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Short Communication

Natural frequencies of a functionally graded anisotropic rectangular plate

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1. Introduction

We use the first-order shear deformation theory (FSDT) coupled with the finite element method (FEM) to study free vibrations of a functionally graded (FG) anisotropic rectangular plate with the objective of maximizing one of its first five natural frequencies. The following edge conditions are considered: (i) all edges clamped, (ii) all edges simply supported, (iii) two opposite edges clamped and the other two free, and (iv) two opposite edges simply supported and the other two free. An advantage of a functionally graded plate over a laminated plate is that material properties vary continuously through the plate thickness. Thus no sudden discontinuities in stresses occur across an interface between any two adjoining laminae thereby eliminating the delamination mode of failure. Whereas there have been numerous works on studying the response of FG plates made of isotropic elastic constituents with the homogenized material also modeled as isotropic elastic (e.g., see Refs. [1,2] and references cited therein), the only other study on FG anisotropic plate [3] has assumed that all elastic constants vary exponentially through the plate thickness at the same rate. It is highly unlikely that elastic moduli of a FG anisotropic plate will exhibit this property. Here we consider a FG anisotropic plate in which the fiber orientation varies smoothly through the plate thickness.

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2. Brief description of the method

As shown in Fig. 1, deformations of the plate are described with respect to rectangular Cartesian coordinate axes aligned along the mutually orthogonal edges of the plate and the origin located on the midsurface. The length, $a = 200\,\text{mm}$, of the plate along the x-axis equals twice that along the y-axis. Results have been computed for $a/h = 10$ and 20 where h is the plate thickness along the z-axis. The FSDT may not give good results for $a/h < 10$; hence natural frequencies are computed only for $a/h > 10$. The desired gradation of material properties in the thickness direction is attained by varying the fiber orientation angle, θ, in a graphite/epoxy plate according to the relation

$$\theta = \frac{\pi}{2}\left(\frac{1}{2}+\frac{z}{h}\right)^p, \quad -\frac{h}{2} \leqslant z \leqslant \frac{h}{2}. \tag{1}$$

Thus, the fiber orientation angle varies continuously from 0° at the bottom surface to 90° at the top surface; $p = 0$ in Eq. (1) implies that the plate is made of a homogeneous material with fibers

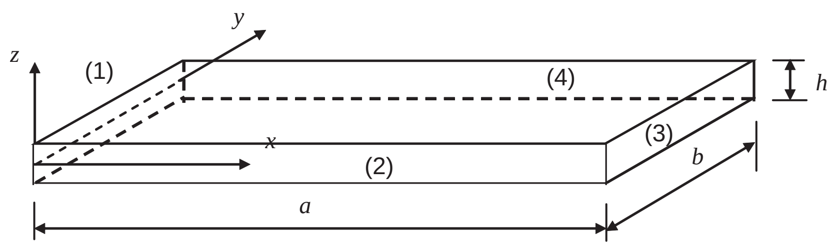

Fig. 1. Schematic sketch of the problem studied.

