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Vibration of beams with double delaminations

Christian N. Della, Dongwei Shu*

School of Mechanical and Production Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore

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Abstract

The vibration of beams with double delaminations has been solved analytically without resorting to numerical approximation. The beam is analyzed as five interconnected beams using the delaminations as their boundaries. The continuity and boundary conditions are satisfied between adjoining beams. Classical beam theory is applied to each of the beams. A new slenderness ratio specific for vibration is introduced, which is shown to dominate the vibration behavior of the beam. Global, mixed and local vibration modes occur depending upon the slenderness ratios of the delaminated beams. Different vibration behaviors emerge for different sizes, depths, spanwise locations and relative slenderness ratios of the delaminations.

1. Introduction

Delaminations are one of the most common defects in composite laminates due to their relatively weak interlaminar strengths. They may arise during fabrication, such as incomplete wetting, or during service, such as low-velocity impact. The presence of delaminations may cause changes to the vibration parameters of a composite laminate, such as the natural frequency and the mode shape. In particular, delaminations reduce the natural frequency, which may cause resonance if the reduced frequency is close to the working frequency. It is imperative that we should be able to predict the change in the frequency, as well as the mode shape, in a dynamic environment.

*Corresponding author. Tel.: +65-67904440; fax: +65-67911859.
E-mail address: mdshu@ntu.edu.sg (D. Shu).

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Wang et al. [1] examined the free vibrations of an isotropic beam with a through-width delamination by using four Euler–Bernoulli beams that are joined together. By applying appropriate boundary and continuity conditions, the response of the beam was obtained as a whole. However, the vibration modes are physically inadmissible for off-midplane delaminations because the delaminated layers were assumed to deform 'freely' without touching each other and thus have different transverse deformations ('free mode'). Mujumdar and Suryanarayan [2] then proposed a model based on the assumption that the delaminated layers are 'constrained' to have identical transverse deformations ('constrained mode'). Similar 'constrained mode' approach was used by Tracy and Pardoen [3] on a simply supported composite beam, Hu and Hwu [4] on a sandwich beam and Shu and Fan [5] on a bimaterial beam. The 'constrained mode' analysis, however, failed to explain the opening in the mode shapes found in the experiments by Shen and Grady [6]. To capture the opening in the mode shapes found in the experiments [6], Luo and Hanagud [7] proposed an analytical model based on the Timoshenko beam theory, which uses piecewise-linear springs to simulate the 'open' and 'closed' behavior between the delaminated surfaces.

The above works are on one-dimensional beam plates with a single delamination. Two-dimensional plates with a single delamination have mostly been numerically investigated. Zak et al. [8,9] used the finite element methods in their study. They modeled the delaminated region by using additional boundary conditions at the delamination fronts. Chattopadhyay et al. [10], Radu and Chattopadhyay [11] and Hu et al. [12] presented finite element methods using higher-order deformation theories.

In practice, multiple delaminations, as well as double delaminations appear frequently in composite laminates, such as those resulting from transverse impact [13–15]. The multiple delaminations have been studied by a number of researchers. Shu [16] presented an analytical solution to study a sandwich beam with double delaminations. His study emphasized on the influence of the contact mode, 'free' and 'constrained', between the delaminated layers and the local deformation at the delamination fronts. Works by Shu and Mai [17–19] on delamination buckling investigated the local deformation near the two fronts of delamination and identified the rigid connector and the soft connector conditions. The cross-section in a rigid connector remains perpendicular to the deformed midplane of the beam, and thus takes account of the differential stretching between the delaminated beams. The cross-section of a soft connector remains perpendicular to the undeformed beam, and thus neglects the differential stretching. Lestari and Hanagud [20] studied a composite beam with multiple delaminations using the Euler–Bernoulli beam theory with piecewise-linear springs to simulate the 'open' and 'closed' behavior between the delaminated surfaces. Lee et al. [21] studied a composite beam with arbitrary lateral and longitudinal multiple delaminations using the 'free mode' analysis and assumed a constant curvature at the multiple-delamination tip. Finite element analyses have been presented by Ju et al. [22] using the Timoshenko beam theory and Lee [23] using the layerwise theory.

Similarly with the single delamination case, two-dimensional plates with multiple delaminations have been numerically investigated. Finite element methods have been developed by Ju et al. [24] using the Mindlin plate theory, Cho and Kim [25] using the higher-order zig-zag theory and Kim et al. [26,27] using the layerwise theory.

The current work presents an analytical solution to the free vibration of beams with double delaminations by treating the beam as five interconnected beams using the delaminations as their boundaries. Classical beam theory is applied to each of the beams. A new slenderness ratio specific for vibration is introduced. The analysis shows complicated vibration modes depending upon the slenderness ratio of the delaminated surface layer. The results further show that the natural frequency is sensitive to the depth, length and spanwise location of the delaminations and the relative slenderness ratio after some threshold values of the depth, the length, the spanwise location or the relative slenderness ratio.

2. Formulation

Fig. 1(a) shows a beam with length L and thickness H with two delaminations. The two delaminations are of identical length a and located at a distance d from the center of the beam. The beam is analyzed as five interconnected beams. The beam is divided into virgin beams 1 and 5, and the delaminated beams 2, 3 and 4 (Fig. 1b).

Two assumptions were made in the literature on delamination buckling and vibrations. The first concerns the complicated changing contact between the delaminated layers. Wang et al. [1] assumed that the delaminated layers deformed 'freely' without touching each other, which was shown to be physically inadmissible [2]. Mujumdar and Suryanarayan [2] then proposed a 'constrained' mode where the delaminated layers are assumed to be in touch along their whole length all the time, but are allowed to slide over each other. The second assumption concerns the deformation of the delamination fronts. Two possibilities were examined by Shu and Mai [17–19], the 'rigid' and the 'soft' connectors. A rigid connector takes full account of the differential-stretching between the delaminated layers while a soft connector disregards the

