Geodesics, nonlinear normal modes of conservative vibratory systems and decomposition method ☆

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Abstract

In this paper, the relationship between the nonlinear normal modes (NNMs) of conservative vibratory systems and the geodesics of the corresponding Riemannian manifolds with the Jacobi metric is investigated and a modified Adomian decomposition method for constructing the NNMs is proposed. It is indicated that NNMs in the configuration space are special geodesics of the corresponding Riemannian manifold. These geodesics pass through the origin of the configuration space and with specific directions determined by the total energy of the system. Geodesic equations describing the NNMs become singular at the intersection of the Riemannian manifold and the energy level plane. To solve these singular geodesic equations, the Adomian decomposition method is used with a slight modification. The NNMs of strongly nonlinear vibration systems are constructed via the analytic approximation of the solution of the geodesics. Higher-order approximate NNMs can be constructed by a recursive procedure and the solution series is convergent rapidly. Finally, two examples of nonlinear vibratory systems with two and four degrees of freedom (dof), respectively, are given as illustration. Simulation results verified the effectiveness of this method.
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1. Introduction

For multi-degree-of-freedom or continuous nonlinear vibratory systems, how to analyze their qualitative and quantitative behavior is still a problem to be explored. As a possible approach, the concept and method of nonlinear normal mode (NNM) were proposed and studied by Rosenberg in 1960s [1], and especially, attracted extensive studies since 1990. For conservative vibratory systems, there are basically two different approaches for constructing NNMs. One is the modal configuration space method which uses one-dimensional manifolds to approximate each of the NNMs; see, for example, Refs. [2–7]. The other approach, proposed by Shaw and Pierre [8], is the modal state space method which uses two (for non-internal-resonant case) or $2k$ (internal resonant case)-dimensional manifolds to approximate each of the NNMs. For further reference, see Refs. [9,10].

With the modal configuration space method, the NNMs of nonlinear conservative systems are proved to be solutions of a nonlinear singular boundary value problem (BVP). Generally, singular BVPs are global and can be solved by singular perturbation methods. Constructing the NNMs in this way is not an easy task, because it involves much complicated manipulations. Although the exact series solutions of the NNMs are obtainable in certain cases [11], it is still believed that the scheme of exact series solution is impractical. This is true especially for high-dimensional nonlinear systems. On the other hand, the modal configuration space method has such an advantage that, theoretically, it is applicable to strongly nonlinear systems. With the modal state space method, the velocity and the displacement of a specific mass point are chosen as reference variables to express the NNMs as their Taylor series. This method is constructive and can be used to obtain the approximate NNMs of conservative as well as damped vibratory systems. The drawback of this approach is that, it is local and not applicable to strongly nonlinear systems.

In this paper, based on the Jacobi's theorem [12], the NNMs are reformulated as the special geodesics of the corresponding Riemannian manifold with the Jacobi metric. The dynamical problem to construct the NNMs is converted to a purely geometric problem to determine the special geodesics corresponding to NNMs, among infinite ones passing through the origin of the configuration space. The equations of the geodesics are more suitable for constructing the NNMs, although they are essentially identical to the motion equations of the systems. The obtained geodesic equations become singular at the intersections of the Riemannian manifold and the total energy level plane, so how to solve effectively the geodesic equations is still a problem to be studied.

Customarily, such singular boundary value problems are solved by singular perturbation methods, but in this paper the Adomian decomposition method (ADM) [13,14], which was developed by Adomian since the beginning of 1980s, will be adopted. The traditional Adomian decomposition method will be slightly modified and used to investigate the singular geodesic equations which described the NNMs. In the past decade, a lot of attention has been devoted to the application of ADM to a wide variety of stochastic and deterministic problems involving (ordinary or partial) differential, integral, integro-differential, algebraic and systems of such equations. For the application of the ADM to BVPs, see, for example, Refs. [15,16]. With the ADM, firstly, the original solution of the differential equation is decomposed into an infinite sum of components. Then a recursive procedure is built and each of the components can be easily calculated by simple integration. The noticeable advantage of the ADM is that, analytic,

approximate or even exact, solution could be obtained for rather general nonlinear systems with a rapidly convergent series; it involves much less computation; no linearization and perturbation methods which result in intensive computation are necessary.

In the following, first the NNMs are reformulated as special geodesics on a Riemannian manifold with the Jacobi metric. Then the ADM is used for solving the geodesic equations. Finally, two examples with two degrees of freedom (dof) and four, respectively, are given to illustrate the effectiveness of the decomposition method.

2. NNMs as geodesics

For a given n-dimensional Riemannian manifold, suppose its metric is g_{ij}, then the geodesics on it would satisfy the following geodesic equation [17]:

$$\frac{d^2 u_i}{ds^2} + \sum_{j,k=1}^{n} \Gamma_{jk}^i(u_1, \ldots, u_n) \frac{du_j}{ds} \frac{du_k}{ds} = 0, \tag{1}$$

where s is the arc parameter, u_1, u_2, ..., u_n are the generalized coordinates on the Riemannian manifold. The Γ_{jk}^i stands for the Christoffel symbol and can be computed from the given metric g_{ij} via the following formula:

$$\Gamma_{jk}^i = \frac{1}{2} \sum_l g^{li} \left(\frac{\partial g_{kl}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_k} - \frac{\partial g_{jk}}{\partial u_l} \right). \tag{2}$$

If the arc parameter s is replaced by some generalized coordinate, for example, the first coordinate u_1, then the other generalized coordinates u_i's can be expressed as functions of u_1. That is

$$u_i = u_i(u_1), \quad i = 2, \ldots, n. \tag{3}$$

Therefore

$$\frac{du_i}{ds} = \frac{du_i(u_1)}{du_1} \frac{du_1}{ds}, \quad \frac{d^2 u_i}{ds^2} = \frac{d^2 u_i(u_1)}{du_1^2} \left(\frac{du_1}{ds} \right)^2 + \frac{du_i(u_1)}{du_1} \frac{d^2 u_1}{ds^2}, \quad i = 2, \ldots, n. \tag{4}$$

Substitute these equations into Eq. (1), canceling the common factor of $(du_1/ds)^2$, one obtains the following ODEs for geodesics:

$$\frac{d^2 u_i(u_1)}{du_1^2} - \frac{du_i(u_1)}{du_1} \sum_{j,k=1}^{n} \Gamma_{jk}^1(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1}$$

$$+ \sum_{j,k=1}^{n} \Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1} = 0, \quad i = 2, \ldots, n. \tag{5}$$

This is another form of geodesic equations on a Riemannian manifolds.

For an n-dimensional conservative vibratory system, define the Jacobi metric [12] by

$$g_{jk}(h) = (h - V(u_1, \ldots, u_n)) m_{jk}, \tag{6}$$

where h and $V(u_1,\ldots,u_n)$ are the total energy and the potential function, respectively. m_{jk} is the original metric of the system and it is also the coefficients of the kinetic energy. That is, the kinetic energy T of the vibratory system can be expressed as follows:

$$T = \frac{1}{2}\sum_{j,k=1}^{n} m_{jk}\dot{u}_j\dot{u}_k. \tag{7}$$

According to Jacobi's theorem [12], motions in a potential $V(u_1,\ldots,u_n)$ are actually geodesic motions on a Riemannian manifold with the Jacobi metric.

