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Robust partial eigenvalue assignment problem for the second-order system

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Abstract

In this paper, we study the partial eigenvalue assignment problem for the second-order system, where only a small part of eigenvalues of the open-loop system is to be reassigned, and the rest are required to remain unchanged. It is desirable that the feedback controller not only assigns specific eigenvalues to the second-order closed-loop system but also that the system is robust, or insensitive to perturbations. We propose a numerical method such that the condition number of the matrix of the eigenvectors of the closed-loop system is minimized. In the method, we only need the knowledge of the eigenvalues to be altered and the corresponding eigenvectors, while we do not need the knowledge of the eigenstructures that are required to remain unchanged and are often unknown. Numerical examples show that the present method often leads to better conditioned closed-loop system.

1. Introduction

Consider the second-order dynamical system

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{C}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are $n \times n$ real symmetric matrices with $\mathbf{M}$ positive definite and $\mathbf{K}$ nonsingular, and $\dot{\mathbf{z}}(t)$ and $\ddot{\mathbf{z}}(t)$ denote the first and second derivatives of the $n \times 1$ time-dependent real vector

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$\mathbf{z}(t)$, respectively. Models in the form of Eq. (3) arise frequently in a wide variety of applications in vibration and structural analysis, and the matrices $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are known, respectively, as the mass, damping and stiffness matrices.

Upon separation of variables, system (1) gives rise to the quadratic eigenvalue problem of finding the eigenvalues λ_k and the associated eigenvectors $\mathbf{x}_k \neq \mathbf{0}$, which satisfy

$$Q(\lambda_k)\mathbf{x}_k = \mathbf{0}, \quad k = 1, 2, \ldots, 2n, \tag{2}$$

where

$$Q(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}. \tag{3}$$

In general, the $2n$ eigenvalues are called the poles of system (1).

It is well known that if the $2n$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ of the open-loop quadratic pencil (3) satisfy $\mathrm{Re}(\lambda_j) \leqslant 0$ for all $j = 1, 2, \ldots, 2n$, then the response of system (1) is bounded for arbitrary initial conditions. The response of the system to initial conditions is required in some applications to diminish rapidly. This objective can be achieved by relocating poles of the system in the complex plane. Suppose we wish to alter the location of the poles by applying the control force $\mathbf{f}(t) = \mathbf{B}\mathbf{u}(t)$, where $\mathbf{B}$ is an $n \times m\,(m \leqslant n)$ real matrix and $\mathbf{u}(t)$ is a time-dependent $m \times 1$ real vector. Matrix $\mathbf{B}$ is known as the control matrix, and without loss of generality, we assume that $\mathbf{B}$ has full column rank, that is, $\mathrm{rank}(\mathbf{B}) = m$. The special choice

$$\mathbf{u}(t) = \mathbf{F}^{\mathrm{T}}\dot{\mathbf{z}} + \mathbf{G}^{\mathrm{T}}\mathbf{z},$$

where $\mathbf{F}$ and $\mathbf{G}$ are $n \times m$ real matrices, is called state feedback control, and leads to the closed-loop quadratic eigenvalue problem

$$Q_c(\lambda)\mathbf{x} = \left(\lambda^2\mathbf{M} + \lambda(\mathbf{C} - \mathbf{B}\mathbf{F}^{\mathrm{T}}) + (\mathbf{K} - \mathbf{B}\mathbf{G}^{\mathrm{T}})\right)\mathbf{x} = \mathbf{0}. \tag{4}$$

Mathematically, the problem is then to choose the matrices $\mathbf{F}$ and $\mathbf{G}$ such that the eigenvalues of the closed-loop quadratic eigenvalue problem (4) can be altered as required.

In most practical situations, however, only a few eigenvalues of the open-loop pencil (2) are undesirable, so it makes more sense to alter only those undesirable eigenvalues, while keeping the rest of the spectrum invariant. This leads to the following problem, known as the partial eigenvalue assignment problem for the second-order system.

Problem 1. Given $n \times n$ real symmetric matrices $\mathbf{M}, \mathbf{C}, \mathbf{K}$ with $\mathbf{M}$ positive definite and $\mathbf{K}$ nonsingular, the $n \times m$ real control matrix $\mathbf{B}$, the self-conjugate subset $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ $(p < n)$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the corresponding eigenvector set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$, and given a self-conjugate set $\{\mu_1, \mu_2, \ldots, \mu_p\}$ of numbers, find $n \times m$ real feedback matrices $\mathbf{F}$ and $\mathbf{G}$ such that the spectrum of the closed-loop pencil $Q_c(\lambda)$ in (4) is $\{\mu_1, \mu_2, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

It is known that when $m > 1$, the solution to Problem 1 is essentially undetermined, with many degrees of freedom. Therefore, the question arises as to how this freedom is to be parameterized and how it is to be exploited in practice. It is well known that for any system design a desirable property is that the eigenvalues of the closed-loop system should be insensitive to perturbations in matrix $\mathbf{M}_c = \mathbf{M}, \mathbf{C}_c = \mathbf{C} - \mathbf{B}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{K}_c = \mathbf{K} - \mathbf{B}\mathbf{G}^{\mathrm{T}}$. This leads to the following robust partial eigenvalue assignment problem for the second-order system.