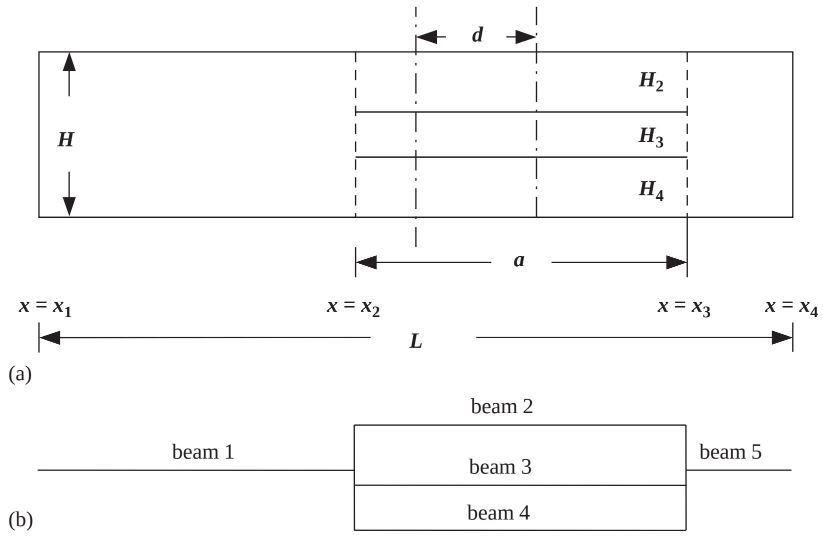

Fig. 1. (a) Geometry of a beam with double delaminations; (b) the beam which is modeled as five interconnected beams.

differential-stretching between the delaminated layers. Studies by Shu and Mai [18] show that the real delamination fronts lie between the two connectors, but closer to the rigid connector (Fig. 2). Further studies show that the natural frequencies obtained using the rigid connector were closer to the experimental values than the frequencies obtained using the soft connector [16]. In this research, the ‘free’ and ‘constrained’ modes and the rigid connector are considered in the analysis.

When the beam vibrates, the thinnest amongst beams 2, 3 and 4 tends to have a higher magnitude of deformation than the thicker ones. However, the thinnest beam may be constrained by the others, and they have to vibrate together in a constrained mode. This greatly complicates the problem, the following cases are then considered:

1. *Free mode*: $H_4 > H_3 > H_2$ during upward motion, or $H_2 > H_3 > H_4$ during downward motion. Without loss of generality, $H_4 > H_3 > H_2$ is chosen (Fig. 3a). In this case, the delaminated beams 2, 3 and 4 vibrate freely and have different transverse deformations.
2. *Partially constrained mode*: $H_4 > H_2 > H_3$ during upward motion, or $H_2 > H_4 > H_3$ during downward motion. $H_4 > H_2 > H_3$ is chosen (Fig. 3b). Although the thinnest beam 3 vibrates first, it will impinge on and be stopped by the thicker beam 2. The two beams vibrate together which will be followed by the thickest beam 4.
3. *Constrained mode*: $H_2 > H_3 > H_4$ or $H_2 > H_4 > H_3$ during upward motion, or $H_4 > H_3 > H_2$ or $H_4 > H_2 > H_3$ during downward motion. Both $H_4 > H_3 > H_2$ (Fig. 3c) and $H_4 > H_2 > H_3$ are chosen and the results will be compared with the results of cases 1 and 2. The two thinner beams 2 and 3 are constrained by the thickest beam 4 and the three beams vibrate together. The formulation of the ‘partially constrained mode’ is similar to the free mode and the constrained mode and is omitted for the sake of brevity. The following sections present the formulation of the free mode and the constrained mode, respectively, for the vibration of a beam with double delaminations.

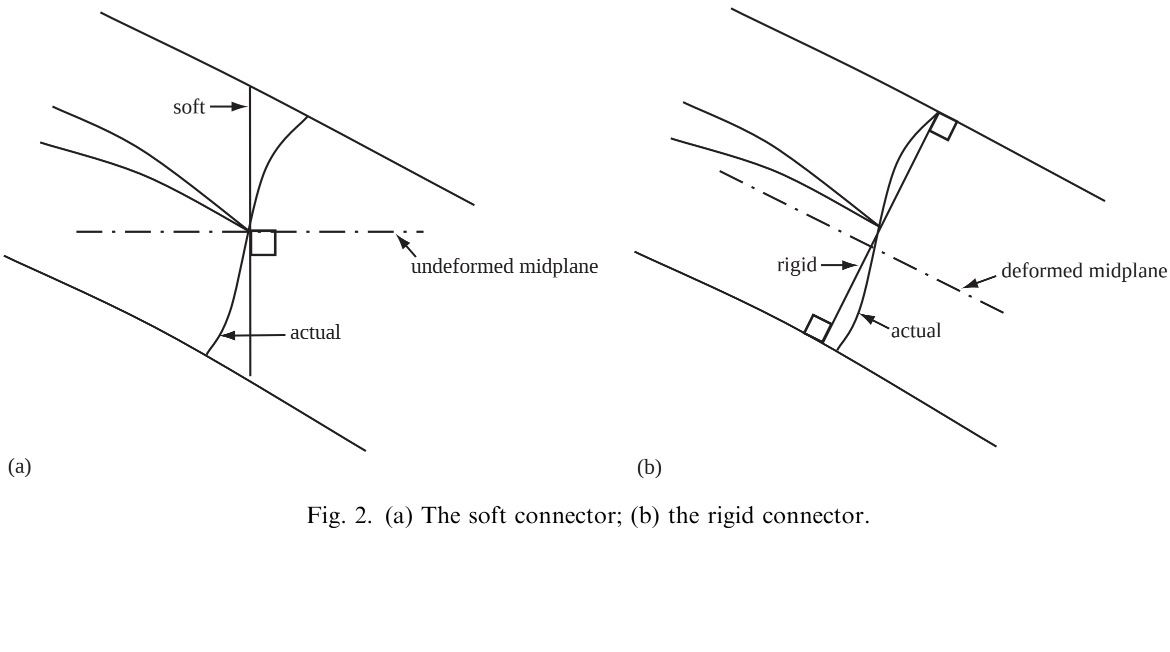

Fig. 2. (a) The soft connector; (b) the rigid connector.