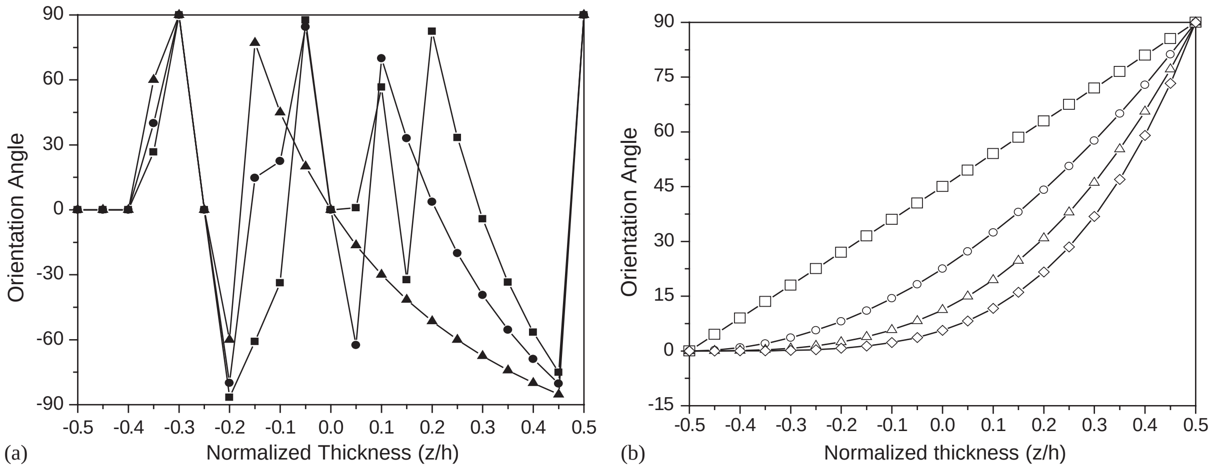

Fig. 2. For different values of the exponent p, variation of the fiber orientation angle through the plate-thickness: (a) —△—, $p = -1$; —○—, $p = -2$; —□—, $p = -3$; (b) —□—, $p = 1$; —○—, $p = 2$; —△—, $p = 3$; —◇—, $p = 4$.

Table 1
First 10 natural frequencies (Hz) of a FG anisotropic rectangular plate for different values of p and the plate divided into 10 and 20 layers (frequencies computed with 20 layers are indicated in parentheses; sides 1 and 3 are clamped, and 2 and 4 are free)

Mode no.	$p=1$		$p=2$		$p=3$		$p=4$	
1	1135.0	(1135.3)	1170.1	(1169.5)	1198.3	(1198.1)	1222.4	(1222.8)
2	1525.7	(1525.6)	1534.5	(1534.6)	1552.7	(1553.9)	1567	(1569.1)
3	2520.2	(2520.7)	2571.5	(2570.7)	2612.1	(2611.7)	2646.7	(2647)
4	3092.4	(3092.7)	3114.7	(3115.1)	3145.2	(3147.1)	3168.4	(3171.8)
5	3539.5	(3541.8)	3550.1	(3553.2)	3520.1	(3523.7)	3486.7	(3491.1)
6	4144.4	(4144.9)	4208.3	(4207.2)	4259.6	(4259.1)	4303.3	(4303.7)
7	4731.1	(4732.3)	4686.2	(4688.5)	4637.5	(4641.1)	4598.4	(4602.1)
8	4779.6	(4779.1)	4768.6	(4769.3)	4807.1	(4809.5)	4837	(4841)
9	5630.5	(5629.4)	5587.5	(5588.7)	5568.8	(5571.7)	5550	(5553.4)
10	5872.7	(5873.2)	5943.2	(5941.9)	6000.1	(5999.5)	6048.1	(6048.6)

Table 2
First five non-dimensional frequencies (f^*) of a simply supported FG anisotropic rectangular plate with $a/h=10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h=10$					
$f_1=2785$, $f_2=3456.2$, $f_3=4625.9$, $f_4=6111.1$, $f_5=6377.7$					
−3	0.92068	1.0565	1.1056	0.9776	1.0384
−2	0.9335	1.0564	1.1008	0.98781	1.0445
−1	0.95978	1.0413	1.0727	1.0088	1.0254
1	0.92248	1.0208	1.0647	0.96811	1.0201
2	0.90524	1.0238	1.0744	0.95809	1.0148
3	0.8944	1.0276	1.0829	0.9525	1.0127
4	0.88506	1.0295	1.0889	0.94778	1.0103
$a/h=20$					
$f_1=1868.6$, $f_2=2313.2$, $f_3=3159.3$, $f_4=4356.8$, $f_5=5302.6$					
−3	0.85262	1.0591	1.1854	1.0277	0.96132
−2	0.87113	1.0583	1.1737	1.0519	0.97482
−1	0.91352	1.0356	1.1155	1.1061	0.93554
1	0.85834	1.0012	1.1022	1.0131	0.93022
2	0.83292	1.0055	1.123	0.9905	0.91819
3	0.81778	1.0131	1.1428	0.97833	0.91374
4	0.80542	1.0179	1.1579	0.96828	0.90937

oriented along the y-axis, and for $p=\infty$, fibers are aligned along the x-axis. Note that the mass density of the plate is unaffected by the value of p. For different values of p, Fig. 2 exhibits the variation of the fiber orientation through the plate thickness. This can be achieved experimentally by laying successively thin layers of different fiber orientations. Even though we have considered integer values of p, it could be any real number. We assume that these layers are perfectly bonded together. Whereas the gradient of the fiber orientation is continuous for $p>0$, it may be discontinuous for $p<0$. For $p<0$, Eq. (1) gives $\theta > 2\pi$ for z/h close to -0.5. At such points, $2n\pi$, where n is an integer, should be subtracted from θ to get the fiber orientation.