Problem 2. Find a solution $\mathbf{F}, \mathbf{G}$ to Problem 1 defined above, such that the closed-loop system is robust, in the sense that the eigenvalues of the quadratic pencil $Q_c(\lambda)$ in Eq. (4) are as insensitive to perturbations in the matrices $\mathbf{M}_c, \mathbf{C}_c$ and $\mathbf{K}_c$ as possible.

Datta et al. [1] give an algorithm to find a solution for Problem 1. In that paper, three orthogonality relations between the eigenvectors of a symmetric definite quadratic pencil are derived, and then an algorithm is proposed based on one of these relations. Different from some other algorithms developed before, the algorithm in Ref. [1] works directly with the data matrices $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ of the second-order system, rather than the $2n \times 2n$ first-order linearization of the second-order system. This allows the exploitation of matrix structural properties, such as symmetry, sparsity and bandedness. Furthermore, the algorithm does not require knowledge of the unchanged eigenvalues and their corresponding eigenvectors of the open-loop pencil.

In this paper, based on the orthogonality relations derived in Ref. [1], we develop a numerical method to solve Problem 2, that is, we choose a solution $\mathbf{F}, \mathbf{G}$ to Problem 1, in the sense that the condition number of the matrix of the eigenvectors of the closed-loop system is minimized. In the method we use a similar technique as used in Method 0 of Kautsky et al. [2] (see also Ref. [3] for details) for the first-order system. At each iteration in the algorithm, one column of the matrix of the eigenvectors is made closest to the orthogonal complement of the subspace spanned by the other columns by being forced to remain in a certain subspace, while the other columns are kept unchanged. Numerical examples show that the method often leads to better conditioned closed-loop system.

This paper is organized as follows. In Section 2, we introduce some notations and take a deeper look at the three orthogonality relations between the eigenvectors of a symmetric definite quadratic pencil. The algorithm is derived in Section 3. In Section 4, we give some examples to illustrate the performance of the present algorithm. Conclusions are finally drawn in Section 5.

2. Preliminaries

Throughout this paper, the following notations will be used. The $2n$ eigenvalues of the open-loop pencil $Q(\lambda)$ are $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$, and corresponding eigenvectors are $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2n}$, and we let

$$\begin{aligned}
\mathbf{\Lambda} &= \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}),\\
\mathbf{\Lambda}_1 &= \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p), \text{ whose diagonal elements are the poles to be altered,}\\
\mathbf{\Lambda}_2 &= \mathrm{diag}(\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}), \text{ whose diagonal elements are the poles kept unchanged}\\
\mathbf{X} &= [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2n}],\\
\mathbf{X}_1 &= [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p],\\
\mathbf{X}_2 &= [\mathbf{x}_{p+1}, \mathbf{x}_{p+2}, \ldots, \mathbf{x}_{2n}],\\
\mathbf{D}_1 &= \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_p), \text{ whose diagonal elements are eigenvalues of } Q_c(\lambda),\\
\mathbf{Y}_1 &= [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p], \text{ whose columns are corresponding eigenvectors of } Q_c(\lambda).
\end{aligned}$$

In addition, $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{A}$, $\bar{\mathbf{A}}$ the conjugate of $\mathbf{A}$, and $\mathbf{A}^{*}$ the conjugate transpose of $\mathbf{A}$.

We first state the three orthogonality relations between the eigenvectors of a symmetric definite quadratic pencil derived in Ref. [1] without proof (for details see Ref. [1]).

Theorem 2.1. *Let* $Q(\lambda)=\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$, *where* $\mathbf{M},\mathbf{C},\mathbf{K}$ *are real symmetric and* $\mathbf{M}$ *is positive definite*, $\mathbf{K}$ *is nonsingular, and* $\mathbf{\Lambda}_1,\mathbf{\Lambda}_2,\mathbf{X}_1,\mathbf{X}_2$ *are defined as above. Suppose the sets* $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$ *and* $\{\lambda_{p+1},\lambda_{p+2},\ldots,\lambda_{2n}\}$ *are disjoint, then*

1. $\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2-\mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{X}_2=\mathbf{0}$,
2. $\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{C}\mathbf{X}_2\mathbf{\Lambda}_2+\mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{X}_2\mathbf{\Lambda}_2+\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{X}_2=\mathbf{0}$,
3. $\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2+\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{X}_2+\mathbf{X}_1^{\mathrm{T}}\mathbf{C}\mathbf{X}_2=\mathbf{0}$.

Actually, it is easy to prove that the three orthogonality relations are equivalent under the conditions in Theorem 2.1.