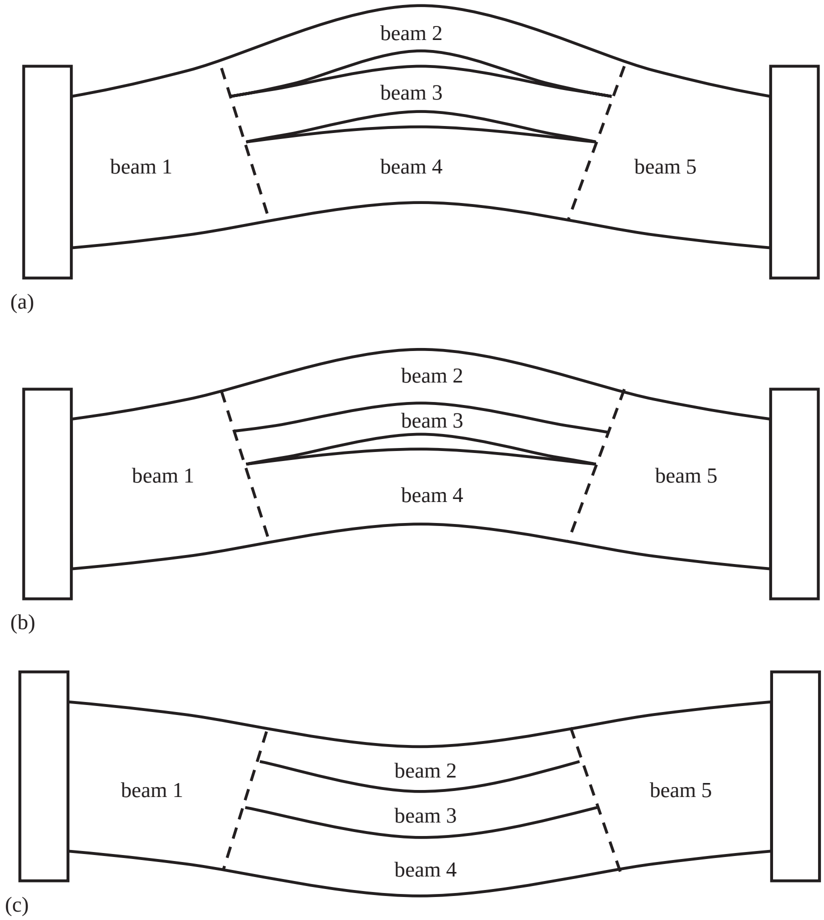

Fig. 3. The beam can either vibrate freely or in a constrained manner. (a) Free mode; (b) partially constrained mode; (c) constrained mode.

2.1. Free mode

The governing equations for the free vibrations of a delaminated beam using the classical Euler–Bernoulli beam theory are

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1, \ldots, 5), \tag{1}$$

where EI_i is the bending stiffness of the ith beam, ρ_i is the mass density and A_i is the cross-sectional area. For plain-stress problem, E is the Young's modulus. For plain-strain problem, E would be replaced by an equivalent Young's modulus $E = E/(1 - \upsilon^2)$, where υ is Poisson's ratio.

For free vibrations

$$w_i(x_i, t) = W_i(x_i) \sin(\omega t), \tag{2}$$

where ω is the natural frequency and W_i is the mode shape. Substituting Eq. (2) in Eq. (1) and eliminating the trivial solution $\sin(\omega t) = 0$, one can obtain the generalized solutions of the differential equation in Eq. (1) as

$$W_i(x) = C_i \cos\left(\lambda_i \frac{x}{L}\right) + S_i \sin\left(\lambda_i \frac{x}{L}\right) + CH_i \cosh\left(\lambda_i \frac{x}{L}\right) + SH_i \sinh\left(\lambda_i \frac{x}{L}\right), \tag{3}$$

where

$$\lambda_i^4 = \frac{\omega^2 \rho_i A_i}{EI_i} L^4 \tag{4}$$

and where λ_i is the non-dimensional frequency. The 20 unknown coefficients C_i, S_i, CH_i and SH_i are determined by 4 boundary conditions and 16 continuity conditions.

The appropriate boundary conditions that can be applied at the supports, $x = x_1$ and $x = x_4$, are $W_i = 0$ and $W'_i = 0$, if the end of the beam is clamped; $W_i = 0$ and $W''_i = 0$, if simply supported; $W''_i = 0$ and $W'''_i = 0$, if free where $i = 1$ and 5 and prime $'$ denotes differentiation with respect to the x-coordinate.

The continuity conditions for deflection and slope at $x = x_2$ are

$$W_1 = W_2, \quad W_1 = W_3, \quad W_1 = W_4, \tag{5}$$

$$W'_1 = W'_2, \quad W'_1 = W'_3, \quad W'_1 = W'_4. \tag{6}$$

From Fig. 4, the continuity for shear and bending moments at $x = x_2$ are

$$V_1 = V_2 + V_3 + V_4, \tag{7}$$

$$M_1 = M_2 + M_3 + M_4 + P_2\left(\frac{H_2}{2}\right) + P_3\left(H_2 + \frac{H_3}{2}\right) + P_4\left(H_2 + H_3 + \frac{H_4}{2}\right), \tag{8}$$

where

$$V_i = -EI_i W'''_i, \tag{9}$$

$$M_i = -EI_i W''_i \quad (i = 1, \ldots, 4). \tag{10}$$

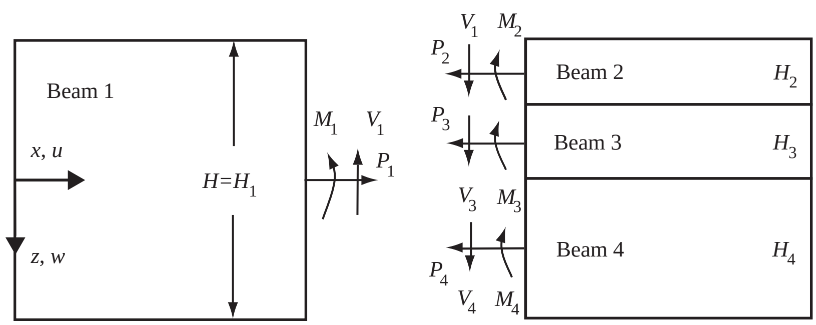

Fig. 4. Continuity of shear and moment at the delamination boundary $x = x_2$.