In nonlinear vibration theory, for conservative systems, NNMs are defined as special motions where all coordinates vary equi-periodically, reaching their extreme values at the same instants of time. As indicated above, all motions in the potential $V(u_1,\ldots,u_n)$ are geodesic motions and therefore satisfy the geodesic equation (5). With no exception, as special geodesic motions, the nonlinear normal modal motions should also satisfy the geodesic equation (5) with the Jacobi metric $g_{jk}(h)$. All the geodesics describing NNMs pass through the origin of generalized coordinate space; however, there are infinite geodesics passing through the origin, so the essential problem is how to identify the nonlinear normal modal geodesics among the infinite candidates described by Eq. (5).

Customarily, the NNMs are constructed by solving Eq. (5) as a BVP. Note that, for this type of BVPs, singularities would always arise on the boundary of $h = V(u_1,\ldots,u_n)$. Because of the global property of the BVP, it is not an easy task to find an analytic approximate solution, even a local series solution near the origin where the BVP is regular locally. Singular perturbation methods are often used to solve Eq. (5). Another approach is to find the exact solution to Eq. (5) by the series method [11].

In principle, Eq. (5) can also be solved as an initial-value problem by the shooting method through supplementing the following initial conditions:

$$u_i(0) = 0, \quad \frac{\mathrm{d}u_i}{\mathrm{d}u_1}(0) = a_i, \tag{8a,b}$$

where the a_i's are unknown parameters to be determined. The target conditions are as follows:

$$h - V(u_1, u_2, \ldots, u_n) = 0, \tag{9a}$$

$$\frac{\mathrm{d}u_i(u_1)}{\mathrm{d}u_1} = \frac{\sum_{j,k=1}^{n} \Gamma_{jk}^1(u_1, u_2(u_1), \ldots, u_n(u_1))(\mathrm{d}u_j(u_1)/\mathrm{d}u_1)(\mathrm{d}u_k(u_1)/\mathrm{d}u_1)}{\sum_{j,k=1}^{n} \Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1))(\mathrm{d}u_j(u_1)/\mathrm{d}u_1)(\mathrm{d}u_k(u_1)/\mathrm{d}u_1)}, \quad i = 2,\ldots,n. \tag{9b}$$

Note that, in linear cases, the a_i's are just the linear normal modes' components; however, in nonlinear cases, these initial 'velocity components' are generally dependent of the total energy h of the system and thus need to be determined.

The existence of the singularities on the energy surface $h = V(u_1,\ldots,u_n)$ makes it hard to solve Eq. (5), even by the shooting method. In the following, the ADM is used to solve Eq. (5), together with the boundary conditions (8a) and (9a,b).

3. Constructing NNMS via the ADM

3.1. ADM

The ADM has been used for solving various equations in mathematics and physics. Extensive studies have already shown that this method has significant advantages: it provides a recursive procedure by which the original solution could be approximated analytically via a rapid convergent series and each of the components could be easily computed. In the following, the standard ADM will be simply reviewed. Consider the following differential equation:

$$Lu + Ru + Nu = g(x), \tag{10}$$

where L is an operator with the highest order derivative which is assumed to be easily invertible, R is the remainder linear differential operators of less order than L, Nu represents the nonlinear operators, and $g(x)$ is the external exciting terms. Applying the inverse operator L^{-1}, an integral operator, to both sides of Eq. (10), and using the given initial and/or boundary conditions, one obtains

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{11}$$

where the function $f(x)$ stands for the response terms arising from integrating the external exciting term $g(x)$ and from the given initial and/or boundary conditions, all of which are assumed to be prescribed. For the convenience of constructing the approximate solution by a recursive procedure, Eq. (11) is modified as follows:

$$u = f(x) - \lambda(L^{-1}(Ru) + L^{-1}(Nu)) = f(x) - \lambda G(u), \tag{12}$$

where the formal parameter λ is artificially introduced and has no specific meaning.

In the standard ADM, the solution u is decomposed into an infinite sum of components as follows:

$$u = \sum_{i=0}^{\infty} u_i \lambda^i, \tag{13}$$

where the formal parameter λ is the same as the one in Eq. (12). Substitute Eq. (13) into Eq. (12) and expand the right hand side function $G(u_i, \lambda)$ of Eq. (12) with respect to λ as follows:

$$G(u, \lambda) = \sum_{i=0}^{\infty} A_i(u_0, u_1, \cdots, u_i) \lambda^i. \tag{14}$$

Substituting Eq. (14) into Eq. (12) and setting the coefficients of λ of the same order to be zero, respectively, the following recursive formula can be easily deduced:

$$
\begin{aligned}
u_0 &= f(x), \\
u_1 &= A_0(u_0), \\
u_2 &= A_1(u_0, u_1), \\
&\;\vdots \\
u_m &= A_{m-1}(u_0, u_1, \ldots, u_{m-1}),
\end{aligned}
\tag{15}
$$

where the A_i's are the so-called Adomian polynomials. Note that, the Adomian polynomial A_i depends only on the components u_0, u_1, ..., u_i, so it can always be explicitly calculated via the first components of u. Finally one gets the approximate solution as follows:

$$
u = \sum_{i=0}^{m} u_i,
\tag{16}
$$

Note that, in Eq. (16), the formal parameter λ is set to be 1.

3.2. Constructing NNMs via the ADM

First of all, note that the singularities appearing in Eq. (5) are caused just by the inverse of the Jacobi metric $g_{jk}(h)$ in the Christoffel symbol (2), and this would lead to a divisor of the form of $(h - V(u_1, \ldots, u_n))$ in Christoffel symbol Γ_{jk}^i. Obviously, these Γ_{jk}^i's would become singular near the energy surface of $h = V(u_1, \ldots, u_n)$. To cancel this factor in Γ_{jk}^i's, simply multiply Eq. (5) by $(h - V(u_1, \ldots, u_n))$, thus one obtains

$$
(h - V(u_1, \ldots, u_n(u_1))) \frac{\mathrm{d}^2 u_i(u_1)}{\mathrm{d}u_1^2} - \frac{\mathrm{d}u_i(u_1)}{\mathrm{d}u_1} \sum_{j,k=1}^{n} \Gamma\Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{\mathrm{d}u_j(u_1)}{\mathrm{d}u_1} \frac{\mathrm{d}u_k(u_1)}{\mathrm{d}u_1}
$$
$$
+ \sum_{j,k=1}^{n} \Gamma\Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{\mathrm{d}u_j(u_1)}{\mathrm{d}u_1} \frac{\mathrm{d}u_k(u_1)}{\mathrm{d}u_1} = 0, \quad i = 2, \ldots, n,
\tag{17}
$$

where $\Gamma\Gamma_{jk}^i = (h - V(u_1, \cdots, u_n(u_1)))\Gamma_{jk}^i$. Thus, there would no singularities appearing in $\Gamma\Gamma_{jk}^i$'s; however, $(h - V(u_1, \ldots, u_n))$, the coefficient of the highest derivative term, would become zero on the energy surface $h = V(u_1, \ldots, u_n(u_1))$. Eq. (17), together with the boundary conditions (8a) and (9a,b), still constitutes a singular BVP.