For the quadratic eigenvalue problem

$$Q(\lambda)\mathbf{x}=(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})\mathbf{x}=\mathbf{0},$$

under the condition that both $\mathbf{M}$ and $\mathbf{K}$ are nonsingular, it is equivalent to the following generalized eigenvalue problem:

$$\begin{pmatrix}\mathbf{0}&-\mathbf{K}\\-\mathbf{K}&-\mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\\lambda\mathbf{x}\end{pmatrix}=\lambda\begin{pmatrix}-\mathbf{K}&\mathbf{0}\\\mathbf{0}&\mathbf{M}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\\lambda\mathbf{x}\end{pmatrix}. \tag{5}$$

In what follows, we assume that all eigenvalues of $Q(\lambda)$ are nondefective, which means that all eigenvalues of Eq. (5) are nondefective. Thus, it follows that $\binom{\mathbf{X}}{\mathbf{X}\mathbf{\Lambda}}$ can be chosen nonsingular, that is

$$\begin{pmatrix}\mathbf{X}_1&\mathbf{X}_2\\\mathbf{X}_1\mathbf{\Lambda}_1&\mathbf{X}_2\mathbf{\Lambda}_2\end{pmatrix}$$

is nonsingular. Notice that the first orthogonality relation of Theorem 2.1 can be rewritten as

$$\begin{pmatrix}-\mathbf{K}\mathbf{X}_1\\\mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\mathbf{X}_2\\\mathbf{X}_2\mathbf{\Lambda}_2\end{pmatrix}=\mathbf{0}.$$

We know that if a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{\mathrm{T}}\binom{\mathbf{X}_2}{\mathbf{X}_2\mathbf{\Lambda}_2}=\mathbf{0}$, then there must exist a matrix $\mathbf{\Psi}$ such that

$$\mathbf{A}=\begin{pmatrix}-\mathbf{K}\mathbf{X}_1\\\mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\end{pmatrix}\mathbf{\Psi}, \tag{6}$$

under the hypothesis of Theorem 2.1.

We know that $\mathbf{F}$ and $\mathbf{G}$ are solutions to Problem 1, if they satisfy that

$$\mathbf{M}\mathbf{Y}_1\mathbf{D}_1^2+(\mathbf{C}-\mathbf{B}\mathbf{F}^{\mathrm{T}})\mathbf{Y}_1\mathbf{D}_1+(\mathbf{K}-\mathbf{B}\mathbf{G}^{\mathrm{T}})\mathbf{Y}_1=\mathbf{0}, \tag{7}$$

$$\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2^2+(\mathbf{C}-\mathbf{B}\mathbf{F}^{\mathrm{T}})\mathbf{X}_2\mathbf{\Lambda}_2+(\mathbf{K}-\mathbf{B}\mathbf{G}^{\mathrm{T}})\mathbf{X}_2=\mathbf{0}. \tag{8}$$

Since $\mathbf{\Lambda}_2$ and $\mathbf{X}_2$ satisfy that

$$\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2^2 + \mathbf{C}\mathbf{X}_2\mathbf{\Lambda}_2 + \mathbf{K}\mathbf{X}_2 = \mathbf{0},$$

Eq. (8) becomes

$$\mathbf{B}(\mathbf{F}^{\mathrm{T}}\mathbf{X}_2\mathbf{\Lambda}_2 + \mathbf{G}^{\mathrm{T}}\mathbf{X}_2) = \mathbf{0}. \tag{9}$$

Because $\mathbf{B}$ is of full column rank, Eq. (9) implies that

$$\mathbf{F}^{\mathrm{T}}\mathbf{X}_2\mathbf{\Lambda}_2 + \mathbf{G}^{\mathrm{T}}\mathbf{X}_2 = \mathbf{0},$$

that is,

$$\begin{pmatrix} \mathbf{G} \\ \mathbf{F} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{pmatrix} = \mathbf{0}. \tag{10}$$

Then from Eq. (6), $\mathbf{F}$ and $\mathbf{G}$ must have the following form:

$$\mathbf{F} = \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{\Phi}, \quad \mathbf{G} = -\mathbf{K}\mathbf{X}_1\mathbf{\Phi}, \tag{11}$$

where $\mathbf{\Phi}$ is a $p \times m$ matrix. Substituting Eq. (11) into Eq. (7) gives

$$\mathbf{M}\mathbf{Y}_1\mathbf{D}_1^2 + \mathbf{C}\mathbf{Y}_1\mathbf{D}_1 + \mathbf{K}\mathbf{Y}_1 = \mathbf{B}\mathbf{\Phi}^{\mathrm{T}}(\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{Y}_1\mathbf{D}_1 - \mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{Y}_1). \tag{12}$$

Notice that in Eq. (12) both $\mathbf{Y}_1$ and $\mathbf{\Phi}$ are unknown. We can randomly choose p vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p$ satisfying that if $\mu_i = \bar{\mu}_k$, then $\mathbf{r}_i = \bar{\mathbf{r}}_k$, and solve the following p linear systems for $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$, respectively:

$$(\mu_j^2\mathbf{M} + \mu_j\mathbf{C} + \mathbf{K})\mathbf{y}_j = \mathbf{B}\mathbf{r}_j, \quad j = 1, 2, \ldots, p.$$

In this way we can get a matrix $\mathbf{Y}_1 = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p]$. Then we compute

$$\mathbf{Z}_1 = \mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{Y}_1\mathbf{D}_1 - \mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{Y}_1, \tag{13}$$

and solve

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{Z}_1 = \mathbf{\Gamma}_1 \tag{14}$$

for $\mathbf{\Phi}$, where $\mathbf{\Gamma}_1 = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p]$. Of course, here we need the matrix $\mathbf{Z}_1$ is nonsingular. Substitute $\mathbf{\Phi}$ into Eq. (11), then we get $\mathbf{F}$ and $\mathbf{G}$, which is a pair of solutions to Problem 1. These steps show how Datta et al.'s algorithm is derived in Ref. [1], and it is proved in Ref. [1] that such $\mathbf{F}$ and $\mathbf{G}$ must be real.