The response of the 0° layer is assumed to be orthotropic with the material principal axes coincident with the coordinate axes, and elastic constants of other orthotropic layers are derived by applying tensor transformation rules to those of the 0° layer. Values assigned to different material parameters for the 0° AS4/3501 layer are

$$\begin{aligned} &E_{11} = 132.38\,\text{GPa}, \quad E_{22} = E_{33} = 10.76\,\text{GPa}, \quad G_{23} = 3.61\,\text{GPa}, \quad G_{13} = G_{12} = 5.65\,\text{GPa}, \\ &\nu_{12} = \nu_{13} = 0.24, \;\; \nu_{23} = 0.49. \end{aligned} \tag{2}$$

Here, E_{11}, E_{22} and E_{33} are Young's moduli along the x-, the y- and the z-axis respectively, ν's are Poisson's ratios, and G's are shear moduli.

The FE code [4] based on the FSDT and used to find natural frequencies has been validated by comparing the computed frequencies of a laminated anisotropic graphite/epoxy plate with the corresponding experimental values reported in the literature, and also with those obtained by analysing the three-dimensional problem with the commercial FE code ANSYS. The FSDT based

Table 3
First five non-dimensional frequencies (f^*) of a clamped FG anisotropic rectangular plate with $a/h = 10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz, are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h = 10$					
$f_1 = 3363.2$, $f_2 = 3983.1$, $f_3 = 5047.1$, $f_4 = 6416.2$, $f_5 = 6495.6$					
−3	0.97598	1.0386	1.0709	0.97385	1.0562
−2	0.98121	1.0381	1.0674	0.97957	1.0585
−1	0.98909	1.0277	1.0481	0.99135	1.0386
1	0.96289	1.0113	1.0401	0.96746	1.0352
2	0.95837	1.015	1.0479	0.96178	1.0385
3	0.95644	1.0191	1.0549	0.95856	1.0377
4	0.95475	1.022	1.0602	0.95592	1.0368
$a/h = 20$					
$f_1 = 2835.7$, $f_2 = 3203.3$, $f_3 = 3948.8$, $f_4 = 5044.9$, $f_5 = 5819.7$					
−3	0.88941	1.0067	1.1038	1.0497	0.99254
−2	0.90433	1.0095	1.0979	1.0641	0.9845
−1	0.9348	1.003	1.0638	1.0911	0.94976
1	0.87167	0.95807	1.0397	1.0401	0.93721
2	0.85806	0.96017	1.0529	1.0258	0.95204
3	0.85196	0.96653	1.0673	1.0177	0.965
4	0.84716	0.97162	1.0789	1.0112	0.96273

code employs 8-node isoparametric elements with 2×2 integration rule to evaluate various integrals over an element.

3. Results and remarks

For $p = 1, 2, 3$ and 4 in Eq. (1), $a/h = 10$, and edges 1 and 3 of the plate clamped and the other two edges free, first ten natural frequencies computed by dividing the plate into 10 and 20 layers of equal thickness are listed in Table 1. It is evident that the two sets of frequencies are essentially identical. Results presented below have been obtained by dividing the plate thickness into 20 uniform layers with the fiber orientation in a layer computed from Eq. (1) by setting z equal to the coordinate of a point on the midsurface of the layer.

Results included in Tables 2–7, clearly establish that the natural frequencies of a plate with different edge conditions can be maximized by smoothly changing the fiber-orientation through the plate thickness. The through-the-thickness variation of the orientation angle depends upon the frequency to be optimized. The value of p that maximizes a frequency can be determined by plotting the frequency vs. p and estimating the value of p. For example, for $a/h = 20$ and all edges simply supported, the fourth frequency is maximum when $p = -1.0$.