By a comparison between Eqs. (17) and (10), one can easily identify that

$$
L = \frac{\mathrm{d}^2}{\mathrm{d}u_1^2} \mathrm{diag}[1, ..., 1]_{(n-1) \times (n-1)},
\tag{18a}
$$

$$F_i(u) = \frac{du_i(u_1)}{du_1} \sum_{j,k=1}^{n} \Gamma_{jk}^1(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1}$$

$$- \sum_{j,k=1}^{n} \Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1}, \quad i = 2, \ldots, n. \tag{18b}$$

Theoretically, following the standard ADM, the corresponding Adomian polynomials A_i's could be calculated by procedure (14) and consequently the NNMs, as the approximate solutions of Eq. (17) together with Eqs. (8a) and (9a,b), could be obtained, but in such cases, since the functions $F_i(u)$'s take the form of a polynomial fraction, the actual computation for A_i's is much tedious and the resulting solution series (16) converges slowly. A close observation of Eq (17) suggests that, here, being different from the standard ADM scheme, other choices for the operator L and the functions $F_i(u)$'s are possible; for example, the following choice is made:

$$L_1 = h \frac{d^2}{du_1^2} \mathrm{diag}[1, \ldots, 1]_{(n-1) \times (n-1)}, \tag{19a}$$

$$F_{1i}(u) = V(u_1, \ldots, u_n(u_1)) \frac{d^2 u_i(u_1)}{du_1^2} + \frac{du_i(u_1)}{du_1} \sum_{j,k=1}^{n} \Gamma\Gamma_{jk}^1(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1}$$

$$- \sum_{j,k=1}^{n} \Gamma\Gamma_{jk}^i(u_1, u_2(u_1), \ldots, u_n(u_1)) \frac{du_j(u_1)}{du_1} \frac{du_k(u_1)}{du_1}, \quad i = 2, \ldots, n. \tag{19b}$$

The main difference between the standard scheme (18) and the modified scheme (19) is that the functions $F_{1i}(u)$'s in Eq. (19) contain partially the highest order linear derivative terms but the functions $F_i(u)$'s in Eq. (18) do not contain such terms.

When applying the modified ADM (19) to concrete models, firstly Eqs. (12)–(16) are used to construct a formal approximate solution. Then conditions (8a,b) and (9a,b) are used to determine the unknown parameters, such as the parameters a_i's, etc. The obtained approximate analytic solutions are just the NNMs of the system.

It should be noted that, in Eqs. (5) and (17), the first mass point is actually chosen as the base point of the NNMs. Similarly, any other mass point could be chosen as the base point, too. In fact, as discussed in Ref. [9], even abstract points could be chosen as the base point of the NNMs. The displacement of the abstract base point is defined as a weighted sum of the displacements of all the mass points of the system. To the first-order approximation, the components of the linear normal modes can be chosen as the weights. Since the abstract base point scheme is superior to any other schemes, it is worthy to deduce the geodesic equations with respect to the abstract base point. Let v denote the displacement of the abstract base point, then for the ith NNM, define $v_i = \sum_{j=1}^{n} c_{ij} u_j$, where the coefficients c_{ij} represents the components of the ith linear normal mode and satisfies the normalization condition that $\sum_{j=1}^{n} m_{ij} c_{ij}^2 = 1$ ($i = 1, \ldots, n$). The geodesic equations with respect to the abstract

base point v are given as follows:

$$(h - V(u_1(v), \ldots, u_n(v))) \frac{\mathrm{d}^2 u_i(v)}{\mathrm{d}v^2} - \frac{\mathrm{d}u_i(v)}{\mathrm{d}v} \sum_{j,k,l=1}^{n} c_{il} \Gamma \Gamma_{jk}^1 (u_1(v), u_2(v), \ldots, u_n(v)) \frac{\mathrm{d}u_j(v)}{\mathrm{d}v} \frac{\mathrm{d}u_k(v)}{\mathrm{d}v}$$

$$+ \sum_{j,k=1}^{n} \Gamma \Gamma_{jk}^i (u_1(v), u_2(v), \ldots, u_n(v)) \frac{\mathrm{d}u_j(v)}{\mathrm{d}v} \frac{\mathrm{d}u_k(v)}{\mathrm{d}v} = 0, \quad i = 1, \ldots, n, \tag{20}$$

where for the ith NNM, $v = v_i$. Similarly, the corresponding operator L_2 and the function $F_2(u)$ for the modified ADM, and the related boundary conditions, can be easily identified and hence they are omitted here for simplicity. It should be noted that, when doing actual computation, it is not necessary to compute the terms $\Gamma \Gamma_{jk}^i$'s via the definition for Christoffel symbol. In fact, such terms can be obtained directly via the original equations of motion.

In the next section, two examples are given to verify the effectiveness of the modified ADM.

4. Examples

To demonstrate the modified ADM for solving the NNMs, in the following, two nonlinear vibratory models with two-dof and four-dof, respectively, are considered. In the first example, two base-point schemes—the second mass-point scheme and the abstract base-point scheme—for the NNMs are used here for comparison; in the second example, for simplicity, but not accuracy, the first mass-point scheme is adopted.

Example 1. The physical model to be considered here is a two-dof nonlinear vibratory system [8] which is as follows:

$$\ddot{x}_1 = -x_1 - k(x_1 - x_2) - g x_1^3,$$
$$\ddot{x}_2 = -x_2 - k(x_2 - x_1), \tag{21}$$

where k, g are parameters. The potential function $V(x_1, x_2)$ is as follows:

$$V(x_1, x_2) = \tfrac{1}{2}(x_1^2 + x_2^2) + \frac{k}{2}(x_1 - x_2)^2 + \frac{g}{4}x_1^4. \tag{22}$$

Here, the mass matrix $[m_{ij}]$ is just a 2×2 unit matrix. Let h denote the total energy of the system, then the geodesic equations, which are equivalent to Eq. (21), on the Riemannian manifold with the Jacobi metric $g_{ij}(h) = (h - V(x_1, x_2)) m_{ij}$, can be deduced as follows:

$$[h - V(x_1(x_2), x_2)] \frac{\mathrm{d}^2 x_1(x_2)}{\mathrm{d}x_2^2} - \frac{1}{2} \left[1 + \left(\frac{\mathrm{d}x_1(x_2)}{\mathrm{d}x_2} \right)^2 \right]$$

$$\times \left[(x_2 + k(x_2 - x_1(x_2))) \frac{\mathrm{d}x_1(x_2)}{\mathrm{d}x_2} - (x_1(x_2) + k(x_1(x_2) - x_2) + g(x_1(x_2))^3) \right] = 0, \tag{23}$$

where the displacement x_2 is taken as the reference variable. Although this equation can be deduced in other way [6], the formulation of geodesics which provides us with more geometrical

intuition is still adopted. According to Eq. (19), the linear operator L_1 and the function $F_1(u)$ are chosen as follows:

$$L_1 = h \frac{d^2}{dx_2^2}, \tag{24a}$$

$$F_1(x_2) = V(x_1(x_2), x_2) \frac{d^2 x_1(x_2)}{dx_2^2} + \frac{1}{2} \left[1 + \left(\frac{dx_1(x_2)}{dx_2} \right)^2 \right]$$
$$\times \left[(x_2 + k(x_2 - x_1(x_2))) \frac{dx_1(x_2)}{dx_2} - (x_1(x_2) + k(x_1(x_2) - x_2) + gx_1(x_2)) \right]. \tag{24b}$$

By procedure (15), the components of the solution series (16), $x_{10}, x_{11}, \ldots, x_{1m}$, can be obtained recursively:

$$x_{10} = a \, x_2,$$

$$x_{11} = -\frac{1}{120h}(1 + a^2)(3ga^3 x_2^2 + 10ka^2 - 10k)x_2^3,$$

$$x_{12} = \frac{1}{13,440h^2}(1 + a^2)x_2^5(7g^2a^5 x_2^4 + 7g^2a^7 x_2^4 + 144ga^6 x_2^2 k + 104ga^4 x_2^2 k$$
$$- 96ga^3 x_2^2 - 96ga^3 x_2^2 k - 40ga^2 x_2^2 k - 96ga^5 x_2^2 - 96ga^5 x_2^2 k$$
$$+ 280a^5 k^2 - 224ka^4 - 224k^2 a^4 - 280k^2 a + 224k + 224k^2). \tag{25}$$

Since the ADM involves only simple calculations such as polynomial integration and expansion, the other components $x_{13}, x_{14}, \ldots, x_{1m}$, can also be easily calculated. It is predictable that these expressions would become comparatively complicated as the number of the solution components increases, so only the expression for x_{13} is listed in Appendix A and the rest is omitted. Then, the approximate solution with 4 components is $x_1(a) = x_{10} + x_{11} + x_{12} + x_{13}$, where the unknown parameter a is to be determined by the boundary conditions (9).