If Eq. (12) is true, then we can see that for each j ($j = 1, 2, \ldots, p$), there exists an $\mathbf{r}_j$ such that

$$\mathbf{y}_j = (\mu_j^2\mathbf{M} + \mu_j\mathbf{C} + \mathbf{K})^{-1}\mathbf{B}\mathbf{r}_j = (Q(\mu_j)^{-1}\mathbf{b}_1, \ldots, Q(\mu_j)^{-1}\mathbf{b}_m)\mathbf{r}_j,$$

where $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m]$. So if we let $\mathscr{W}_j = \mathrm{span}\{Q(\mu_j)^{-1}\mathbf{b}_1, \ldots, Q(\mu_j)^{-1}\mathbf{b}_m\}$, $\mathbf{y}_j$ ($j = 1, 2, \ldots, p$) must satisfy that

$$\begin{aligned} &\mathbf{y}_j \in \mathscr{W}_j, \quad j = 1, 2, \ldots, p, \\ &\mathbf{y}_i = \bar{\mathbf{y}}_k, \quad \text{if } \mu_i = \bar{\mu}_k. \end{aligned} \tag{15}$$

In summary, we have the following theorem:

Theorem 2.2. *For any set of linearly independent vectors* $\mathbf{y}_j\,(j=1,2,\ldots,p)$ *satisfying condition* (15), *if* $\mathbf{Z}_1=\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}\mathbf{Y}_1\mathbf{D}_1-\mathbf{X}_1^{\mathrm{T}}\mathbf{K}\mathbf{Y}_1$ *is nonsingular, then* $\mathbf{F}=\mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{\Phi}$ *and* $\mathbf{G}=-\mathbf{K}\mathbf{X}_1\mathbf{\Phi}$ *are a pair of solutions to Problem* 1, *where* $\mathbf{\Phi}$ *is defined by Eq.* (16).

In the next section, we shall use this result to develop an algorithm for solving Problem 2.

3. Robust eigenvalue assignment

Now we consider robust eigenvalue assignment problem for the second-order system. Since the quadratic eigenvalues problem (4) is equivalent to the following generalized eigenvalue problem:

$$\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{K}-\mathbf{B}\mathbf{G}^{\mathrm{T}} & \mathbf{C}-\mathbf{B}\mathbf{F}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \lambda\mathbf{x}\end{pmatrix}=\lambda\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \lambda\mathbf{x}\end{pmatrix},$$

a measure of robustness is the condition number of the matrix of the eigenvectors

$$\begin{pmatrix}\mathbf{Y}_1 & \mathbf{X}_2\\ \mathbf{Y}_1\mathbf{D}_1 & \mathbf{X}_2\mathbf{\Lambda}_2\end{pmatrix}. \tag{16}$$

The robust eigenvalue assignment is then concerned with choosing the matrix of the eigenvectors as in Eq. (16) in such a way that the condition number of the matrix of the eigenvectors is as small as possible. Since the matrices $\mathbf{X}_2$ and $\mathbf{\Lambda}_2$ are to remain unaltered, and $\mathbf{D}_1$ is given, the problem then reduces to choosing $\mathbf{Y}_1=[\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_p]$ in such a way that the condition number of the matrix as in Eq. (16) is minimized, where $\mathbf{y}_j\,(j=1,2,\ldots,p)$ must satisfy condition (15). Since $\mathbf{y}_j\in\mathscr{W}_j$, $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}$ must be in the space

$$\tilde{\mathscr{W}}_j=\mathrm{span}\left\{\begin{pmatrix}(Q(\mu_j)^{-1}\mathbf{b}_1 & (Q(\mu_j)^{-1}\mathbf{b}_2 & \cdots & Q(\mu_j)^{-1}\mathbf{b}_m)\\ \mu_j(Q(\mu_j)^{-1}\mathbf{b}_1 & \mu_j(Q(\mu_j)^{-1}\mathbf{b}_2 & \cdots & \mu_jQ(\mu_j)^{-1}\mathbf{b}_m)\end{pmatrix}\right\}. \tag{17}$$

Thus, the problem becomes choosing $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\in\tilde{\mathscr{W}}_j\ (j=1,2,\ldots,p)$ satisfying $\begin{pmatrix}\mathbf{y}_i\\ \mu_i\mathbf{y}_i\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{y}}_k\\ \mu_k\bar{\mathbf{y}}_k\end{pmatrix}$, where $\mu_i=\bar{\mu}_k$, such that the condition number of the matrix as in Eq. (16) is minimized.

In summary, the computation of the solution to Problem 2 requires three major steps:

Step 1: Construct an orthogonal basis, comprised by the columns of matrix $\tilde{\mathbf{W}}_j$, for the space $\tilde{\mathscr{W}}_j, j=1,2,\ldots,p$.