The axial forces P_i can be solved from the compatibility between the stretching/shortening of the delaminated layers and axial equilibrium [2], thus

$$\frac{P_3 a}{E_3 A_3} - \frac{P_2 a}{E_2 A_2} = (W_1''(x_2) - W_5''(x_3))\frac{H_2 + H_3}{2}, \tag{11}$$

$$\frac{P_4 a}{E_4 A_4} - \frac{P_3 a}{E_3 A_3} = (W_1''(x_2) - W_5''(x_3))\frac{H_3 + H_4}{2}, \tag{12}$$

$$P_1 = P_2 + P_3 + P_4 = 0. \tag{13}$$

Eqs. (7) and (8) can then be expressed as

$$EI_1 W_1''' = EI_2 W_2''' + EI_3 W_3''' + EI_4 W_4''', \tag{14}$$

$$\begin{aligned} &EI_1 W_1'' + \frac{(H + H_3)^2 E_2 A_2 E_4 A_4 + (H_2 + H_3)^2 E_2 A_2 E_3 A_3 + (H_3 + H_4)^2 E_3 A_3 E_4 A_4}{4a(E_2 A_2 + E_3 A_3 + E_4 A_4)} \\ &\times (W_1'(x_2) - W_5'(x_3)) = EI_2 W_2'' + EI_3 W_3'' + EI_4 W_4''. \end{aligned} \tag{15}$$

The second term on the left of Eq. (15) represents the contribution to the bending moment from the differential stretching between beams 2, 3 and 4 and contributes to the bending stiffness of the beam. This term is considered in the rigid connector assumption but neglected in the soft connector.

Similarly, we can derive the continuity conditions at $x = x_3$. The boundary conditions and the continuity conditions provide 20 homogeneous equations for 20 unknown coefficients C_i, S_i, CH_i and SH_i. A non-trivial solution for the coefficients exists only when the determinant of the coefficient matrix vanishes. All the components in the matrix are expressed in terms of the frequency ω, which must be input to the computer before an LDU decomposition is performed. The input of the expressions in the 20×20 matrix and the subsequent LDU decomposition have to be done with care to ensure successful operation.

2.2. Constrained mode

For the constrained mode the governing equations are

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad (i = 1 \text{ and } 5). \tag{16}$$

For delaminated beams 2, 3 and 4, we have

$$(EI_2 + EI_3 + EI_4)\frac{\partial^4 w_2}{\partial x^4} + (\rho_2 A_2 + \rho_3 A_3 + \rho_4 A_4)\frac{\partial^2 w_2}{\partial t^2} = 0. \tag{17}$$

The generalized solutions for the constrained mode are identical in form to the free mode. The unknown coefficients C_i, S_i, CH_i and SH_i, however, are reduced to 12 coefficients which can be determined by 4 boundary conditions and 8 continuity conditions.

The continuity conditions for deflection, slope, shear and bending moments at $x = x_2$ are

$$W_1 = W_2, \tag{18}$$

$$W_1' = W_2', \tag{19}$$

$$EI_1 W_1''' = (EI_2 + EI_3 + EI_4)W_2''', \tag{20}$$

$$EI_1 W_1'' + \frac{(H + H_3)^2 E_2A_2E_4A_4 + (H_2 + H_3)^2 E_2A_2E_3A_3 + (H_3 + H_4)^2 E_3A_3E_4A_4}{4a(E_2A_2 + E_3A_3 + E_4A_4)} \times (W_1'(x_2) - W_5'(x_3)) = (EI_2 + EI_3 + EI_4)W_2''. \tag{21}$$

Similarly, we can derive the continuity conditions at $x = x_3$. For the constrained mode, the boundary conditions and the continuity conditions provide 12 homogeneous equations for 12 unknown coefficients C_i, S_i, CH_i and SH_i.

3. Results and discussions

This section presents the results obtained using the analytical model described above to study a clamped–clamped homogeneous beam with double delaminations. To verify the accuracy of the present results, a comparison with published results on a single delamination is made. The first three non-dimensional natural frequencies of a clamped–clamped beam with a midplane and central delamination having various lengths are compared with the analytical results of Wang et al. [1] and finite element method results of Lee [23]. Tables 1–3 show good agreement between the present results and the analytical and finite element method results.

Fig. 5 shows the influence of the surface layer thickness, H_2, on the fundamental frequency of the beam. The delaminations are symmetrically located about the beam center ($d = 0.0$). The fundamental frequency is normalized with respect to the frequency of an undelaminated beam. For the constrained mode, ω/ω_0 decreases in phase I, with the lowest frequency occurring at

Table 1
Non-dimensional fundamental frequency (λ^2) of a clamped–clamped isotropic beam with a midplane delamination

Delamination length, a/L	Present cons and free	Analytical [1]	FEM [23]
0.00	22.37	22.39	22.36
0.10	22.37	22.37	22.36
0.20	22.36	22.35	22.35
0.30	22.24	22.23	22.23
0.40	21.83	21.83	21.82
0.50	20.89	20.88	20.88
0.60	19.30	19.29	19.28
0.70	17.23	17.23	17.22
0.80	15.05	15.05	15.05
0.90	13.00	13.00	12.99

Table 2
Non-dimensional mode 2 frequency (λ^2) of a clamped–clamped isotropic beam with a midplane delamination

Delamination length, a/L	Present cons and free	Analytical [1]	FEM [23]
0.00	61.67	61.67	61.61
0.10	60.81	60.76	60.74
0.20	56.00	55.97	55.95
0.30	49.00	49.00	48.97
0.40	43.89	43.87	43.86
0.50	41.52	41.45	41.50
0.60	41.04	40.93	41.01
0.70	40.82	40.72	40.80
0.80	39.07	39.01	39.04
0.90	35.39	35.38	35.38

Table 3
Non-dimensional mode 3 frequency (λ^2) of a clamped–clamped isotropic beam with a midplane delamination

Delamination length, a/L	Present cons and free	Analytical [1]	FEM [23]
0.00	120.90	120.91	120.68
0.10	120.83	120.81	120.62
0.20	118.87	118.76	118.69
0.30	109.16	109.04	109.03
0.40	93.59	93.57	93.51
0.50	82.29	82.29	82.23
0.60	77.69	77.64	77.64
0.70	77.18	77.05	77.12
0.80	75.43	75.33	75.39
0.90	69.19	69.17	69.16