Next, the NNMs and the corresponding modal oscillator equations of model (21) for two specific cases, the low-energy case with $k = 1$, $g = 0.5$, $h = 1$; and the high-energy case with $k = 1$, $g = 0.5$, $h = 3$, are calculated, respectively.

Low-energy case ($k = 1$, $g = 0.5$, $h = 1$)

  Mode-I: Parameter $a = 0.8174918014$, modal amplitude $(x_2)_{max} = 1.022283479$,

$$x_1 = 0.8174918014x_2 + 0.0461153894x_2^3 + 0.0036575128x_2^5 + 0.0007712539x_2^7$$
$$- 0.0044021603x_2^9 + 0.0001445457x_2^{11} - 0.156026666 \times 10^{-5}x_2^{13}, \tag{26a}$$

Modal oscillator I:

$$\ddot{x}_2 + 1.182508199x_2 - 0.0461153894x_2^3 - 0.0036575128x_2^5 - 0.0007712539x_2^7$$
$$+ 0.0044021603x_2^9 - 0.0001445457x_2^{11} + 0.156026666 \times 10^{-5}x_2^{13} = 0, \tag{26b}$$

Mode-II: Parameter $a = -1.070706302$, modal amplitude $(x_2)_{\max} = 0.5511812782$,

$$
\begin{aligned}
x_1 = {}& -1.070706302 x_2 - 0.0261883699 x_2^3 - 0.0045975704 x_2^5 - 0.0078638712 x_2^7 \\
& + 0.1452059425 x_2^9 - 0.0032813377 x_2^{11} + 2.196999165 \times 10^{-5} x_2^{13},
\end{aligned} \tag{26c}
$$

Modal oscillator II:

$$
\begin{aligned}
& \ddot{x}_2 + 3.070706302 x_2 + 0.0261883699 x_2^3 + 0.0045975704 x_2^5 + 0.0078638712 x_2^7 \\
& - 0.1452059425 x_2^9 + 0.0032813378 x_2^{11} - 2.196999 \times 10^{-5} x_2^{13} = 0.
\end{aligned} \tag{26d}
$$

High-energy case ($k = 1$, $g = 0.5$, $h = 3$)

Mode-I: Parameter $a = 0.6527160761$, modal amplitude $(x_2)_{\max} = 1.806758063$,

$$
\begin{aligned}
x_1 = {}& 0.6527160761 x_2 + 0.0227358719 x_2^3 + 0.0009071058 x_2^5 + 0.595081636 \times 10^{-4} x_2^7 \\
& - 0.7898780689 \times 10^{-4} x_2^9 + 0.1406169847 \times 10^{-5} x_2^{11} - 0.7466173435 \times 10^{-8} x_2^{13}.
\end{aligned} \tag{27a}
$$

Modal oscillator I:

$$
\begin{aligned}
& \ddot{x}_2 + 1.347283924 x_2 - 0.0227358719 x_2^3 - 0.0009071058 x_2^5 - 0.595081636 \times 10^{-4} x_2^7 \\
& + 0.7898780689 \times 10^{-4} x_2^9 - 0.1406169847 \times 10^{-5} x_2^{11} + 0.7466173435 \times 10^{-8} x_2^{13} = 0.
\end{aligned} \tag{27b}
$$

Mode-II: Parameter $a = -1.253650016$, modal amplitude $(x_2)_{\max} = 0.8486280685$,

$$
\begin{aligned}
x_1 = {}& -1.253650016 x_2 - 0.0408346429 x_2^3 - 0.0038603627 x_2^5 - 0.002294722 x_2^7 \\
& + 0.0175055657 x_2^9 - 0.0005047968 x_2^{11} + 0.4221805517 \times 10^{-5} x_2^{13}.
\end{aligned} \tag{27c}
$$

Modal oscillator II:

$$
\begin{aligned}
& \ddot{x}_2 + 3.253650016 x_2 + 0.0408346429 x_2^3 + 0.0038603627 x_2^5 + 0.002294722 x_2^7 \\
& - 0.0175055657 x_2^9 + 0.0005047968 x_2^{11} - 0.4221805517 \times 10^{-5} x_2^{13} = 0.
\end{aligned} \tag{27d}
$$

Note that, Eqs. (26) and (27) are the NNMs approximated with a four-component series and with the x_2 as their reference variable. As a comparison, the NNMs approximated with a three-component series and with abstract points as their base points, for the high-energy case, are also listed in the following:

Mode-I: ($k = 1$, $g = 0.5$, $h = 3$): Parameters $a = 0.55764767$; $b = 0.856565888$, $u_{\max} = 2.21003$,

$$
x_1 = 0.5576476743 u + 0.0086742416 u^3 - 0.41005 \times 10^{-5} u^5 - 0.386278 \times 10^{-4} u^7 - 0.7044 \times 10^{-6} u^9,
$$
$$
x_2 = 0.8565658883 u - 0.0086742416 u^3 + 0.41005 \times 10^{-5} u^5 + 0.386278 \times 10^{-4} u^7 + 0.7044 \times 10^{-6} u^9. \tag{28a}
$$

Modal oscillator I:

$$
\begin{aligned}
& \ddot{u} + \sqrt{2}(0.7071067815 u + 0.0433530538 u^3 + 0.0020230777 u^5 + 0.305127 \times 10^{-4} u^7 \\
& - 0.88756769 \times 10^{-5} u^9 - 0.44478 \times 10^{-6} u^{11} - 0.71582396 \times 10^{-8} u^{13}).
\end{aligned} \tag{28b}
$$

Mode-II: ($k = 1$, $g = 0.5$, $h = 3$) : Parameters $a = 0.78971492$; $b = -0.62449864$, $u_{max} = 1.3682345$,

$$x_1 = 0.789714921u + 0.004651977u^3 + 0.10313328 \times 10^{-3}u^5 - 0.26159489 \times 10^{-3}u^7 - 0.4739 \times 10^{-5}u^9,$$
$$x_2 = -0.6244986416u + 0.004651977u^3 + 0.10313329 \times 10^{-3}u^5 - 0.00026159u^7 - 0.4739 \times 10^{-5}u^9.$$
(28c)

Modal oscillator II:

$$\ddot{u} + \sqrt{2}(2.121320344u + 0.1231263598u^3 + 0.002175903u^5 + 0.610568 \times 10^{-4}u^7$$
$$- 0.1217641 \times 10^{-3}u^9 - 0.365022 \times 10^{-5}u^{11} - 0.622827 \times 10^{-7}u^{13} + 0.39687 \times 10^{-7}u^{15}).$$
(28d)

Theoretically, all the NNMs and their modal dynamical equations obtained above could be expressed by the parameters k, g, h, etc. of the system; however, these expressions, especially for the high energy cases, are very complicated, so not pursued here.

