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Short Communication

On the representation of a cantilevered beam carrying a tip mass by an equivalent spring–mass system

M. Gürgöze*

Faculty of Mechanical Engineering, Technical University of Istanbul, 80191 Gümüşsuyu, Istanbul, Turkey

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1. Introduction

It is a well-known fact since Rayleigh that the mass of a linear spring can be taken into account approximately if one-third of the spring mass is added to the mass M at the end of the spring. In a short paper [1] Yamamoto has revisited this problem. Recently, in the context of a research study on the effect of the spring mass on the frequency spectrum of a combined system consisting of a cantilever to which a spring–mass system is attached at the free end, the need arose to estimate the error caused by using an equivalent massless spring. The present note presents some related results of these efforts and represents to some extent the counterpart of Ref. [1] for bending vibrations.

Although it is acknowledged that the contribution of this study does not solve a very complex problem, it is nevertheless thought that the numerical results collected in one table can be helpful to design engineers working in this area.

2. Theory

The exact frequency equation of the vibrational system in Fig. 1, i.e., a cantilevered Bernoulli–Euler beam carrying a tip mass M can be found in the literature [2,3] in the following

*Fax: +90-212-245-07-95.
E-mail address: gurgozem@itu.edu.tr (M. Gürgöze).

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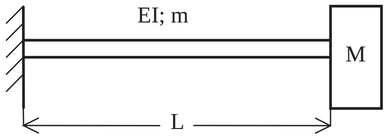

Fig. 1. Cantilevered beam carrying a tip mass.

form:

$$1 + \cos\,\overline{\beta}\,\cosh\,\overline{\beta} + \beta_M \overline{\beta}(\cos\,\overline{\beta}\,\sinh\,\overline{\beta} - \sin\,\overline{\beta}\,\cosh\,\overline{\beta}) = 0, \tag{1}$$

where the abbreviations

$$\beta_M = \frac{M}{mL}, \quad \omega = \overline{\beta}^2 \sqrt{\frac{EI}{mL^4}} \tag{2}$$

are used. Here, β_M and $\overline{\beta}$ denote the non-dimensional mass and frequency parameter. Further, m, EI and ω represent the mass per unit length and bending rigidity of the beam and the eigenfrequency of the combined system, respectively.

On the other hand, when the mass of the beam mL is small in comparison to the tip mass, the restoring effect of the beam on the tip mass can be represented by a massless spring with an equivalent spring stiffness coefficient of [4]

$$k_{eq} = 3EI/L^3. \tag{3}$$

Hence, the fundamental eigenfrequency of the combined system in Fig. 1 can be approximated by

$$\omega_{eq} = \sqrt{\frac{3EI}{ML^3}}. \tag{4}$$

It is reasonable to pose the question of how to express the exact fundamental eigenfrequency ω_1 of the system in terms of ω_{eq}. It can easily be shown that

$$\omega_1 = \psi \omega_{eq}, \tag{5}$$

where the factor ψ is defined as

$$\psi = \overline{\beta}_1^2 \sqrt{\frac{\beta_M}{3}}. \tag{6}$$

As mentioned in the Introduction, the mass of a linear spring can approximately be accounted for in that one-third of its mass is added theoretically to that at the end of the spring. In the context of bending vibrations of the cantilever carrying a tip mass, the number $\alpha = \frac{33}{140}$ is a well known value [5,6]. Its derivation is based on the assumption that during the vibrations the shape of the deflection curve of the beam is the same as the one produced by a load statically applied at the free end [6].

It is in order to investigate the degree of accuracy of the value $\frac{33}{140}$. The task is to represent the combined system in Fig. 1 by an equivalent massless spring–mass system. Let us pose the question

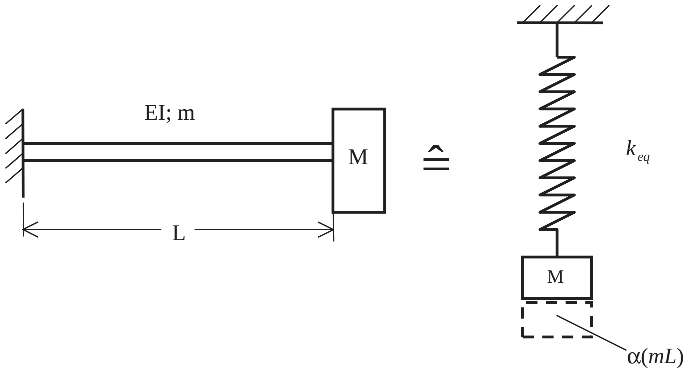

Fig. 2. Equivalent spring–mass system for obtaining the fundamental frequency of the system in Fig. 1.

of how the coefficient α should be determined in the simple system in Fig. 2 in order that the eigenfrequency of this system is equal to that of the one in Fig. 1.

Requiring

$$\omega_1 = \overline{\beta}_1^2 \sqrt{\frac{\beta_M}{3}} \omega_{eq} = \sqrt{\frac{k_{eq}}{M + \alpha(mL)}} \tag{7}$$

one obtains

$$\alpha = \frac{3}{\overline{\beta}_1^4} - \beta_M. \tag{8}$$

Here and in Eq. (6), $\overline{\beta}_1$ represents the first root of the transcendental Eq. (1) for the corresponding β_M value which is to be obtained numerically.