$H_2 = H/4$. This is because, the bending stiffness of the delaminated beams, $EI_2 + EI_3 + EI_4$, decreases in phase I and has the lowest value when $H_2 = H/4$. The decrease, however, is less significant for $a < 0.5L$. In phase II, ω/ω_0 increases which is due to the increasing bending stiffness of the delaminated beams. For the free mode (phase I) with thin surface layers ($H_3 > H_2$), ω/ω_0 increases rapidly. A kink is observed for the curves $a > 0.4L$ at a location near $H_2 = H/4$, after which ω/ω_0 increases relatively slowly (phase II). The reason is that, for $H_2 > H/4$, $H_3 < H/4$, thus beam 2 is thicker than beam 3 (partially constrained mode). Beam 3 vibrates first and pushes beam 2. The push weakens beam 2 and slows the increase in the frequency. The partially constrained mode and the constrained mode frequencies converge at $H_2 = H/2$. This is because, when $H_2 = H/2$, $H_3 = 0$, which results in a single midplane delamination where the free mode and the 'constrained mode' frequencies are identical [2,7].

The free mode mode shapes ($H_2 < H/4$) and the partially constrained mode mode shapes ($H_2 > H/4$) for the circled geometries in Fig. 5 are computed and shown in Fig. 6. Three types of vibration modes are observed, local, mixed and global vibration modes. In a local vibration mode,

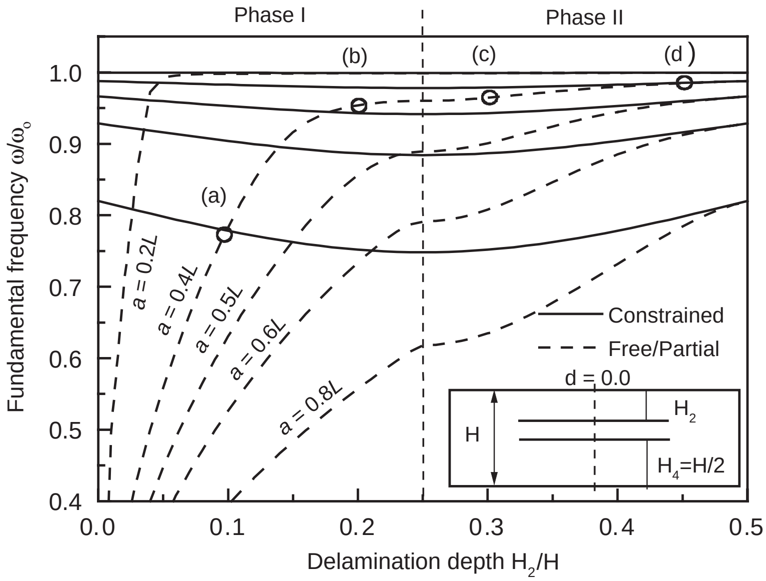

Fig. 5. Variation of the fundamental frequency ω/ω_0 with the surface delamination depth H_2 in two phases.

one or two of the beams 2, 3 and 4 will have much higher amplitude than the rest of the beams, in a mixed vibration mode, the amplitudes of the beams 2, 3 and 4 are comparable, and in a global vibration mode, the amplitudes of beams 2, 3 and 4 are almost equal. When the delamination is located before the kink in Fig. 5, a local vibration mode is observed (Fig. 6a). When the delamination is located near the kink but before the transition point $H_2 = H/4$, a mixed vibration mode is observed (Fig. 6b), where beams 2, 3 and 4 vibrates freely and have different transverse deformations. When the delamination is located near the kink but after the transition point, a mixed vibration mode is observed (Fig. 6c), however, beam 3 is constrained by the thicker beam 2 and the two beams vibrate in a constrained manner. When the delamination is near midplane, a global vibration mode is observed (Fig. 6d), this is because the thickness of beams 2 and 4 are almost the same. The opening in the mode shape further explains the difference between the constrained mode frequency and the free mode or partially constrained mode frequency. For a large opening, the difference between the two frequencies is large (Fig. 5). For a small opening, the difference between the two frequencies is small.

Fig. 7 compares the present double delaminations analysis with earlier single delamination analysis. Single midplane delamination ($H_2 = H_3 = H/2$) is chosen to compare with the double delaminations. The formulation of the single delamination has been done before [1,2]. Their analysis is reproduced to calculate the fundamental frequency. Two cases $H_2 = H_3 = H/4$, $H_4 = H/2$; and $H_2 = H_3 = H_4 = H/3$ for the double delaminations are chosen for the present analysis. For $H_2 = H_3 = H/4$, $H_4 = H/2$, the frequencies for the free and partially constrained modes are identical since the transverse deformations of beams 2 and 3 are identical. For $H_2 = H_3 = H_4 = H/3$, the frequency for the free and partially constrained and constrained modes are identical

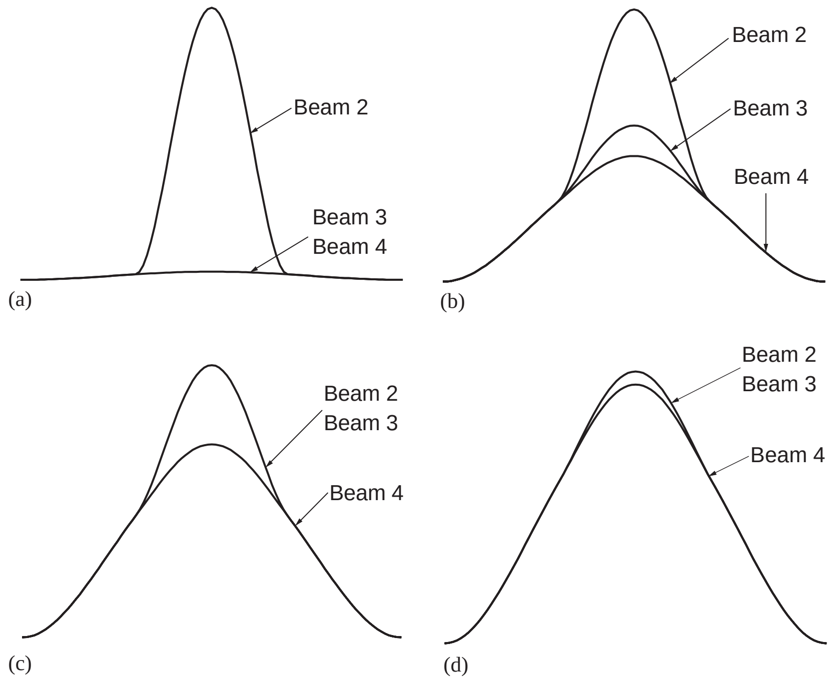

Fig. 6. Vibration modes at $a = 0.4L$ and $d = 0.0$ for various delamination depth H_2/H. (a) $H_2 = 0.1H$; (b) $H_2 = 0.2H$; (c) $H_2 = 0.3H$; (d) $H_2 = 0.45H$.