Example 2. The physical model to be considered is a 4-dof nonlinear vibratory system which is as follows:

$$\ddot{x}_1 = -x_1 - (x_1 - x_2) - \alpha x_1^3 = f_1,$$
$$\ddot{x}_2 = -(x_2 - x_1) - (x_2 - x_3) = f_2,$$
$$\ddot{x}_3 = -(x_3 - x_2) - (x_3 - x_4) = f_3,$$
$$\ddot{x}_4 = -(x_4 - x_3) - x_4 = f_4,$$
(29)

where α is a parameter describing the nonlinearity of the system. The potential function $V(x_1, x_2, x_3, x_4)$ is as follows:

$$V(x_1, x_2, x_3, x_4) = \tfrac{1}{2}(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2) + \frac{\alpha}{4}x_1^4.$$
(30)

Here, the mass matrix $[m_{ij}]$ is just a 4×4 unit matrix. Let h denote the total energy of the system, then the geodesic equations, which are equivalent to Eq. (29), on the Riemannian manifold with the Jacobi metric $g_{ij}(h) = (h - V(x_1, x_2, x_3, x_4))m_{ij}$, can be deduced as follows:

$$[h - V(x_1, x_2(x_1), x_3(x_1), x_4(x_1))]\frac{d^2 x_i(x_1)}{dx_1^2}$$
$$- \frac{1}{2}\left[1 + \sum_{j=2}^{4}\left(\frac{dx_j(x_1)}{dx_1}\right)^2\right] \cdot \left[f_i - f_1\frac{dx_i(x_1)}{dx_1}\right] = 0, \quad i = 2, \ldots, 4,$$
(31)

where the displacement x_1 is taken as the reference variable and the f_i's are the right-hand side functions of Eq. (29). According to Eq. (19), the linear operator L_1 and the function $F_{1i}(u)$ can be chosen as follows:

$$L_1 = h\frac{d^2}{dx_1^2}\text{diag}[1, 1, 1],$$
(32a)

$$F_{1i}(x_1) = V(x_1, x_2(x_1), x_3(x_1), x_4(x_1)) \frac{d^2 x_i(x_1)}{dx_1^2}$$

$$+ \frac{1}{2} \left[1 + \sum_{j=2}^{4} \left(\frac{dx_j(x_1)}{dx_1} \right)^2 \right] \cdot \left[f_i - f_1 \frac{dx_i(x_1)}{dx_1} \right], \quad i = 2, \dots, 4. \tag{32b}$$

By procedure (15), the components of the solution series (16) for $m = 3$, $X_0, X_1, \dots, X_3$, can be obtained recursively:

$$X_0 = [a_2, a_3, a_4]^T u, \tag{33a}$$

$$X_1 = \frac{1}{480h}(1 + a_2^2 + a_3^2 + a_4^2)u^3[(3a_2 u^2 + 40 + 40a_3 - 40a_2^2),$$
$$(3a_3 u^2 + 40a_2 + 40a_4 - 40a_2 a_3),$$
$$(3a_4 u^2 + 40a_3 - 40a_2 a_4)]^T. \tag{33b}$$

The expression for X_2 is listed in Appendix B. The expression for X_3 is also obtained, but it is comparatively complicated and so omitted here for saving the space. Thus, the approximate solution with 4 components is $X(a_2, a_3, a_4) = X_0 + X_1 + X_2 + X_3$, where the unknown parameters a_i's are to be determined by the boundary conditions (9).

Next, the NNMs and the corresponding modal oscillator equations of model (29) for the case that $h = \frac{1}{2}$, $\alpha = \frac{1}{4}$ are calculated.

Mode-I: Parameters $\{a_2 = 1.52955328, \ a_3 = 1.768595637, \ a_4 = 0.9453675745\}$, modal amplitude $(x_1)_{max} = 0.5675187856$,

$$x_1 = u,$$
$$x_2 = 1.52955328u + 0.5264011867u^3 + 0.173262781u^5 + 0.084211887u^7$$
$$+ 0.6738555034u^9 + 0.286871147u^{11} + 0.02864469946u^{13}, \tag{34a}$$

$$x_3 = 1.768595637u - 0.2824736471u^3 - 0.072765137u^5 - 0.04376138u^7$$
$$+ 0.2897657336u^9 + 0.2262207098u^{11} + 0.0331213637u^{13}, \tag{34b}$$

$$x_4 = 0.9453675745u + 0.3957931308u^3 + 0.1253004463u^5 + 0.0578606967u^7$$
$$+ 0.4895348259u^9 + 0.192678508u^{11} + 0.0177043653u^{13}. \tag{34c}$$

Modal oscillator I:

$$\ddot{u} + 0.47044672u - 0.2764011867u^3 - 0.173262781u^5 - 0.084211887u^7$$
$$- 0.6738555034u^9 - 0.286871147u^{11} - 0.02864469946u^{13} = 0. \tag{34d}$$

Mode-II: Parameters $\{a_2 = 0.6553822099, \ a_3 = -0.6122097864, \ a_4 = -1.021722089\}$, modal amplitude $(x_1)_{max} = 0.5040802619$,

$$x_2 = 0.6553822099u - 0.01981219933u^3 - 0.00095996516u^5 - 0.00409812468u^7$$
$$+ 0.1122482289u^9 + 0.01752191178u^{11} + 0.0007109880997u^{13}, \quad (35a)$$

$$x_3 = -0.6122097864u + 0.01656325458u^3 + 0.00136333401u^5 + 0.00479270749u^7$$
$$- 0.113832331u^9 - 0.01694241071u^{11} - 0.0006641527126u^{13}, \quad (35b)$$

$$x_4 = -1.021722089u + 0.0272523155u^3 + 0.00230438139u^5 + 0.00795843942u^7$$
$$- 0.1899005228u^9 - 0.0282829997u^{11} - 0.00110841u^{13}. \quad (35c)$$

Modal oscillators II:

$$\ddot{u} + 1.34461779u + 0.2698121993u^3 + 0.00095996516u^5 + 0.00409812468u^7$$
$$- 0.1122482289u^9 - 0.01752191178u^{11} - 0.0007109880997u^{13} = 0. \quad (35d)$$

Mode-III: Parameters $\{a_2 = -0.60021718, \ a_3 = -0.6238747751, \ a_4 = 0.9924507463\}$, modal amplitude $(x_1)_{max} = 0.3736857474$,

$$x_2 = -0.60021718u + 0.0072301124u^3 + 0.00234120461u^5 + 0.0105257157u^7$$
$$- 0.4126884838u^9 - 0.03029564654u^{11} - 0.00057616887u^{13}, \quad (36a)$$

$$x_3 = -0.6238747751u + 0.0080999585u^3 + 0.002225873u^5 + 0.009915613u^7$$
$$- 0.4122770137u^9 - 0.03095088614u^{11} - 0.0005988786u^{13}, \quad (36b)$$

$$x_4 = 0.9924507463u - 0.0128467524u^3 - 0.00354226u^5 - 0.0157550635u^7$$
$$+ 0.6557256951u^9 + 0.0492369089u^{11} + 0.0009526872u^{13}. \quad (36c)$$

Modal oscillators III:

$$\ddot{u} + 2.60021718u + 0.2427698876u^3 - 0.0023412046u^5 - 0.0105257157u^7$$
$$+ 0.4126884838u^9 + 0.0302956465u^{11} + 0.00057616887u^{13} = 0. \quad (36d)$$