Table 4
First five non-dimensional frequencies (f^*) of a FG anisotropic rectangular plate with sides 2 and 4 simply supported and the other two sides free and for $a/h = 10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz, are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h = 10$					
$f_1 = 2594.2$, $f_2 = 2727.5$, $f_3 = 3225.5$, $f_4 = 4200.6$, $f_5 = 5579.7$					
−3	0.82272	0.8906	1.0384	1.1229	1.0244
−2	0.84515	0.90552	1.039	1.1161	1.0381
−1	0.90336	0.93903	1.0253	1.0795	1.0671
1	0.86042	0.89745	0.99687	1.0663	1.0214
2	0.83151	0.87127	0.99721	1.0787	1.0074
3	0.80931	0.85606	1.0001	1.0901	0.99855
4	0.79207	0.84495	1.001	1.0983	0.99177
$a/h = 20$					
$f_1 = 1741.2$, $f_2 = 1831.1$, $f_3 = 2164.7$, $f_4 = 2832.5$, $f_5 = 3866.1$					
−3	0.73823	0.8141	0.99921	1.1559	1.094
−2	0.76752	0.8354	1.0032	1.1457	1.1251
−1	0.84729	0.88728	0.99515	1.0915	1.1365
1	0.78785	0.82912	0.94771	1.0643	1.0911
2	0.7504	0.79428	0.94179	1.0813	1.0615
3	0.72261	0.7738	0.94295	1.0993	1.0441
4	0.70141	0.759	0.94267	1.1127	1.0307

Table 5
First five non-dimensional frequencies (f^*) of a FG anisotropic rectangular plate with sides 2 and 4 clamped and the other two sides free and for $a/h = 10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz, are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h = 10$					
$f_1 = 3812$, $f_2 = 3245.9$, $f_3 = 3562.4$, $f_4 = 4371.7$, $f_5 = 5662.4$					
−3	0.93457	0.95705	1.0372	1.1127	1.063
−2	0.94371	0.96414	1.0367	1.1059	1.0707
−1	0.96464	0.97773	1.0247	1.0724	1.0847
1	0.93344	0.94649	1.0023	1.0621	1.0612
2	0.92341	0.93792	1.0068	1.0754	1.0532
3	0.91706	0.93327	1.011	1.0874	1.048
4	0.91201	0.92945	1.0135	1.0964	1.044
$a/h = 20$					
$f_1 = 2740.7$, $f_2 = 2775.1$, $f_3 = 2935.8$, $f_4 = 3370.5$, $f_5 = 4201.2$					
−3	0.83719	0.85903	0.94291	1.0733	1.1715
−2	0.85759	0.87669	0.95173	1.0692	1.1589
−1	0.9059	0.91791	0.96464	1.0403	1.1022
1	0.83636	0.84912	0.90503	1.0015	1.0843
2	0.81578	0.83129	0.89897	1.0123	1.1068
3	0.80352	0.82084	0.89775	1.0257	1.1291
4	0.79385	0.81244	0.89614	1.0362	1.1468

Except when edges 1 and 3 are either simply supported or clamped, the homogeneous plate with 90° fiber orientation has the maximum first frequency. However, when edges 1 and 3 are either simply supported or clamped, the first frequency is maximum for $p>4$. The first frequency of an anisotropic plate with edges 1 and 3 clamped can be increased by 30% and 46% for $a/h = 10$ and 20, respectively by making it of a FG material with $p \simeq 4$, and the corresponding increases are 54% and 62% when edges 1 and 3 are simply supported.

For a simply supported plate, the second frequency is maximum for $p \simeq -3$ and the improvement over the second frequency of a homogeneous plate is 5.65% for $a/h = 10$ and 5.9% for $a/h = 20$. The maximum increase in the second frequency of a clamped FG plate is about 4% for $a/h = 10$ and virtually zero for $a/h = 20$. The maximum gain of 18.5% is achieved in the third frequency of a simply supported FG plate with $p = -3$ and $a/h = 20$.

When edges 2 and 4 of the FG plate are either simply supported or clamped, the first two frequencies are maximum for a homogeneous plate with 90° fibers. The largest increase in the third, fourth and fifth frequencies equal 3.8%, 15.6% and 17.2% for $a/h = 10$, 20 and 20, respectively.