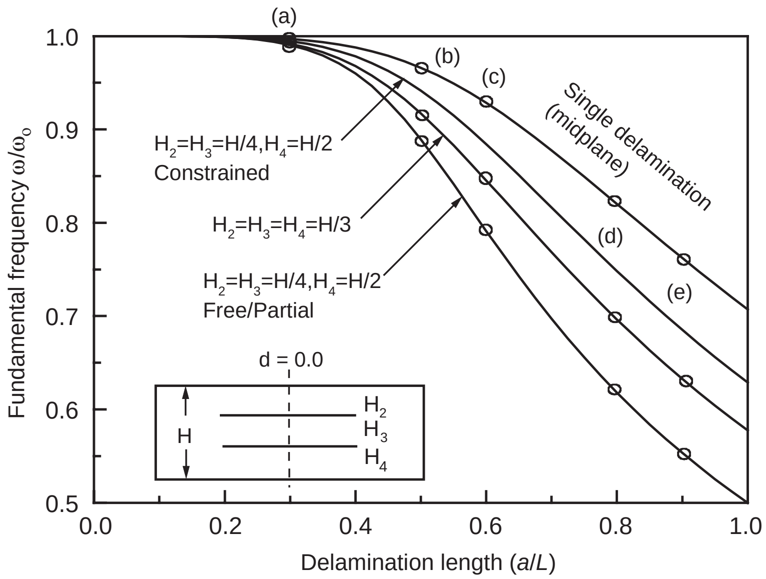

Fig. 7. Compared with a single delamination, double delaminations further reduce the fundamental frequency.

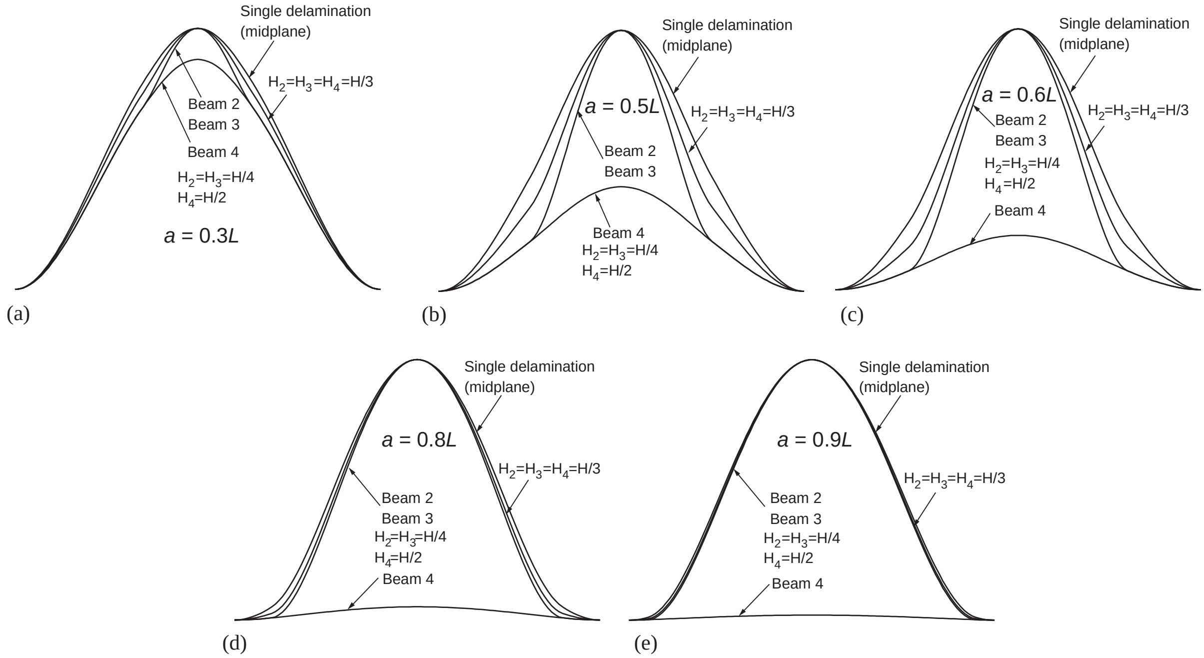

Fig. 8. Vibration modes of a clamped–clamped beam with central delaminations ($d = 0.0$). (a) $a = 0.3L$; (b) $a = 0.5L$; (c) $a = 0.6L$; (d) $a = 0.8L$; (e) $a = 0.9L$.

since the transverse deformations of beams 2, 3 and 4 are identical. The delaminations are located at the center of the beam ($d = 0.0$). For short delaminations ($a < 0.3L$), ω/ω_0 is not sensitive to a. ω/ω_0 decreases slowly up to a value a/L of about 0.5, after which ω/ω_0 decreases rapidly.

The mode shapes for the circled geometries in Fig. 7 are shown in Fig. 8. Fig. 8(a) compares the three types of delaminations for $a = 0.3L$, while Figs. 8(b–e) for $a = 0.5L$, $a = 0.6L$, $a = 0.8L$ and $a = 0.9L$, respectively. The configurations are scaled to similar magnitudes to facilitate comparison. For single delamination $H_2 = H_3 = H/2$, beams 2 and 3 deforms identically. For double delaminations $H_2 = H_3 = H/4$, $H_4 = H/2$, beams 2 and 3 deforms identically. For double delaminations $H_2 = H_3 = H_4 = H/3$, beams 2, 3 and 4 deforms identically. Figs. 8(b, c) show the largest difference in the configuration amongst Figs. 8(a–e), which corresponds to medium length delaminations ($a = 0.5L$, $0.6L$) in Fig. 7. Furthermore, Figs. 8(b, c) display mixed vibration modes, since the magnitudes of the beams 2, 3 and 4 are comparable. Figs. 8(d, e) display local vibration modes, which corresponds to very long delaminations ($a = 0.8L$, $0.9L$), since the amplitudes of beams 2 and 3 are much higher than the amplitude of beam 4.

