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Short Communication

Comments on the MAC and the NCO, and a linear modal correlation coefficient

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1. Introduction

Remarks on the MAC [1,2] and the NCO [2,3] are presented. A modal correlation coefficient is proposed whose development is based on working on deficiencies of the widespread MAC and the NCO.

The modal assurance criterion, MAC, is an extensively used parameter to estimate the degree of correlation between modal vectors. It is commonly used to pair experimentally extracted eigenvectors with theoretically derived ones. Naturally, this criterion has advantages and drawbacks which are discussed in detail next; it suffices in this preface (1) to point out that this *function* is simple to apply and does not demand experimental-eigenvector completeness or system matrices and (2) to note three shortcomings: it is not an orthogonality check, does not have a meaningful domain for its [0,1] rank and is nonlinear.

The normalized cross orthogonality, NCO, is a modification of the MAC tailored to solve its first noted shortcoming. Interestingly, it is herein shown that the NCO also fixes the second MAC defect. Nevertheless, the NCO is still a nonlinear function.

In this technical letter, a critical review and new insights on modal correlation coefficients are presented, both analytically and through examples. Moreover, a linear modal correlation coefficient is suggested.

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2. Critical review and insight on the MAC

The MAC between two modal vectors from different modal matrix estimates has been defined as [2]

$$\mathrm{MAC}(\mathbf{u}_{x,i}, \mathbf{u}_{a,j}) = \frac{\left|\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{u}_{a,j}\right|^2}{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{u}_{x,i})(\mathbf{u}_{a,j}^{\mathrm{T}}\mathbf{u}_{a,j})}. \tag{1}$$

It is commonly used to pair experimentally extracted eigenvectors, $\mathbf{u}_x$, with analytically derived ones, $\mathbf{u}_a$, and in this case the $\mathbf{u}_a$ have to be *reduced* to the elements that correspond to the measurement locations. Its main advantages are that it is easy to apply and it does not require coordinate-complete experimental eigenvectors, or system matrices.

First of all, this function can be written as

$$\mathrm{MAC}(\mathbf{u}_{x,i}, \mathbf{u}_{a,j}) = \frac{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{u}_{a,j})^2}{u_{x,i}^2 u_{a,j}^2}, \tag{2}$$

where u denotes the norm of $\mathbf{u}$; more importantly, through expansion of the numerator, it can be easily shown that the MAC is nothing else but a simple function of the angle α between the eigenvectors,

$$\mathrm{MAC}(\alpha) = \cos^2\alpha. \tag{3}$$

This MAC expression has not been previously considered and is the starting point of this insightful review of correlation coefficients.

The first disadvantage of the MAC is that it only serves as a correlation or collinearity test between eigenvectors; that is, the superior limit of its [0,1] rank means that the two vectors represent the same vibration mode; however, no absolute number in the rank, including 0, signifies orthogonality in the *vibrations* sense (Eq. (17)) [4]. An example may not be necessary to illustrate this point, but because an example is important in the next remarks anyway, it does no harm to introduce it at this point.

It is simply the 3-degree-of-freedom undamped system restrained at one end; it is defined and shown in Fig. 1. The MAC among the three eigenvectors are: 0.238, 0.051 and 0.140; thus, it is shown that there is no unique number in the [0,1] rank that corresponds to orthogonality, particularly 0; in other words, the MAC is not an orthogonality check.

The second MAC drawback is that the corresponding domain for its [0,1] rank is meaningless; that is, out of $[0,\pi/2]$ solely $\alpha = 0$ indicates an absolute physical fact (two vectors representing the same mode). This insignificance of the domain is because its superior limit does not signify anything in practice, which implies that any α $(0 < \alpha < \pi/2)$ does not indicate *absolutely* where the

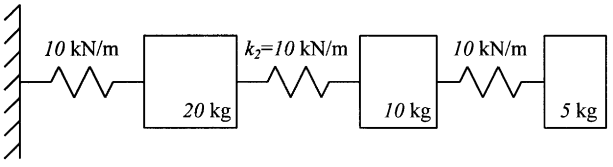

Fig. 1. Sample and analytical system.

