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Short Communication

On the approximate solution of a piecewise nonlinear oscillator under super-harmonic resonance

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Abstract

An approximate solution for the super-harmonic resonance response of a periodically excited nonlinear oscillator with a piecewise nonlinear–linear characteristic is constructed using both a matching method and a modified averaging method. The validity of the developed analysis is confirmed by comparing the approximate solutions with the results of direct numerical integration of the original equation.

1. Introduction

Piecewise linear systems are mathematically characterized by a set of piecewise linear ordinary differential equations, together with the switching conditions of changes in displacement and/or velocity. The dynamics of piecewise linear systems has been the subject of many studies [1–15]. However, piecewise linear models may not always be well representative of some physical systems. This shortcoming can be avoided by the introduction of additional nonlinearities, thereby resulting in piecewise nonlinear systems with a piecewise nonlinear–linear characteristic [16,17].

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The piecewise nonlinear oscillator with a piecewise nonlinear–linear characteristic considered here is modelled by the following set of nonlinear and linear differential equations:

$$\ddot{y} + (c+d)\dot{y} + \omega^2 y + \alpha y^3 = k \cos \Omega t \quad \text{for } |y| \leqslant y_s, \tag{1a}$$

$$\ddot{y} + c\dot{y} + \omega^2 y + k_0\,\mathrm{sgn}(y) = k \cos \Omega t \quad \text{for } |y| \geqslant y_s, \tag{1b}$$

where y is the displacement, c and d are the damping coefficients, ω is the natural frequency, α is the coefficient of nonlinear terms, k and Ω are the amplitude and frequency of the excitation, respectively, $k_0 = \alpha y_s^3$, $\pm y_s$ are the switching boundaries between linear and nonlinear regions, and an overdot denotes differentiation with respect to the time t.

The nonlinear oscillator given by Eq. (1) may exhibit a symmetric periodic response with the maximum amplitude less than y_s (i.e. small amplitude motion) or larger than y_s (i.e. large amplitude motion). The small amplitude motion is not considered here, as it is determined only by Eq. (1a) and can be approximately located using a regular perturbation method. Unlike a piecewise linear oscillator, for the nonlinear oscillator with a piecewise nonlinear–linear characteristic considered here, an exact analytical solution for the large amplitude response cannot be obtained as no exact analytical solution exists to the nonlinear differential equation. A simple but powerful procedure, which has been developed to construct analytical approximations to the primary resonance response of a piecewise nonlinear oscillator [17], is to seek the individual general solutions to Eq. (1a) and (1b) corresponding to different regions $|y(t)| \leqslant y_s$, and $|y(t)| \geqslant y_s$, and then to combine these solutions at the transition points of non-smooth nonlinearity by implementing an appropriate set of matching conditions. This procedure of constructing the approximate solution, which is different from the method of harmonic balance and the averaging procedure, will be referred to here as the matching method for brevity.

The main goal of the work outlined here is to construct an approximate analytical solution for the super-harmonic resonance response of a piecewise nonlinear oscillator using the matching method and a modified averaging method, respectively. The super-harmonic resonance response is interesting because it appears across the working frequency domain. In addition, the procedure of locating an approximate solution for the super-harmonic resonance response can be easily extended to construct an approximate solution for the sub-harmonic resonance response.

To proceed, Eq. (1) is rewritten in a suitable form for applying the method of averaging, by introducing the dummy perturbation parameter ε

$$\ddot{y} + 9\Omega^2 y = k \cos \Omega t - \varepsilon(c+d)\dot{y} - \varepsilon\sigma y - \varepsilon\alpha y^3 \quad \text{for } |y| \leqslant y_s, \tag{2a}$$

$$\ddot{y} + 9\Omega^2 y = k \cos \Omega t - \varepsilon c\dot{y} - \varepsilon\sigma y - \varepsilon k_0\,\mathrm{sgn}(y) \quad \text{for } |y| \geqslant y_s, \tag{2b}$$

where $\varepsilon = 1$, and the external detuning σ in terms of $\omega^2 = 9\Omega^2 + \varepsilon\sigma$ has been introduced to study the super-harmonic resonance response [18,19].

2. Approximate solutions obtained using the matching method

For a symmetric periodic response of large amplitude motion, as shown in Fig. 1, the periodic solution is made up of four distinct segments according to the following four time intervals; $[t_0, t_1]$,

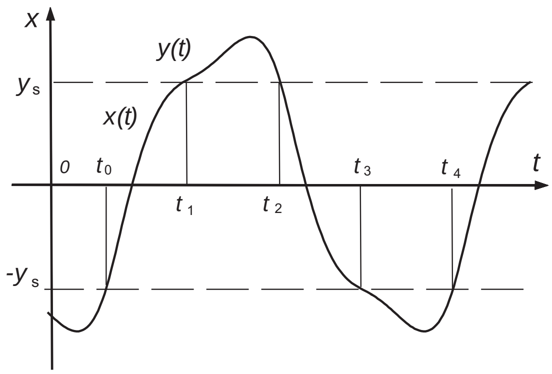

Fig. 1. Symmetric periodic motion of large amplitude response of the overall system, where $x(t)$ denotes the first segment of the motion in the region $|x(t)|\mathrm{d}x \leqslant y_s$, $y(t)$ represents the second segment of the motion in the region $y(t) \geqslant y_s$, t_0 denotes the starting time, and t_i ($i = 1, 2, 3, 4$) represent the time instants that the non-smooth nonlinearities take place at $\pm y_s$.