Mode-IV: Parameters $\{a_2 = -1.611493199, a_3 = 1.609022582, a_4 = -0.9937461995\}$, modal amplitude $(x_1)_{max} = 0.1963277966$

$$x_2 = -1.611493199u + 0.01448099628u^3 + 0.0170242355u^5 + 0.262046692u^7$$
$$- 38.3056787u^9 - 2.040748273u^{11} - 0.02792797937u^{13}, \quad (37a)$$

$$x_3 = 1.609022582u - 0.01471795446u^3 - 0.016648292u^5 - 0.254607342u^7$$
$$+ 37.84347485u^9 + 2.028425613u^{11} + 0.0278851623u^{13}, \quad (37b)$$

$$x_4 = -0.9937461995u + 0.0090950764u^3 + 0.0102820048u^5 + 0.157283956u^7$$
$$- 23.37337744u^9 - 1.252772482u^{11} - 0.0172221163u^{13}, \tag{37c}$$

Modal oscillator IV:

$$\ddot{u} + 3.611493199u + 0.235519u^3 - 0.017024235u^5 - 0.262046692u^7$$
$$+ 38.3056787u^9 + 2.040748273u^{11} + 0.02792797937u^{13} = 0. \tag{37d}$$

Note that Eqs. (34)–(37) are the NNMs approximated with a four-component series and with the x_1 as their reference variable.

In the next section, various numerical simulations will be done for models (21) and (29).

5. Simulation results and remarks

To confirm the effectiveness of the NNMs obtained above, numerical simulations for different base-point schemes and/or different energy levels are done in the following. For models (21) and
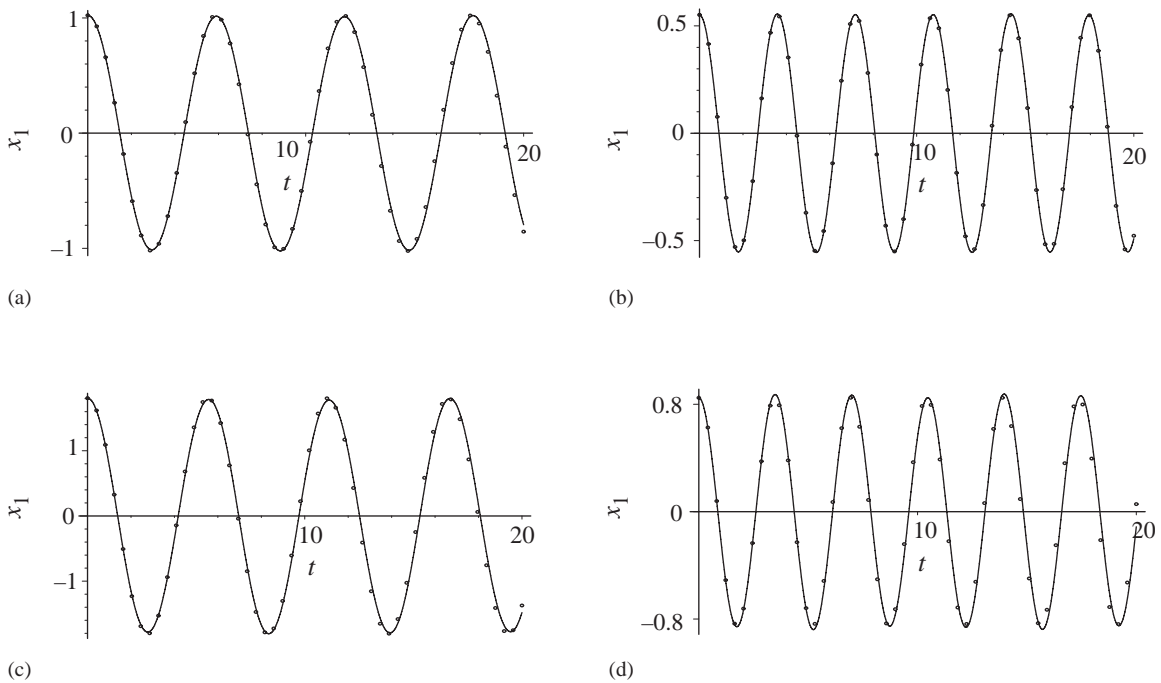

Fig. 1. Numerical simulation of modal dynamical responses of model (21). All the curves are plotted for x_1 vs. t. ———, The simulation based on the original Eq. (21); ∘∘∘∘∘∘, the simulation based on the NNM oscillator Eqs. (26) and (27). (a) Mode-I, the low-energy case, $h = 1$, x_2 as the reference variable of NNMs; (b) Mode-II, same as (a); (c) same as (a), high-energy case, $h = 3$; (d) same as (b), high-energy case.
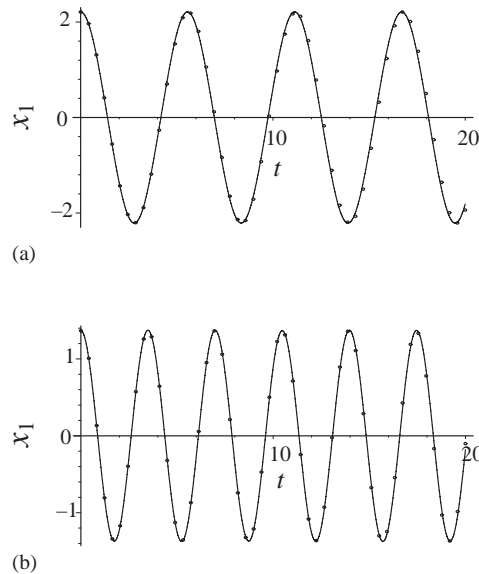
Fig. 2. Numerical simulation of model (21). All the curves are plotted for x_1 vs. t. ——, The simulation based on the original Eq. (21); ∘∘∘∘∘∘∘∘, the simulation based on the NNM oscillator Eqs. (28). (a) Mode-I, the high-energy case, $h = 3$, abstract point as the base point of NNMs; (b) Mode-II, same as (a).

(29) with specific energy level h, the parameter a or a_i's and the maximum modal displacements $(x_2)_{\max}$ or $(x_1)_{\max}$ can be easily calculated by using the Newton–Raphson iteration method. The linear normal modes' components could be suitably chosen as the initial values for the parameter(s) a or a_i's. Similarly, the linear maximum modal displacement could be chosen as the initial value for $(x_2)_{\max}$ or $(x_1)_{\max}$.

In Fig. 1, model (21) is simulated. The second coordinate x_2 is chosen as the reference variable of NNMs and the modified ADM with a four-component-solution series is used to approximate the modal manifolds and the corresponding modal dynamics. The modal dynamics on mode-I and mode-II are simulated for two different energy levels, $h = 1$ and 3, respectively. The results indicate that, even in the high energy case, the NNMs constructed by the modified ADM are quite accurate.

As a comparison, in Fig. 2, the abstract base point scheme is adopted and the modified ADM with a three-component-solution series is used for constructing the NNM manifolds and approximating the modal dynamics. For simplicity, only the results for high-energy case, $h = 3$, are presented here. It can be seen that, this scheme which uses merely three components is more accurate than the one stated above which uses four components. It should be noted that, for n-dof conservative vibratory systems, the abstract base point scheme usually concerns n equations for determining NNMs, but the x_2-(or any other x_i-)base point scheme involves only $n-1$ equations for NNMs.