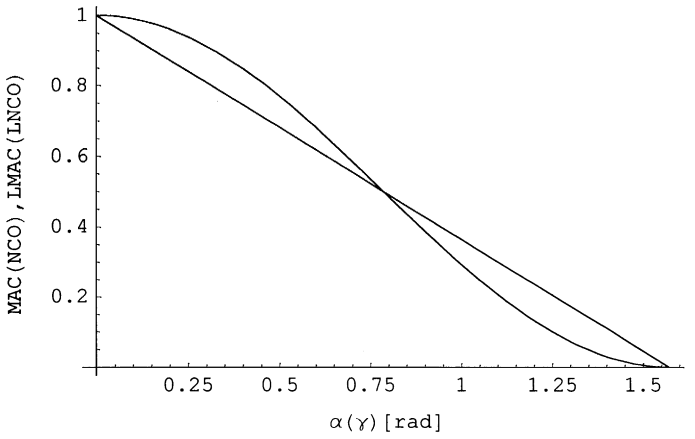

Fig. 2. Nonlinear (MAC and NCO) and linear (LMAC and LNCO) correlation functions.

actual situation is *between* the limiting states of full and null correlation; these points are, of course, related to the first MAC drawback explained previously.

The third important MAC deficiency is that it is not a linear function. The MAC graph is shown in Fig. 2; to understand the practical implication of this nonlinearity, a linear MAC is defined

$$\mathrm{LMAC}(\alpha) = 1 - 2/\pi \;\; \alpha = 1 - 2/\pi \;\; \mathrm{acos}\left|\frac{\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{u}_{a,j}}{u_{x,i}u_{a,j}}\right|, \tag{4}$$

whose graph is also shown in Fig. 2. The fact to note is that in the neighbourhood of $\alpha = 0$, the important one in model updating, as the analytical and experimental modes separate, the MAC is misleadingly yielding values still close to *1*. This fact has been previously noticed; indeed, it has led to define other, more sensitive, correlation coefficients as the normalized modal difference NMD [5,6], which, in short, can be written as

$$\mathrm{NMD} = \sqrt{\frac{1 - \mathrm{MAC}}{\mathrm{MAC}}}. \tag{5}$$

The disadvantage of the NMD is that it is not a bounded function; thus, it is absolutely not an orthogonality check (in the sense that MAC is an approximate check).

The example system (Fig. 1) can serve to further clarify these points; it is considered to be the analytical system. The actual or experimental structure is defined by assuming that the intermediate stiffness k_2 is not correct. MAC and LMAC results are shown in Table 1 for the *second eigenvectors* (corresponding to both second natural frequencies), where $k_{2,x}$ is the experimental middle stiffness. First, note that at a significant separation or *uncorrelation* of 10 (20) degrees, the MAC is misleadingly indicating a high correlation of 97 (88) %. Furthermore, it is observed that the MACs corresponding to an *uncorrelated* case (10°) and a *twice-as-uncorrelated* case (20°) do not express that actual proportionality; in fact, those indicate that the second case is 4 times as uncorrelated (3.9). This deficiency of the MAC is critically consequential when

Table 1
Second mode correlation coefficients and corresponding separation angles

$k_{2,x}$ (kN/m)	10	15.3	24.2
α (rad)	0	0.177 (10°)	0.354 (20°)
MAC	1	0.969	0.880
LMAC	1	0.887	0.774
γ (rad)	0		0.387
NCO	1		0.857
LNCO	1		0.754

comparing MAC values because such comparisons can be, as shown, deceiving; note that one main MAC application is as a *comparative* tool [2]. In contrast, the linear LMAC does not, of course, suffer from these drawbacks; still, the LMAC bears the first two disadvantages of the MAC.

3. Critical review and insight on the NCO

The NCO is regarded as a concept that solves the first MAC drawback [2,7]; interestingly, in the original paper it is not mentioned that this was an objective in developing the NCO [3]. Here it is in its common version,

$$\mathrm{NCO}(\mathbf{u}_{x,i}, \mathbf{u}_{a,j}) = \frac{\left|\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{M}\mathbf{u}_{a,j}\right|^2}{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{M}\mathbf{u}_{x,i})(\mathbf{u}_{a,j}^{\mathrm{T}}\mathbf{M}\mathbf{u}_{a,j})}, \tag{6}$$

where $\mathbf{M}$ is the analytical mass matrix. It can be observed that the NCO is an orthogonality test because in the event of perfectly correlated eigenvectors and in the case of considering two system modes, NCO = 0 (Eq. (17)); nonetheless, to gain more insight on the NCO, the eigenvalue problem in *standard* form is considered [4].