Step 2: Select vectors $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\in\tilde{\mathscr{W}}_j$ with $\left\|\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\right\|_2=1$ satisfying $\begin{pmatrix}\mathbf{y}_i\\ \mu_i\mathbf{y}_i\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{y}}_k\\ \mu_k\bar{\mathbf{y}}_k\end{pmatrix}$, if $\mu_i=\bar{\mu}_k$, such that the matrix as in Eq. (16) is as well-conditioned as possible.

Step 3: From the matrix $\mathbf{Y}_1$ and compute the matrices $\mathbf{Z}_1,\mathbf{\Gamma}_1,\mathbf{\Phi},\mathbf{F}$ and $\mathbf{G}$, which are defined in Section 2.

In Step 1, we do not need to compute an orthogonal basis for the space $\tilde{\mathscr{W}}_j$ directly. Instead of that, we can first compute an orthogonal basis for the space $\mathscr{W}_j$, comprised by the columns of matrix $\mathbf{W}_j$, then the columns of matrix

$$\tilde{\mathbf{W}}_j=\frac{1}{\sqrt{1+|\mu_j|^2}}\begin{pmatrix}\mathbf{W}_j\\ \mu_j\mathbf{W}_j\end{pmatrix} \tag{18}$$

are an orthogonal basis for the space $\tilde{\mathscr{W}}_j$. As for the computation of $\mathbf{W}_j$, here we use QR decomposition, that is, if the QR decomposition of $(Q(\mu_j)^{-1}\mathbf{b}_1,\ldots,Q(\mu_j)^{-1}\mathbf{b}_m)$ is

$$(Q(\mu_j)^{-1}\mathbf{b}_1,\ldots,Q(\mu_j)^{-1}\mathbf{b}_m)=(\mathbf{W}_j,\hat{\mathbf{W}}_j)\begin{pmatrix}\mathbf{R}_j\\ \mathbf{0}\end{pmatrix} \tag{19}$$

with $\mathbf{R}_j$ of full row rank, then $\mathbf{W}_j$ is the required matrix.

Step 2 is the key step of this method. Minimizing the condition number of the matrix of the eigenvectors as in Eq. (16) using optimal methods are usually difficult. Here we use the technique described in Ref. [2]. The objective here is to choose $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\in\tilde{\mathscr{W}}_j$ $(j=1,2,\ldots,p)$ satisfying $\begin{pmatrix}\mathbf{y}_i\\ \mu_i\mathbf{y}_i\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{y}}_k\\ \mu_k\bar{\mathbf{y}}_k\end{pmatrix}$, if $\mu_i=\bar{\mu}_k$, such that each column of the matrix as in Eq. (16) is as orthogonal as possible to the space spanned by the remaining columns. Let

$$\mathscr{X}_j=\operatorname{span}\begin{pmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_{j-1} & \mathbf{y}_{j+1} & \cdots & \mathbf{y}_p & \mathbf{X}_2\\ \mu_1\mathbf{y}_1 & \cdots & \mu_{j-1}\mathbf{y}_{j-1} & \mu_{j+1}\mathbf{y}_{j+1} & \cdots & \mu_p\mathbf{y}_p & \mathbf{X}_2\mathbf{\Lambda}_2\end{pmatrix},$$

and let $\mathbf{z}_j$ be a normalized vector orthogonal to $\mathscr{X}_j$. Then choosing $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\in\tilde{\mathscr{W}}_j$ to be as orthogonal as possible to $\mathscr{X}_j$ is equivalent to choosing $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}\in\tilde{\mathscr{W}}_j$ to minimize the angle between $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}$ and $\mathbf{z}_j$.

The solution can be found by an iteration as follows. Take any set of independent vectors $\mathbf{y}_j\in\mathscr{W}_j$ $(j=1,2,\ldots,p)$ to form an initial matrix $\mathbf{T}=\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_1\mathbf{D}_1\end{pmatrix}$, where $\mathbf{Y}_1=[\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_p]$. The method is then to sweep through the columns of $\mathbf{T}$ replacing $\begin{pmatrix}\mathbf{y}_j\\ \mu_j\mathbf{y}_j\end{pmatrix}$ in turn with $\begin{pmatrix}\tilde{\mathbf{y}}_j\\ \mu_j\tilde{\mathbf{y}}_j\end{pmatrix}$ which is the normalized projection of $\mathbf{z}_j$ onto the space $\tilde{\mathscr{W}}_j$. This ensures that the angel between $\mathbf{z}_j$ and $\begin{pmatrix}\tilde{\mathbf{y}}_j\\ \mu_j\tilde{\mathbf{y}}_j\end{pmatrix}$ is minimized. We must first find $\mathbf{z}_j$. Since $\mathbf{z}_j$ satisfies that

$$\mathbf{z}_j^*\begin{pmatrix}\mathbf{X}_2\\ \mathbf{X}_2\mathbf{\Lambda}_2\end{pmatrix}=\mathbf{0},$$

from Eq. (6), there exists a vector $\mathbf{s}_j$ such that $\bar{\mathbf{z}}_j=\begin{pmatrix}-\mathbf{K}\mathbf{X}_1\\ \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\end{pmatrix}\bar{\mathbf{s}}_j$, that is

$$\mathbf{z}_j=\begin{pmatrix}-\mathbf{K}\bar{\mathbf{X}}_1\\ \mathbf{M}\bar{\mathbf{X}}_1\bar{\mathbf{\Lambda}}_1\end{pmatrix}\mathbf{s}_j. \tag{20}$$