For further comparison, the state space method with the abstract base point scheme is used for the same model (21). Similarly, only the high-energy case is presented here for saving the space. From Fig. 3, it can be easily seen that, in this case, the error of the approximate solution based on
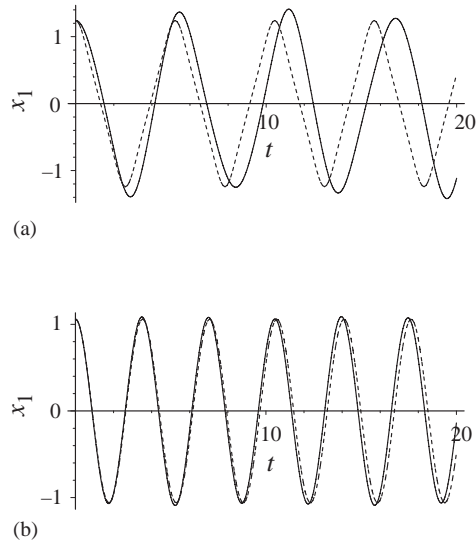
(a)

(b)

Fig. 3. Numerical simulation of model (21). All the curves are plotted for x_1 vs. t. ——, The simulation based on the original Eq. (21); ——, the simulation based on the NNM oscillator equations (cf. Ref. [9]). (a) Mode-I, high-energy case, $h = 3$, abstract base point scheme, state space method; (b) Mode-II, same as (a).
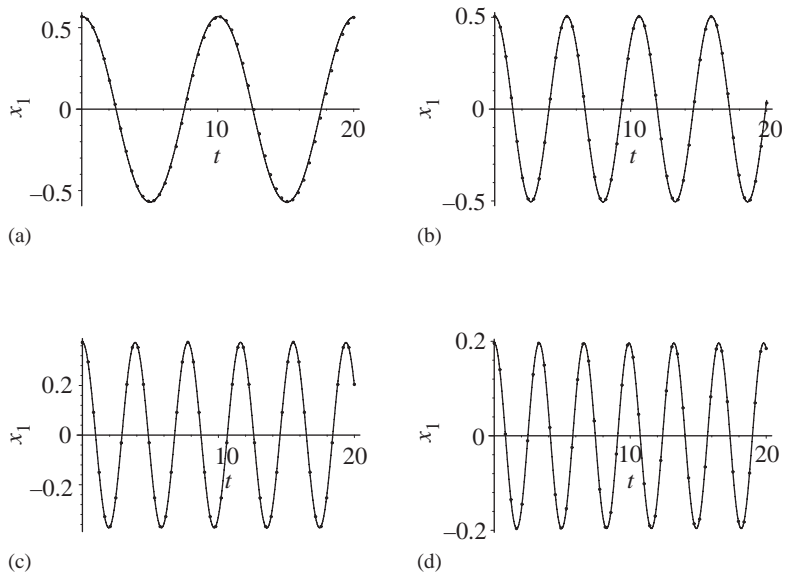

(a)   (b)

(c)   (d)

Fig. 4. Numerical simulation of modal dynamical responses of model (29). All the curves are plotted for x_1 vs. t. ——, The simulation based on the original Eq. (29); ∘∘∘∘∘∘, the simulation based on the NNM oscillator Eqs. (34)–(37). In all of the figures, $h = 1/2$, $\alpha = 1/4$, x_1 is taken as the reference variable of the NNMs (a) mode-I; (b) mode-II; (c) mode-III; (d) mode-IV.

the NNMs is significant, this is true especially for the mode-I. This indicates that the state space method is not suitable for high-energy and/or strongly nonlinear vibratory systems.

In Fig. 4, model (29) is simulated. Figs. 4(a)–(d) are the simulation results with the first mass point being chosen as the base point of the NNMs. It can be seen from these figures that, the accuracy of the NNMs is quite good.

These numerical simulations indicate that, the modified ADM for NNMs is more effective than the existing methods. The obtained NNMs are more accurate and less computation volume is required. The obtained solution series with only few terms is convergent rapidly.

It should be indicated that, with the increase of the number m (the solution components) or the number n (the dof of the systems), the resulting Eq. (17) or (20) for determining the parameters a_i's would become more and more complicated.

6. Conclusions

For conservative vibratory systems, based on the analysis and the simulation results above, our conclusions are as follows:

1. Nonlinear normal modal motions are special geodesic motions on the Riemannian manifold with the Jacobi metric. NNMs are special geodesics on the manifold. All these geodesics pass through the origin of the manifold with specific directions which depend on the total energy of the system.
2. The modified ADM is more effective for constructing the NNMs. The NNMs obtained by this method are more accurate and just need less computation volume and simple manipulations. Higher-order approximate NNMs could be easily built in this way.
3. Based on the geodesic formulation, the modified ADM for NNMs is global, rapid convergent, and applicable to strongly nonlinear vibratory systems. It can be applied to high-dimensional conservative vibratory systems.
4. Theoretically, it is possible to combine the modified ADM with the state space method for NNMs. However, since the state space scheme concerns system of partial differential equations for determining NNMs, it is not easy to apply the modified ADM directly.

Appendix A. The expression for x_{13} in Example 1

$$
\begin{aligned}
x_{13} = {} & -\frac{1}{691,891,200h^3}(1+a^2)x_2^7(3,294,720ga^5x_2^2 + 3,294,720ka^6 + 6,589,440k^2a^6 \\
& + 7,687,680k^2a^3 - 6,589,440k^2a^2 + 7,756,320k^3a^6 + 7,687,680k^3a^3 - 7,756,320k^3a^2 \\
& - 7,687,680k^3a^7 + 4,804,800k^3a^8 + 7,687,680k^3a - 1,853,280k^3a^4 - 7,687,680a^5k^3 \\
& + 7623g^3a^7x_2^6 - 198,432g^2a^9x_2^4k + 297,648g^2a^{10}x_2^4k - 396,864g^2a^7x_2^4k \\
& - 198,432g^2a^5x_2^4k + 162,864g^2a^6x_2^4k + 527,904g^2a^8x_2^4k - 67,392g^2a^4x_2^4k
\end{aligned}
$$

$$+ 3,294,720a^7gx_2^2k + 6,580,860a^7gx_2^2k^2 - 2,574,000a^4gx_2^2k^2 + 431,860a^3gx_2^2k^2$$
$$- 4,427,280a^8gux_2^2k^2 + 3,174,600a^9gx_2^2k^2 - 4,427,280a^8gx_2^2k + 926,640a^2gx_2^2k^2$$
$$+ 100,100agx_2^2k^2 + 3,738,020a^5gx_2^2k^2 - 7,927,920a^6gx_2^2k^2 - 7,687,680k^2a^7$$
$$- 2,951,520k^3 - 3,294,720k - 198,432g^2a^5x_2^4 - 396,864g^2a^7x_2^4 + 1,647,360a^7gx_2^2$$
$$- 198,432g^2a^9x_2^4 + 7623g^3a^{11}x_2^6 + 15,246g^3a^9x_2^6 + 1,647,360ga^3x_2^2$$
$$- 3,294,720ka^2 + 3,294,720ka^4 + 7,687,680k^2a - 6,589,440k^2 - 7,927,920ga^6x_2^2k$$
$$- 2,574,000ga^4x_2^2k + 3,294,720ga^3x_2^2k + 6,589,440ga^5x_2^2k + 926,640ga^2x_2^2k$$
$$+ 6,589,440k^2a^4 - 7,687,680a^5k^2).$$