The eigenvalue problem in *two-matrix* form is (e.g. for the theoretical system)

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}, \tag{7}$$

where $\mathbf{K}$ and λ are the analytical stiffness matrix and eigenvalue (square root of natural frequency) and the eigenvector subscript a has been dropped. The mass matrix can be decomposed into [4]

$$\mathbf{M} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}, \tag{8}$$

where $\mathbf{Q}$ is real and nonsingular; thus,

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{u}, \tag{9}$$

Considering the linear transformation

$$\mathbf{v} = \mathbf{Q}\mathbf{u}, \tag{10}$$

the eigenvalue problem can be written as

$$\mathbf{Q}^{\mathrm{T}^{-1}}\mathbf{K}\mathbf{Q}^{-1}\mathbf{v} = \lambda\mathbf{v} \tag{11}$$

or

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \tag{12}$$

where $\mathbf{A}$ is symmetric; such an eigenvalue problem, in terms of a single matrix, is said to be in standard form [4]. In conclusion, the eigenvalue problem of a discrete vibrating system has two forms; certainly, the two forms yield the same eigenvalues; the eigenvectors are different, but closely related by the linear transformation defined by Eq. (10). The concept of *transformed* eigenvectors $\mathbf{v}$ is exploited next; before that, it is noted that from Eq. (12) it can be readily concluded that

$$\hat{\mathbf{v}}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{v}} = \lambda, \tag{13}$$

where the *hat* indicates *unit vector* as usual.

To secure insight on the NCO, it is expressed in terms of the transformed eigenvectors

$$\mathrm{NCO} = \frac{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{u}_{a,j})^2}{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{u}_{x,i})(\mathbf{u}_{a,j}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{u}_{a,j})} = \frac{(\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{v}_{a,j})^2}{v_{x,i}^2 v_{a,j}^2}. \tag{14}$$

Once again, the NCO is nothing else but a simple function of the angle γ between the eigenvectors $\mathbf{v}$,

$$\mathrm{NCO}(\gamma) = \cos^2\gamma. \tag{15}$$

It is emphasized that the eigenvectors $\mathbf{v}$ are no less modal vectors than the eigenvectors $\mathbf{u}$; therefore, there is more similarity between the MAC and the NCO than the fact that these are the same *trigonometric* function of an angle. The main difference, and one with practical (model-updating) implications, between these coefficients is based on the fact that the eigenvectors $\mathbf{v}$ are orthogonal in the usual or *mathematical* sense

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = 0 \quad \text{for } i \neq j, \tag{16}$$

rather than *orthogonal through the mass matrix* as the original eigenvectors $\mathbf{u}$

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_j = 0 \quad \text{for } i \neq j, \tag{17}$$

and that implies that $\mathrm{NCO} = 0$ signifies modal orthogonality (in form (16) or (17) as these are equivalent); therefore, it is confirmed that the NCO is an orthogonality check.

Moreover, the other important implication is that the NCO also solves the second MAC disadvantage; that is, the NCO does have a meaningful domain $[0,\pi/2]$ for its [0,1] rank; $\gamma = 0$ indicates that the two vectors represent the same vibration mode, or perfect correlation, and $\gamma = \pi/2$ signifies that the vectors represent two different mode shapes or null correlation; furthermore, any γ $(0 < \gamma < \pi/2)$ absolutely and linearly indicates how far to *both* limiting states (full or null correlation) the actual situation is. All these facts were not explained at this level in the original paper [3] nor in recent compendia [2,6,7].

The NCO comes in other version [3],

$$\mathrm{NCOK}(\mathbf{u}_{x,i},\mathbf{u}_{a,j})=\frac{\left|\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{K}\mathbf{u}_{a,j}\right|^2}{(\mathbf{u}_{x,i}^{\mathrm{T}}\mathbf{K}\mathbf{u}_{x,i})(\mathbf{u}_{a,j}^{\mathrm{T}}\mathbf{K}\mathbf{u}_{a,j})}. \tag{18}$$