And $\mathbf{s}_j$ must also satisfy that

$$\begin{pmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_{j-1} & \mathbf{y}_{j+1} & \cdots & \mathbf{y}_p\\ \mu_1\mathbf{y}_1 & \cdots & \mu_{j-1}\mathbf{y}_{j-1} & \mu_{j+1}\mathbf{y}_{j+1} & \cdots & \mu_p\mathbf{y}_p\end{pmatrix}^*\begin{pmatrix}-\mathbf{K}\bar{\mathbf{X}}_1\\ \mathbf{M}\bar{\mathbf{X}}_1\bar{\mathbf{\Lambda}}_1\end{pmatrix}\mathbf{s}_j=\mathbf{0}. \tag{21}$$

The matrix in linear system (21) is a $(p-1)\times p$ matrix, and we can find the $\mathbf{s}_j$ such that $\|\mathbf{z}_j\|_2=1$. Thus we have got $\mathbf{z}_j$. Notice that in the computation of $\mathbf{z}_j$, we just need the knowledge of $\mathbf{X}_1$ and $\mathbf{\Lambda}_1$, while we do not need the knowledge of $\mathbf{X}_2$ and $\mathbf{\Lambda}_2$ which are unknown. Then $\begin{pmatrix}\tilde{\mathbf{y}}_j\\ \mu_j\tilde{\mathbf{y}}_j\end{pmatrix}$ is

computed from

$$\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix} = \tilde{\mathbf{W}}_j \tilde{\mathbf{W}}_j^* \mathbf{z}_j / \|\tilde{\mathbf{W}}_j^* \mathbf{z}_j\|_2. \tag{22}$$

Of course, when μ_j is complex, to satisfy requirements (15), $\begin{pmatrix} \tilde{\mathbf{y}}_k \\ \mu_k \tilde{\mathbf{y}}_k \end{pmatrix}$ must be taken to the complex conjugate of $\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix}$, where $\mu_k = \bar{\mu}_j$. Hence, two columns need to be altered simultaneously at this time. The sweeps through the columns of T are continued until the convergence criterion is satisfied or the maximum number of allowed sweeps has been completed. A reasonable criterion for convergence is

$$|\Delta \text{res}| < \text{tol}, \tag{23}$$

where $\text{res} = \|\mathbf{T}^*\mathbf{T} - \mathbf{I}\|_F$ is a sort of measurement of the column orthogonality of $\mathbf{T}$, $\mathbf{I}$ is the $p \times p$ identify matrix, tol is a given tolerance, and Δres is the change of res due to the replacement of $\begin{pmatrix} \mathbf{y}_j \\ \mu_j \mathbf{y}_j \end{pmatrix}$ with $\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix}$, since our objective is that each column of $\mathbf{T}$ is as orthogonal to other columns as possible.

It is worthwhile to point out that when $\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix}$ replaces $\begin{pmatrix} \mathbf{y}_j \\ \mu_j \mathbf{y}_j \end{pmatrix}$, the sensitivities of the other eigenvalues change. Thus, the procedure is not guaranteed to reduce the overall sensitivity and the method may not converge, that is, criterion (23) may not be satisfied. So, to ensure the end of the iteration, we need to set a maximum number of allowed sweeps $k_{\max}$.

As for Step 3, after Step 2 is completed, we have the matrix

$$\begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_p \\ \mu_1 \mathbf{y}_1 & \mu_2 \mathbf{y}_2 & \cdots & \mu_p \mathbf{y}_p \end{pmatrix},$$

then $\mathbf{Y}_1$ is just the first n rows of the matrix. Then we can compute $\mathbf{Z}_1$ by using Eq. (13). As for $\mathbf{\Gamma}_1$, we can store $\tilde{\mathbf{r}}_j = \tilde{\mathbf{W}}_j^* \mathbf{z}_j / \|\tilde{\mathbf{W}}_j^* \mathbf{z}_j\|_2$ at each iteration of Step 2, then we have

$$\mathbf{r}_j = \frac{1}{\sqrt{1 + |\mu_j|^2}} \tilde{\mathbf{r}}_j, \quad j = 1, 2, \ldots, p,$$

and $\mathbf{\Gamma}_1 = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p)$. Then $\mathbf{\Phi}$ is computed by solving Eq. (14), and $\mathbf{F}$ and $\mathbf{G}$ is obtained by substituting $\mathbf{\Phi}$ into Eq. (11).

In summarizing, we have the practical algorithm as follows:

Algorithm 3.1. *Input*: $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathscr{R}^{n \times n}$, $\mathbf{B} \in \mathscr{R}^{n \times m}$, $\mathbf{\Lambda}_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathscr{C}^{p \times p}$, $\mathbf{X}_1 = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \in \mathscr{C}^{n \times p}$, $\mathbf{D}_1 = \text{diag}(\mu_1, \mu_2, \ldots, \mu_p) \in \mathscr{C}^{p \times p}$.