There is no guarantee that the maximum frequency obtained by using a FG plate is higher than the corresponding maximum frequency of a laminated plate of the same weight and made of layers with discontinuous orientation of fibers. For the latter case, some of the stress components will exhibit jumps across an interface between any two adjoining layers which may induce delamination between them.

Table 6
First five non-dimensional frequencies (f^*) of a FG anisotropic rectangular plate with sides 1 and 3 simply supported and the other two sides free and for $a/h = 10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz, are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h = 10$					
$f_1 = 476.77$, $f_2 = 1220.6$, $f_3 = 1704.9$, $f_4 = 2600.6$, $f_5 = 2793$					
−3	1.4662	1.2248	1.32	1.1993	1.399
−2	1.4288	1.2181	1.2953	1.1908	1.416
−1	1.3033	1.1605	1.1975	1.1394	1.3496
1	1.4136	1.1023	1.2208	1.1088	1.2537
2	1.4847	1.1136	1.2571	1.1212	1.2609
3	1.5146	1.1252	1.2917	1.1343	1.2516
4	1.5425	1.1324	1.3234	1.1448	1.2414
$a/h = 20$					
$f_1 = 246.59$, $f_2 = 683.56$, $f_3 = 956.12$, $f_4 = 1548.4$, $f_5 = 2051.4$					
−3	1.5248	1.2828	1.4535	1.3039	1.3899
−2	1.4799	1.2731	1.4133	1.2868	1.3581
−1	1.3338	1.1967	1.2656	1.2009	1.2376
1	1.4651	1.1308	1.3037	1.1666	1.2761
2	1.5457	1.1492	1.3612	1.1911	1.3249
3	1.5819	1.1647	1.4178	1.2142	1.3675
4	1.6163	1.1745	1.4698	1.2334	1.4058

For a FG isotropic plate clamped at one edge only, Qian and Batra [2] found that material properties ought to be graded only in the axial direction to maximize the first frequency. For the FG anisotropic plate studied here, that degree of freedom is not permitted by Eq. (1). For a graphite/epoxy plate comprised of short fibers, one can grade material properties in all three directions by adjusting the orientation and the volume fraction of fibers. However, this is left for future study.

Batra and Aimmanee [5] have pointed out that in-plane pure distortional modes of vibration are admissible in a simply supported laminated orthotropic plate only if the speeds of shear waves of the same amplitude and propagating in the same direction in every two adjoining laminae are equal. Since the mass density is same in each layer and the shear moduli are different, this condition cannot be satisfied in the FG anisotropic plate studied herein. Furthermore, in-plane pure distoritional vibration modes are usually dominant in a thick plate with $a/h < 10$.

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Table 7
First five non-dimensional frequencies of a FG anisotropic rectangular plate with sides 1 and 3 clamped and the other two sides free and for $a/h = 10$ and 20 (each frequency is normalized with respect to its value for the same plate with 90° fibers in all layers; the latter, in Hz, are listed in the top row of the table)

p	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
$a/h = 10$					
$f_1 = 943.1$, $f_2 = 1394.8$, $f_3 = 2214.1$, $f_4 = 2793$, $f_5 = 2858.3$					
−3	1.2898	1.1823	1.1927	1.1787	1.372
−2	1.2684	1.1752	1.1796	1.172	1.3891
−1	1.1829	1.1262	1.1259	1.1311	1.3287
1	1.2038	1.0938	1.1385	1.1073	1.2391
2	1.2401	1.1002	1.1611	1.1153	1.2431
3	1.2704	1.1141	1.1796	1.1268	12328
4	1.2966	1.125	1.1955	1.1356	1.2214
$a/h = 20$					
$f_1 = 539.3$, $f_2 = 848.0$, $f_3 = 1403.9$, $f_4 = 1852.6$, $f_5 = 2580.2$					
−3	1.4417	1.2874	1.3638	1.2722	1.2362
−2	1.4034	1.2713	1.3345	1.2554	1.2677
−1	1.261	1.1854	1.222	1.1754	1.187
1	1.2957	1.1518	1.2495	1.1549	1.2008
2	1.355	1.1719	1.2962	1.1782	1.173
3	1.4079	1.1998	1.3372	1.2044	1.1596
4	1.4555	1.2226	1.3735	1.226	1.1486

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