Considering the linear transformation, the coefficient can be written as

$$\mathrm{NCOK}=\frac{(\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}^{-1}}\mathbf{K}\mathbf{Q}^{-1}\mathbf{v}_{a,j})^2}{(\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}^{-1}}\mathbf{K}\mathbf{Q}^{-1}\mathbf{v}_{x,i})(\mathbf{v}_{a,j}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}^{-1}}\mathbf{K}\mathbf{Q}^{-1}\mathbf{v}_{a,j})}=\frac{(\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{A}\mathbf{v}_{a,j})^2}{\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{A}\mathbf{v}_{x,i}\mathbf{v}_{a,j}^{\mathrm{T}}\mathbf{A}\mathbf{v}_{a,j}}=\frac{(\mathbf{v}_{x,i}^{\mathrm{T}}\lambda\mathbf{v}_{a,j})^2}{\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{A}\mathbf{v}_{x,i}\mathbf{v}_{a,j}^{\mathrm{T}}\lambda\mathbf{v}_{a,j}}; \tag{19}$$

finally,

$$\mathrm{NCOK}=\frac{\lambda}{\hat{\mathbf{v}}_{x,i}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{v}}_{x,i}}\cos^2\gamma=\frac{\lambda}{\mu}\cos^2\gamma, \tag{20}$$

where Eq. (13) was considered in the sense that μ is a *pseudoeigenvalue*, corresponding to the *analytical* system matrix $\mathbf{A}$ and the *experimental* eigenvector; in fact, it is shown in Appendix A that μ is the *value* that makes $\mu\hat{\mathbf{v}}_x$ the *closest* to $\mathbf{A}\hat{\mathbf{v}}_x$.It is first observed that besides being a function of the separation angle, identically as the MAC or NCO, this NCOK is also a function of the analytical eigenvalue to pseudoeigenvalue ratio. This is a disadvantage; to show it, suppose an ideal case, that μ were the experimental eigenvalue; in this case the NCOK would be a weighed NCO, where the factor would be the eigenvalue ratio; that is, NCOK would be a measure of natural frequency and mode shape correlation, all in one. Even in this idealization, the NCOK would not be advantageous because of two reasons: (1) the weighing factor admits values larger than 1 and is not bounded and (2) there are already well-established natural-frequency correlation procedures [2]; but μ is not the experimental eigenvalue, it is an *experimental-analytical* "eigenvalue", which would make it even harder to interpret the range of the NCOK function.

Still, the NCOK could be used as suggested in Ref. [3] where it was indicated, although not as clearly, that if the NCOs are larger (lower) *in average* than the NCOKs in paired modes, $\mathbf{K}$ ($\mathbf{M}$) is the less correct model. This statement was verified by an example therein. To absolutely demonstrate it requires further and future analysis on the physical meaning of the pseudoeigenvalue μ; herein, its mathematical meaning has been only shown (Appendix A), and the statement is only reverified by means of the example structure (Fig. 1). For that purpose, the *actual* structure previously defined by $k_{2,x}=15.3\,\mathrm{kN/m}$ is reconsidered. The average values of the NCOs and NCOKs corresponding to the three paired eigenvectors are

$$\overline{\mathrm{NCO}}=0.979\geqslant 0.962=\overline{\mathrm{NCOK}}, \tag{21}$$

where the inequality sense is as expected; despite that, care must be exercised in using the NCOK in this manner until further study is performed on the pseudoeigenvalue concept. Last, it could be argued that the $\mathbf{v}_x$ are not actual experimental eigenvectors because in applying Eq. (10) in the experimental case, the analytical mass matrix $\mathbf{M}$ is involved, as an $\mathbf{M}_x$ is never available in practice; nevertheless, it is a common assumption in model updating that the finite-element inertia matrix is correct (review books [7,6]; recent research reports [8,9]). In cases in which the mass matrix is updated, the $\mathbf{v}_x$ should be considered as approximate eigenvectors.

4. Linear modal correlation coefficients

The NCO solves the first two MAC drawbacks; however, it is still a nonlinear function as it can be concluded from expression (15); in fact, the graph of this function is already shown in Fig. 2. The practical and negative implications of a nonlinear correlation coefficient have been already explained; consequently, a linear NCO is proposed without further extension:

$$\mathrm{LNCO}(\gamma) = 1 - 2/\pi \;\; \gamma = 1 - 2/\pi \quad \mathrm{acos}\left|\frac{\mathbf{v}_{x,i}^{\mathrm{T}}\mathbf{v}_{a,j}}{v_{x,i}v_{a,j}}\right|, \tag{22}$$

whose graph is also shown in Fig. 2. In practice, this function should be applied as

$$\mathrm{LNCO} = 1 - 2/\pi \;\; \mathrm{acos}\left|\tilde{\mathbf{u}}_{x,i}^{\mathrm{T}}\mathbf{M}\tilde{\mathbf{u}}_{a,j}\right|, \tag{23}$$

where the *wavy hat* indicates the usual normalization,

$$\tilde{\mathbf{u}}_i^{\mathrm{T}}\mathbf{M}\tilde{\mathbf{u}}_i = 1. \tag{24}$$