Output: $\mathbf{F}, \mathbf{G} \in \mathscr{R}^{n \times m}$.

1. For each $j = 1, 2, \ldots, p$, compute the QR decomposition of $[Q(\mu_j)^{-1}\mathbf{b}_1, \ldots, Q(\mu_j)^{-1}\mathbf{b}_m]$:

$$[Q(\mu_j)^{-1}\mathbf{b}_1, \ldots, Q(\mu_j)^{-1}\mathbf{b}_m] = (\mathbf{W}_j, \hat{\mathbf{W}}_j) \begin{pmatrix} \mathbf{R}_j \\ \mathbf{0} \end{pmatrix},$$

with $\mathbf{R}_j$ of full row rank, and compute

$$\tilde{\mathbf{W}}_j = \frac{1}{\sqrt{1+|\mu_j|^2}} \begin{pmatrix} \mathbf{W}_j \\ \mu_j \mathbf{W}_j \end{pmatrix},$$

2. Take any set of linearly independent vectors $\mathbf{y}_j \in \mathscr{W}_j$, $j = 1, 2, \ldots, p$ to form an initial matrix

$$\mathbf{T} = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_p \\ \mu_1 \mathbf{y}_1 & \mu_2 \mathbf{y}_2 & \cdots & \mu_p \mathbf{y}_p \end{pmatrix},$$

and set $k = 0$,

3. While $k < k_{\max}$

for $j = 1, 2, \ldots, p$
find a solution $\mathbf{s}_j$ to Eq. (21), and compute $\mathbf{z}_j$ with Eq. (20) and then normalize $\mathbf{z}_j$,
compute $\tilde{\mathbf{r}}_j = \tilde{\mathbf{W}}_j^* \mathbf{z}_j / \|\tilde{\mathbf{W}}_j^* \mathbf{z}_j\|_2$, and $\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix} = \tilde{\mathbf{W}}_j \tilde{\mathbf{r}}_j$,
compute $\mathrm{res}_1 = \|\mathbf{T}^*\mathbf{T} - \mathbf{I}\|_F$,
update the jth column of $\mathbf{T}$ with $\begin{pmatrix} \tilde{\mathbf{y}}_j \\ \mu_j \tilde{\mathbf{y}}_j \end{pmatrix}$,
compute $\mathrm{res}_2 = \|\mathbf{T}^*\mathbf{T} - \mathbf{I}\|_F$,
if $|\mathrm{res}_2 - \mathrm{res}_1| < \mathrm{tol}$, go to 4,
end for
$k = k + 1$,
end while

4. Let $\mathbf{Y}_1 = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)$ be the first n rows of the matrix obtained after iteration, and compute $\mathbf{Z}_1 = \mathbf{\Lambda}_1 \mathbf{X}_1^{\mathrm{T}} \mathbf{M} \mathbf{Y}_1 \mathbf{D}_1 - \mathbf{X}_1^{\mathrm{T}} \mathbf{K} \mathbf{Y}_1$,
5. Compute $\mathbf{r}_j = \frac{1}{\sqrt{1+|\mu_j|^2}} \tilde{\mathbf{r}}_j$, $j = 1, 2, \ldots, p$, and let $\mathbf{\Gamma}_1 = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p)$,
6. Solve $\mathbf{\Phi}^{\mathrm{T}} \mathbf{Z}_1 = \mathbf{\Gamma}_1$ for $\mathbf{\Phi}$,
7. Compute $\mathbf{F} = \mathbf{M} \mathbf{X}_1 \mathbf{\Lambda}_1 \mathbf{\Phi}$ and $\mathbf{G} = -\mathbf{K} \mathbf{X}_1 \mathbf{\Phi}$.

Although this method does not guarantee convergence, it is simple to implement, and numerical examples show that it is efficient for most applications, and often leads to better conditioned closed-loop systems.

4. Numerical examples

To illustrate the performance of the present algorithm, in this section we give some numerical examples, which were carried out using MATLAB 6.0 with machine epsilon $\varepsilon \approx 2.22 \times 10^{-16}$, and we set $\mathrm{tol} = 10^{-6}$, $k_{\max} = p$, where p is the number of poles to be assigned.

We take the following four problems as our test examples:

P1. In this problem, $n = 3, m = 2, p = 2$. It is given in Refs. [4,5], and is defined by

$$\mathbf{M} = 10\mathbf{I}_3, \quad \mathbf{C} = \mathbf{0}, \quad \mathbf{K} = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}.$$

The system is undamped, and the eigenvalues of the quadratic pencil $Q(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ are

$$\{\pm 3.6039\mathrm{i}, \pm 2.49399\mathrm{i}, \pm 0.8901\mathrm{i}\}.$$

We want to alter the first 2 eigenvalues to -1, and -2, while keeping the others unchanged.

