|\Lambda - \Lambda_{\rm sh}| \ll 1$. In this case the radial $v_r(r,z)$ and transverse $v_z(r,z)$ displacements in the axisymmetric elasticity (z is the transverse coordinate, $-h \le z \le h$) are expressed in terms of amplitude $u_r(r)$ as (see Ref. [2] for details)

$$v_r(r,z) = u_r(r)\sin\left(\Lambda_{\rm sh} z/h\right), \tag{A.3}$$

and

$$v_z(r,z) = u_z(r)\left[\cos\left(\Lambda_{\rm sh} z/h\right) - \frac{2\kappa}{\sin(\kappa\Lambda_{\rm sh})}\cos\left(\kappa\Lambda_{\rm sh} z/h\right)\right] \tag{A.4}$$

with

$$u_z(r) = \frac{h}{\Lambda_{\rm sh}}\left(u_r' + \frac{1}{r}u_r\right).$$

In the case under consideration, the eigenfunctions corresponding to Eq. (A.1) are

$$u_r = CJ_1(\beta r) \tag{A.5}$$

with

$$\beta = \sqrt{\frac{\Lambda^2 - \Lambda_{\rm sh}^2}{h^2 P}}.$$

For a clamped edge, $u_r(a) = 0$, the sought for natural frequencies become

$$\Lambda_i^2 = \Lambda_{\rm sh}^2 + h^2 P \beta_i^2, \tag{A.6}$$

where β_i denotes the ith root of the equation $J_1(\beta a) = 0$.

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