Fig. 9 shows the variation of the fundamental frequency with the surface layer thickness, H_2, for a clamped–clamped beam. A new slenderness ratio, $R = \omega_0/\omega_2$, is introduced, where $\omega_0 = (4.73^2/L^2)\sqrt{EI_0/(\rho_0 A_0)}$ is the fundamental frequency of the undelaminated beam and $\omega_2 = (4.73^2/a^2)\sqrt{EI_2/(\rho_2 A_2)}$ is the fundamental frequency of the surface beam 2 (Fig. 1(a)). The subscripts 2 and o denote the surface beam 2 and the undelaminated beam, respectively. For homogeneous beams, $R = (a/L)^2/(H_2/H)$. $R = 1$ represents the case when the vibration mode is mixed, for $R < 1$, a global vibration mode and for $R > 1$, a local vibration mode. For the

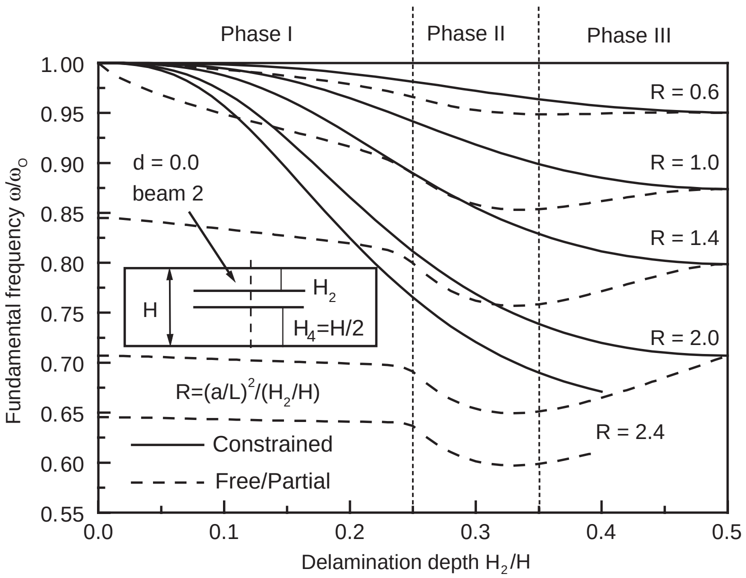

Fig. 9. Fundamental frequency varies with delamination depth.

constrained mode, ω/ω_0 decreases rapidly after a threshold value H_2 of about $0.1H$ and decreases slowly after about $0.3H$. With a higher R, ω/ω_0 decreases more rapidly. For the free and partially constrained modes, the decrease in the frequency varies in three phases. In phase I (free mode), beams 2, 3 and 4 vibrate freely, and R determines an almost constant ω/ω_0. The slight decrease of ω/ω_0 is caused by the relatively weakened constraint at the two ends of beam 2 (Fig. 3a), since beam 2 is thickened in relation to the whole beam. Curve $R = 1$ has the largest decrease in ω/ω_0. In the transition phase II (partially constrained mode), $H_2 > H_3$, beam 3 vibrates first and pushes beam 2 and weakens it, thus ω/ω_0 decreases rapidly. In phase III, the push by beam 3 is weaker and ω/ω_0 stabilizes.

Fig. 10 shows the dominating influence of the slenderness ratio R on the fundamental frequency of the beam. The (H_3 H_4) couples label the crowded five curves by their order. There are two groups of curves $H_3 > H_2$ and $H_2 > H_3$, where curves in group $H_3 > H_2$ represent the free mode and curves in group $H_2 > H_3$ represent the partially constrained mode. The two groups of curves correspond to the two phases in Fig. 5, where phase I (free mode) and phase II (partially constrained mode) have different characteristics. For low R, global vibration modes dominate and the influence of the delaminations vanishes. A good approximation of ω/ω_0 can be obtained for high slenderness ratio ($R > 1.5$). Since for each group, the curves are close to each other. The normalized frequency ω/ω_0 can be assumed to depend on the slenderness ratio only within each group. It is worth noting, however, that the dependence is different for the two groups.

Figs. 11 and 12 show the influence of the spanwise locations, d/L, of the delaminations on the fundamental frequency of a beam. Due to the symmetry of the problem, only $d/L > 0.0$ is shown. At $d/L < 0.05$ ω/ω_0 decreases slowly, after which ω/ω_0 decreases rapidly. This can be explained by the decrease in the differential stretching as the delaminations move towards the beam end.

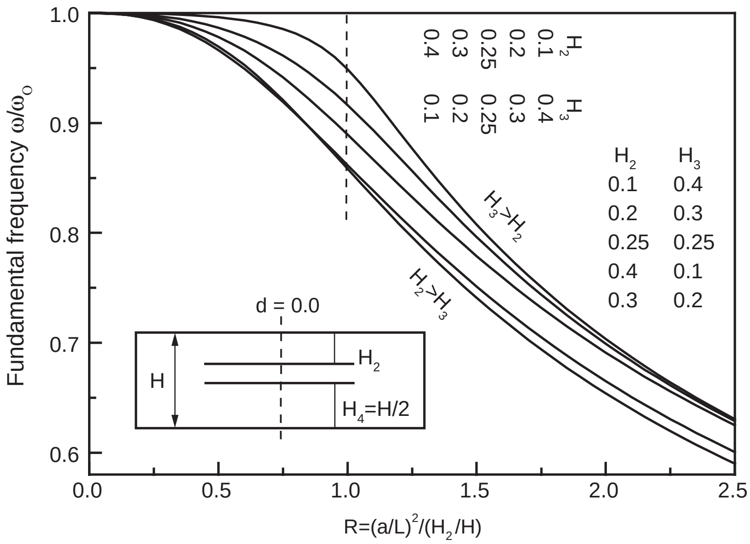

Fig. 10. Slenderness ratio dominates the fundamental frequency for both free and partially constrained modes.