Therefore, this LNCO is an orthogonality check, does have a meaningful domain for its [0,1] rank and is linear. The LNCO has been calculated for one of the exercises ($k_{2,x} = 24.2$ kN/m) related to Table 1 and is shown there. Perhaps a nice end to this technical letter would be to note that this LNCO result (0.754) *absolutely* indicates that the situation is characterized by 75% of (perfect) correlation and 25% of uncorrelation (orthogonality), which neatly summarizes the three advantages of the proposed LNCO over the MAC. Nonetheless, this cannot finish without noting a disadvantage of the LNCO, which is shared by the NCO; it is that experimental-eigenvector completeness is required, which means that an *expansion* procedure must be applied to the experimental vectors [6]. In less demanding cases in which the simplicity of the MAC may be preferred, the proposed LMAC (Eq. (4)),

$$\mathrm{LMAC} = 1 - 2/\pi \;\; \mathrm{acos}\left|\tilde{\mathbf{u}}_{x,i}^{\mathrm{T}}\tilde{\mathbf{u}}_{a,j}\right|, \tag{25}$$

should be selected over the MAC for its linearity, which is fundamental in comparative exercises, which are a main application of correlation coefficients, as discussed.

5. Conclusions

A critical review and new insights on the MAC, NCO and NCOK are presented. A linear modal correlation coefficient LNCO is proposed that solves the defects of the previous coefficients; that is, the LNCO is an orthogonality check, does have a meaningful domain for its [0,1] rank and is linear.

Appendix A. Mathematical meaning of the pseudoeigenvalue

It is shown here that the *pseudoeigenvalue* μ introduced in Eq. (20) is the *particular value* of κ that makes $\kappa\hat{\mathbf{v}}_x$ the *closest* to $\mathbf{A}\hat{\mathbf{v}}_x$. Consider the representation of corresponding solutions of the

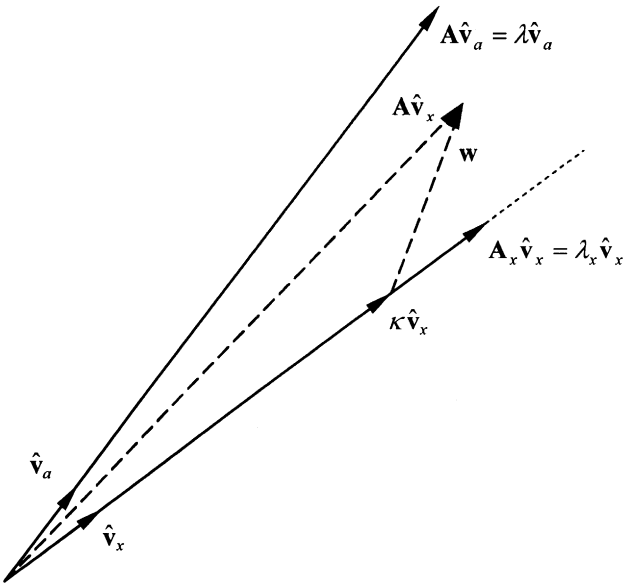

Fig. 3. Transformation of the experimental eigenvector $\hat{\mathbf{v}}_x$ through the analytical matrix $\mathbf{A}$.

analytical and experimental eigenproblems in Fig. 3. Next, note that the transformation of the experimental eigenvector through the analytical system matrix can be expressed as

$$\mathbf{A}\hat{\mathbf{v}}_x = \kappa\hat{\mathbf{v}}_x + \mathbf{w}.$$

Thus, the denominator in Eq. (20) can be written as

$$\hat{\mathbf{v}}_x^{\mathrm{T}}\mathbf{A}\hat{\mathbf{v}}_x = \kappa + \hat{\mathbf{v}}_x^{\mathrm{T}}\mathbf{w}.$$

Now, there is a *value* of κ that minimizes the difference (Euclidean) between $\mathbf{A}\hat{\mathbf{v}}_x$ and $\kappa\hat{\mathbf{v}}_x$, a situation in which $\hat{\mathbf{v}}_x^{\mathrm{T}}\mathbf{w}$ is cancelled; therefore, that pseudoeigenvalue is

$$\mu = \hat{\mathbf{v}}_x^{\mathrm{T}}\mathbf{A}\hat{\mathbf{v}}_x.$$

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