Recognizing that the eigenfrequency obtained with the approximation $\alpha = \frac{33}{140}$ is

$$\omega_{33/140} = \sqrt{\frac{k_{eq}}{M + \frac{33}{140}\, mL}}, \tag{9}$$

it can easily be verified that

$$\delta := \frac{\omega_{33/140}}{\omega_1} = \sqrt{\frac{\beta_M + \alpha}{\beta_M + \frac{33}{140}}}. \tag{10}$$

3. Numerical evaluations

It is obvious that the non-dimensional parameters ψ, α and δ depend (via $\overline{\beta_1}$ or directly) upon the mass parameter β_M. In Table 1, for a wide range of the parameter β_M, the corresponding $\overline{\beta}_1$, ψ, α and δ values are listed. In the second column, the $\overline{\beta}_1$ values are obtained from the numerical solution of the exact frequency Eq. (1). It is seen that for $\beta_M = 0$, i.e., if there is no tip mass M, $\overline{\beta}_1 = 1.875104$ is obtained which is a well-known value for the first dimensionless frequency

Table 1
Collection of $\overline{\beta}_1$, ψ, α and δ values for a wide range of the non-dimensional mass parameter β_M

β_M	$\overline{\beta}_1$	ψ	α	δ
0	1.875104	0	0.242672	1.014652
0.001	1.873233	0.064065	0.242643	1.014529
0.01	1.856787	0.199051	0.242390	1.013494
0.1	1.722742	0.541851	0.240597	1.007246
0.2	1.616400	0.674609	0.239467	1.004297
0.3	1.536143	0.746214	0.238759	1.002838
0.4	1.472408	0.791637	0.238275	1.002012
0.5	1.419964	0.823151	0.237924	1.001500
0.6	1.375669	0.846336	0.237657	1.001161
0.7	1.337499	0.864123	0.237447	1.000926
0.8	1.304087	0.878208	0.237278	1.000755
0.9	1.274462	0.889640	0.237140	1.000627
1	1.247917	0.899106	0.237023	1.000530
2	1.076196	0.945664	0.236435	1.000161
3	0.981231	0.962814	0.236212	1.000077
4	0.917358	0.971734	0.236094	1.000045
5	0.870021	0.977202	0.236021	1.000029
6	0.832826	0.980897	0.235972	1.000021
7	0.802429	0.983562	0.235936	1.000015
8	0.776877	0.985574	0.235909	1.000012
9	0.754937	0.987147	0.235888	1.000009
10	0.735782	0.988411	0.235871	1.000008
15	0.666137	0.992231	0.235820	1.000003
20	0.620512	0.994157	0.235794	1.000002
25	0.587187	0.995318	0.235778	1.000001
50	0.494342	0.997651	0.235746	1.000000
100	0.415934	0.998823	0.235730	1.000000
200	0.349861	0.999411	0.235722	1.000000
300	0.316166	0.999607	0.235720	1.000000
400	0.294240	0.999705	0.235718	1.000000
500	0.278283	0.999764	0.235717	1.000000
600	0.265889	0.999804	0.235717	1.000000
700	0.255840	0.999832	0.235717	1.000000
800	0.247443	0.999853	0.235716	1.000000
900	0.240265	0.999869	0.235716	1.000000
1000	0.234021	0.999882	0.235716	1.000000
2000	0.196793	0.999941	0.235715	1.000000
3000	0.177824	0.999961	0.235715	1.000000
4000	0.165485	0.999971	0.235715	1.000000
5000	0.156507	0.999976	0.235715	1.000000
6000	0.149533	0.999980	0.235715	1.000000
7000	0.143881	0.999983	0.235715	1.000000
8000	0.139157	0.999985	0.235714	1.000000
9000	0.135119	0.999987	0.235714	1.000000
10000	0.131607	0.999988	0.235714	1.000000

parameter of the bare cantilever. Conversely, for extremely large values of $\beta_M, \overline{\beta}_1$ approaches zero which represents the first non-dimensional eigenfrequency parameter of the clamped–simply supported Bernoulli–Euler beam.

Table 1 shows that the coefficient ψ, beginning with zero, goes towards 1 as β_M gets larger. It is seen further that in the range of larger β_M values, i.e., where the tip mass M dominates the mass of the beam mL, the exact fundamental frequency of the combined system in Fig. 1 can be represented more accurately by ω_{eq} as given by Eq. (4).

Before proceeding further, it is in order to calculate the explicit value of $\frac{33}{140}$ which is

$$\alpha = 33/140 = 0.235714. \tag{11}$$

One sees clearly from Table 1 that in a wide β_M-region the $\omega_{33/140}$ as given in Eq. (9) resembles a good approximation for the fundamental eigenfrequency of the vibrational system in Fig. 1, within which region α remains very close to the above value. The last column of Table 1 represents another presentation of this fact.

4. Conclusion

The present note is concerned with the investigation of the degree of approximation in obtaining the eigenfrequency of a cantilever carrying a tip mass M as the eigenfrequency of a massless spring–mass system where the mass is composed of M and α times the mass of the beam. It is thought that the given table can be helpful to design engineers working in this area.

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