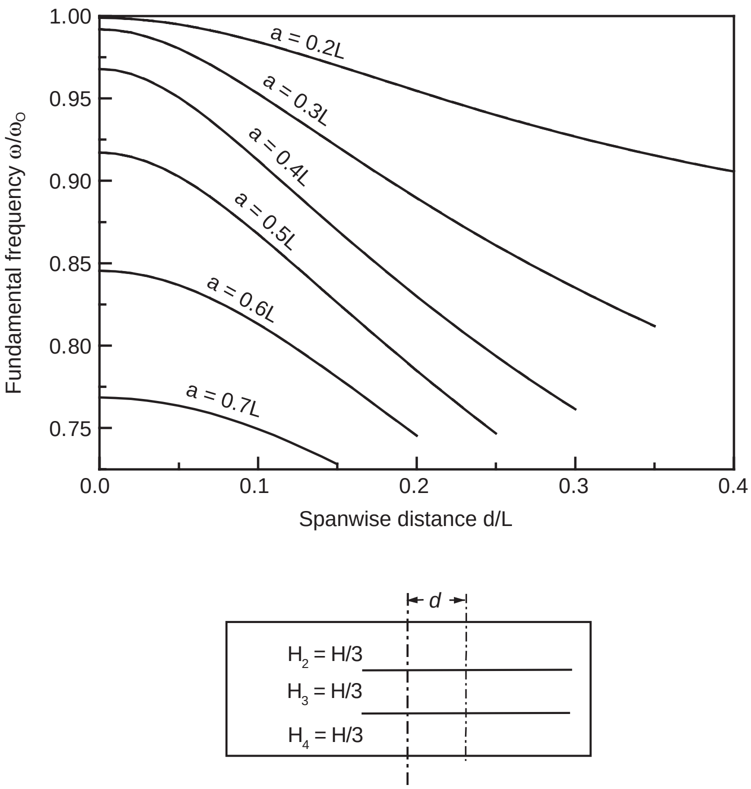

Fig. 11. Fundamental frequency varies with spanwise location, $H_2 = H_3 = H_4 = H/3$.

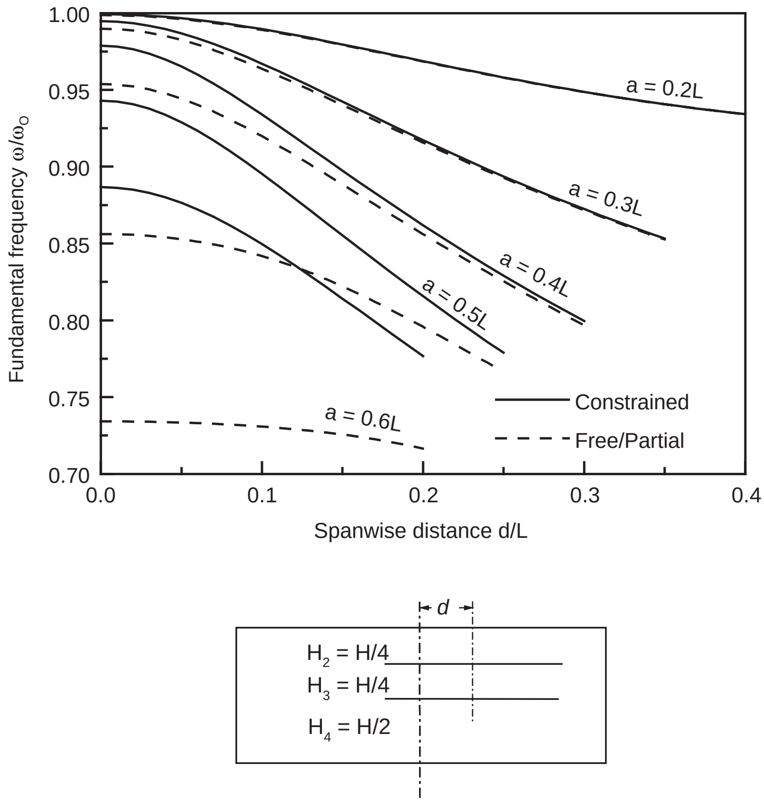

Fig. 12. Fundamental frequency varies with spanwise location, $H_2 = H_3 = H/4, H_4 = H/2$.

The bending moment contribution from the differential stretching represented by the second term on the left-hand side of Eq. (15) is proportional to the difference between the slopes at the two ends of the delaminations $(W_1'(x_2) - W_5'(x_3))$. For a slight movement of the delaminations towards the beam end, the decrease in the difference between the two slopes is less. However, the decrease becomes more significant as the delaminations moves further away from the beam center towards the beam end. This difference also decreases as the delamination lengths increase, which explains the slow decrease in ω/ω_0 for $a/L = 0.6L, 0.7L$. Fig. 12 further shows that the difference between the constrained mode and free mode frequencies decreases as d/L increases, which means that the mode shape displays a larger opening when the delaminations are near the beam center and smaller opening when near the beam end.

Only delaminations of equal lengths are analyzed here. In general, delaminations do not have equal lengths. However, one may envelop another, especially when the delaminations are due to impact, in which case the present solution can be used as either an upper bound or a lower bound solution if the common lengths are taken to be either the short or the long delamination. Apart from being the solution of a basic delamination vibration problem, the solution can also serve as a benchmark test case for other general numerical/approximation schemes for multiple delamination vibration problems.

4. Conclusions

The vibration of beams with double delaminations has been solved analytically without resorting to numerical approximation. The beam is found to vibrate together in a constrained mode, or independently in a free mode, or in a mixed partially constrained mode depending upon the relative thickness of the delaminated layers of the beam. A new slenderness ratio is introduced and is shown to dominate the vibration behavior of the beam. However, the dominance is different for the constrained mode, the free mode, and the partially constrained mode. Global, mixed or local vibration modes occur depending upon the slenderness ratio of the surface layer. The sensitivity of the natural frequency towards the depth, length, and spanwise location of the delamination and the relative slenderness ratio increases rapidly after some threshold values of the depth, the length, the spanwise location or the slenderness ratio.

The results are only for a homogeneous beam. When the laminates are of different materials, it is expected that the quantitative results will be different. However, the trends observed on varying the depth, the length and the spanwise location of the delamination are expected to hold true.

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