Appendix B. The expression for X_2 in Example 2

$$X_2 = \frac{1}{215040h^2}(1 + a_2^2 + a_3^2 + a_4^2)u^5[(7168 + 768a_2u^2 - 4032a_4^2a_2 - 4032a_3^2a_2 + 336a_3u^2$$
$$+ 7168a_3a_2^2 + 448a_2^3 - 736u^2a_2^4 + 336u^2a_3^3 - 416u^2a_2^2 + 448a_2^3 - 736u^2a_2^4 + 336u^2a_3^3$$
$$- 416u^2a_2^2 + 448a_2^2a_4 + 448a_4^3 - 7168a_2^4 + 768u^2a_2^3 + 4480a_2^3a_3^2 + 448a_4a_3^2 - 4928a_3a_2^3$$
$$- 4928a_3^3a_2 + 4480a_2^3a_4^2 + 320u^2a_3^3 + 7168a_3a_4^2 - 7168a_2^2a_4^2 - 7168a_2^2a_3^2 + 7168a_3^3$$
$$+ 336u^2a_3a_2^2 + 768u^2a_3^2a_2 + 768u^2a_4^2a_2 - 4928a_3a_4^2a_2 + 35a_2u^4a_4^2 - 736u^2a_2^2a_4^2 - 4928a_3a_2$$
$$+ 35a_2u^4a_3^2 + 35a_2u^4 - 736u^2a_2^2a_3^2 + 336u^2a_3a_4^2 + 320u^2 + 35a_2^3u^4 + 320a_4^2u^2 - 4032a_2$$
$$+ 7168a_3 + 448a_4 + 7168a_3^2 + 7168a_4^2 + 4480a_2^5),$$
$$(448 + 336a_2u^2 + 7168a_4^2a_2 + 7168a_3^2a_2 + 768a_3u^2 + 4480a_3a_2^2a_4^2 - 4480a_2a_4a_3^2 + 4928a_3a_2^2$$
$$+ 7168a_2^3 + 336u^2a_4 - 736a_3^3a_2u^2 - 736u^2a_2a_3 + 768u^2a_3^3 + 7168a_2^2a_4 + 7168a_4^3$$
$$- 736a_2a_3a_4^2u^2 - 448a_3^2a_4^2 - 4480a_2^4 + 336a_2^3u^2 + 7168a_4a_3^2 - 7168a_3a_2^3 - 7168a_3^3a_2 + 35a_3u^4$$
$$+ 35u^4a_3a_2^2 + 35a_3a_4^2u^4 + 336u^2a_2^2a_4 + 336u^2a_4a_3^2 - 736a_3a_2^3u^2 + 448a_3a_4^2 - 4480a_2^2a_4^2$$
$$- 4928a_2^2a_3^2 + 448a_3^3 - 4480a_2a_4 + 35u^4a_3^3 + 336a_4^3u^2 + 768a_3u^2a_2^2 + 336a_2a_3^2u^2 + 336u^2a_2a_4^2$$
$$- 7168a_2a_3a_4^2 - 7168a_2a_3 + 768u^2a_3a_4^2 - 448a_3^4 - 4480a_2a_4^3 - 4480a_4a_2^3 + 4480a_3a_4^4$$
$$+ 4480a_2^2a_3^3 + 7168a_2 + 448a_3 + 7168a_4 - 4032a_2^2 + 448a_4^2),$$
$$(-448a_3a_4 + 448a_2a_4^2 + 448a_3^2a_2 + 336a_3u^2 - 7168a_2a_3^2a_4 + 7168a_3a_2^2 + 448a_2^3 + 768u^2a_4$$
$$+ 336u^2a_3^3 + 4480a_2^2a_4 - 4480a_2^2a_3 - 4480a_3^3a_2 + 768u^2a_2^2a_4 + 768u^2a_3^2a_4 - 4480a_2^2a_3^2a_4$$
$$+ 35a_4u^4 + 35a_4a_2^2u^4 + 35u^4a_4a_3^2 - 736a_2a_4u^2 - 736a_2a_4^3u^2 - 736a_4a_3^3u^2 - 736u^2a_3^2a_2a_4$$
$$+ 7168a_3a_4^2 + 7168a_3^3 + 4480a_2^2a_4^3 + 4480a_2^4a_4 - 448a_4a_3^3 - 448a_4^3a_3 - 7168a_2a_4$$
$$+ 768u^2a_4^3 + 35u^4a_4^3 + 336a_3a_2^2u^2 - 4480a_2a_3a_4^2 - 448a_2^2a_3a_4 - 4480a_3a_2 + 336a_3a_4^2u^2$$
$$- 7168a_2a_4^3 - 7168a_2^3a_4 + 448a_2 + 7168a_3)]^T.$$

References

[1] R.H. Rosenberg, On nonlinear vibrations of systems with many degrees of freedom, *Advances in Applied Mechanics* 9 (1966) 155–242.

[2] T.K. Caughey, A. Vakakis, V. Sivo, Analytical study of similar normal modes and their bifurcations in a class of strongly nonlinear systems, *International Journal of Non-linear Mechanics* 25 (1990) 521–533.

[3] A. Vakakis, R.H. Rand, Normal modes and global dynamics of a two-degree-of-freedom nonlinear system—I. Low energies, *International Journal of Non-linear Mechanics* 27 (1992) 861–874.

[4] A. Vakakis, R.H. Rand, Normal modes and global dynamics of a two-degree-of-freedom nonlinear system—II. High energies, *International Journal of Non-linear Mechanics* 27 (1992) 875–888.

[5] A.F. Vakakis, T.K. Caughey, A theorem on the exact nonsimilar steady-state motions of a nonlinear oscillator, *Journal of Applied Mechanics* 59 (1992) 418–424.

[6] A.F. Vakakis, Non-similar normal oscillations in a strongly nonlinear discrete system, *Journal of Sound and Vibration* 158 (1992) 341–361.

[7] T.A. Nayfey, E. Emaci, A.F. Vakakis, Application of nonlinear localization to the optimization of a vibration isolation system, *AIAA Journal* 35 (8) (1997) 1378–1386.

[8] S. Shaw, C. Pierre, Normal modes for nonlinear vibratory systems, *Journal of Sound and Vibration* 164 (1993) 85–124.

[9] X.H. Zhang, Effects of base points and normalization schemes on the non-linear normal modes of conservative systems, *Journal of Sound and Vibration* 256 (2002) 447–462.

[10] T.D. Buton, W. Rhee, On the reduction of nonlinear structural dynamics models, *Journal of Vibration and Control* 6 (2000) 531–556.

[11] G.S. Happawana, A.K. Bajaj, M. Azene, An analytical solution to non-similar normal modes in a strongly nonlinear discrete system, *Journal of Sound and Vibration* 183 (2) (1995) 361–367.

[12] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Reading, MA, 1978.

[13] G. Adomian, A review of the decomposition method in applied mathematics, *Journal of Mathematical Analysis and Application* 135 (1988) 501–544.

[14] G. Adomian, Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition, *Journal of Mathematical Analysis and Application* 174 (1993) 118–137.

[15] E. Deeba, S.A. Khuri, S.S. Xie, An algorithm for solving boundary value problems, *Journal of Computational Physics* 159 (2000) 125–138.

[16] A.M. Wazwaz, The numerical solution of fifth-order boundary value problems by the decomposition method, *Journal of Computational and Applied Mathematics* 136 (2001) 259–270.

[17] D.A. Kahn, *Introduction to Global Analysis*, Academic Press, New York, 1980.