P2. In this problem, $n = 10, m = 3, p = 4$. It is from Ref. [6], and

$$\mathbf{M} = \mathbf{I}_{10}, \quad \mathbf{C} = \mathbf{0}, \quad \mathbf{K} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{I}_3 \\ 0 \end{pmatrix}.$$

The 20 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{\pm 1.978\mathrm{i}, \pm 1.911\mathrm{i}, \pm 1.802\mathrm{i}, \pm 1.652\mathrm{i}, \pm 1.466\mathrm{i}, \pm 1.247\mathrm{i}, \pm\mathrm{i}, \pm 0.731\mathrm{i}, \pm 0.445\mathrm{i}, \pm 0.150\mathrm{i}\}.$$

We are to alter the first 4 eigenvalues to $-0.1, -0.2, -0.3, -0.4$, while keeping the others unchanged.

P3. In this problem, $n = 4, m = 2, p = 4$. It is from Ref. [7], and

$$\mathbf{M} = \mathbf{I}_4, \quad \mathbf{C} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 5 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The 8 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{3.525, -3.559, -0.059 \pm 3.732\mathrm{i}, -0.191 \pm 1.489\mathrm{i}, -0.233 \pm 2.692\mathrm{i}\}.$$

We are to alter the first 4 eigenvalues to $-1, -2, -3, -4$, while keeping the others unchanged.

P4. In this problem, $n = 4, m = 2, p = 2$. $\mathbf{M}, \mathbf{C}, \mathbf{K}$ and $\mathbf{B}$ are randomly chosen as

$$\mathbf{M} = \begin{pmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876] \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0.3450 & 0.4578 \\ 0.0579 & 0.7630 \\ 0.5967 & 0.9990 \\ 0.2853 & 0.3063 \end{pmatrix}.$$

It is from Ref. [8], and the 8 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{-0.086 \pm 1.624\mathrm{i}, -0.102 \pm 0.888\mathrm{i}, -0.175 \pm 1.192\mathrm{i}, -0.448 \pm 0.247\mathrm{i}\}.$$

We are to alter the first 2 eigenvalues to $-1 \pm 1.624\mathrm{i}$, while keeping the others unchanged.

We apply Datta et al.'s algorithm in Ref. [1] (for simplicity, we will write it as DER's algorithm) and Algorithm 3.1 presented in this paper to the four problems, respectively. To show the accuracy of the algorithms, we should have computed the differences between the computed solutions and the exact solutions. But unfortunately, the real solutions are unknown for these four problems, so we compute the maximum differences between the computed eigenvalues of the closed-loop systems and the poles to be assigned instead. And to illustrate the performance of the present algorithm, we also compute the condition numbers of the matrices of the eigenvectors of the closed-loop systems generated by both algorithms. Since in both algorithms, the initial matrices are randomly chosen, we run both algorithms 100 times for each problem, and compute the average maximum differences and the average condition numbers, and list them in Table 1.

In Table 1, 'Accuracy' denotes the average maximum difference between the computed eigenvalues of the closed-loop system and the poles to be assigned, while 'Condition number' denotes the average condition number of the matrix of the eigenvectors of the closed-loop system. For each problem, the computed results of the feedback matrices **F** and **G** by each algorithm over 100 trials are often different, so here we do not give these computed results of **F** and **G**.

Since DER's algorithm is proposed for Problem 1, which means that it does not consider robust solutions to Problem 1, we may expect that the present algorithm should improve robustness of the closed-loop systems. From the above table we can see that solutions of both algorithms are accurate enough, since the differences between the computed eigenvalues of the closed-loop system and the poles to be assigned are very small, while the present algorithm does lead to better conditioned closed-loop systems.

Besides, as stated above, the present algorithm may not converge sometimes, and such phenomenon does happen in our numerical experiment. Among the 100 trials of the present algorithm to each problem, the present algorithm stops because the convergence criterion (23) is achieved for 100, 43, 6 and 1 times,respectively. For P3 and P4, the convergence rates are very low indeed, but from Table 1 we can see that even when the convergence criteria (23) is not satisfied, the present algorithm can still give good solutions after a few steps of iterations.

Table 1
Numerical results of applying both algorithms to P1–4

	DER's algorithm		The present algorithm	
	Accuracy	Condition number	Accuracy	Condition number
P1	9.49×10^{-14}	1539.4	1.29×10^{-14}	11.6
P2	6.82×10^{-13}	34665.3	3.63×10^{-13}	1041.1
P3	3.86×10^{-14}	4701.0	1.27×10^{-14}	378.8
P4	4.00×10^{-15}	209.3	3.55×10^{-15}	28.9

5. Conclusions

In this paper, we have developed a numerical method for the robust partial eigenvalue assignment problem of the second-order system. The method computes a solution to the partial eigenvalue assignment problem for the second-order system such that the condition number of the matrix of the eigenvectors of the closed-loop system is as small as possible. At each iteration of the present method, one column of the matrix of the eigenvectors is replaced by the vector that is the normalized projection of $\mathbf{z}_j$ onto the space $\tilde{\mathscr{W}}_j$. It is worthwhile to point out that in the computation of $\mathbf{z}_j$, we do not need the knowledge of $\mathbf{X}_2$ and $\mathbf{\Lambda}_2$, which are required to keep unchanged and are often unknown. Although the method does not guarantee to converge, numerical examples show that the solution of the present method often gives better conditioned matrix of the eigenvectors of the closed-loop system.

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