$[t_1, t_2]$, $[t_2, t_3]$, $[t_3, t_4]$, where t_i denote the time instants at which the non-smooth nonlinearities take place. The motion in each time interval is locally governed by the corresponding Eqs. (2a) or (2b). The overall motion is determined by joining the solutions of Eqs. (2a) and (2b) together. Due to the symmetry of the motion, only two parts of the response need to be considered.

For the first segment of the motion, the first-order approximate solution to represent the exact solution of Eq. (2a) is assumed to have the form

$$x(t) = x_0(t) + \varepsilon x_1(t) \quad \text{for} |x(t)| \leqslant y_s, \tag{3}$$

with

$$x_0(t) = A_1 \sin 3\Omega\tau + B_1 \cos 3\Omega\tau + H \cos \Omega t, \tag{4}$$

$$\begin{aligned} x_1(t) = {} & g_1 \sin 3\Omega\tau + g_2 \cos 3\Omega\tau + h_1\tau \sin 3\Omega\tau + h_2\tau \cos 3\Omega\tau \\ & + h_3 \sin 9\Omega\tau + h_4 \cos 9\Omega\tau + h_5 \sin \Omega t + h_6 \cos \Omega t + h_7 t \sin 3\Omega t \\ & + l_1 \sin(\Omega t - 3\Omega t_0) + l_2 \cos(\Omega t - 3\Omega t_0) + l_3 \sin(5\Omega t - 3\Omega t_0) \\ & + l_4 \cos(5\Omega t - 3\Omega t_0) + l_5 \sin(5\Omega t - 6\Omega t_0) + l_6 \cos(5\Omega t - 6\Omega t_0) \\ & + l_7 \sin(7\Omega t - 6\Omega t_0) + l_8 \cos(7\Omega t - 6\Omega t_0), \end{aligned} \tag{5}$$

where $\tau = t - t_0$, A_1, B_1 and t_0 are constants to be determined, $H = k/8\Omega^2$, and the 17 coefficients in Eq. (5), which can be expressed as functions of A_1, B_1, t_0, and system parameters, are not given here for brevity.

For the second segment of the motion, an exact solution to Eq. (2b) can be expressed in the form:

$$\begin{aligned} y(t) = {} & \mathrm{e}^{-\varepsilon\mu(t-t_1)}[A_2 \cos \beta(t - t_1) + B_2 \sin \beta(t - t_1)] + F_1 \cos \Omega t \\ & + F_2 \sin \Omega t + Y_0 \quad \text{for } y \geqslant y_s, \end{aligned} \tag{6}$$

where A_2, B_2 and t_1 are constants to be determined, $\mu = c/2,$, $\beta = (\omega^2 - \mu^2)^{1/2}$, $F_1 = (\omega^2 - \Omega^2)k/[(\omega^2 - \Omega^2)^2 + 4\mu^2\Omega^2]$, $F_2 = 2\mu\Omega k/[(\omega^2 - \Omega^2)^2 + 4\mu^2\Omega^2]$, $Y_0 = -k_0\,\mathrm{sgn}(y)/\omega^2$.

The four constants A_1, B_1, A_2, B_2, and two crossing times t_0 and t_1 in Eqs. (4) and (6) can be numerically determined by implementing the following set of matching conditions:

$$\begin{aligned} &x(t_0) = -y_s, \quad x(t_1) = y_s, \quad y(t_1) = y_s, \\ &\dot{y}(t_1) = \dot{x}(t_1), \quad y(t_2) = y_s, \quad \dot{y}(t_2) = -\dot{x}(t_0), \end{aligned} \tag{7}$$

where $t_2 = t_0 + \pi/\Omega$.

The corresponding trajectories $x(t)$ and $y(t)$ can be calculated from Eqs. (3) and (6) after obtaining an appropriate value for the four constants A_1, B_1, A_2, B_2, and two crossing times t_0 and t_1.

The stability of the periodic solution can be examined by investigating the asymptotic behaviour of the small perturbations to the steady-state periodic solution over one half of the symmetric response. The dynamic mapping of small perturbations of the symmetric solution over a half period of motion can be written as

$$\begin{bmatrix} \Delta t_2 \\ \Delta v_2 \end{bmatrix} = J \begin{bmatrix} \Delta t_0 \\ \Delta v_0 \end{bmatrix}. \tag{8}$$

where J is a 2×2 matrix. The symmetric periodic motion is asymptotically stable if both eigenvalues λ_1 and λ_2 of matrix J have a modulus less than unity. When either of the two eigenvalues has a modulus greater than one, the solution is unstable.

3. Approximate solutions obtained using a modified averaging method

The approximate solution for the steady-state super-harmonic resonance response of Eq. (2) is assumed to take the form:

$$\begin{aligned} y &= a(t)\cos(3\Omega t + \theta(t)) + H\cos\Omega t, \\ \dot{y} &= -3\Omega a(t)\sin(3\Omega t + \theta(t)) - \Omega H\sin\Omega t, \end{aligned} \tag{9}$$

where $a(t)$ and $\theta(t)$ are assumed to be slowly varying functions of time.

Inserting Eq. (9) into Eq. (2) and then solving the resultant equations leads to the following standard form of the equations governing $a(t)$ and $\theta(t)$:

$$\begin{aligned} \dot{a} &= -\frac{\varepsilon}{3\Omega} f(a,\theta)\sin\varphi, \\ \dot{\theta} &= -\frac{\varepsilon}{3\Omega a} f(a,\theta)\cos\varphi, \end{aligned} \tag{10}$$

where $\varphi = 3\Omega t + \theta$. $f(a,\theta)$ is a nonlinear function of $a(t)$ and $\theta(t)$, which is obtained from the terms containing ε on the right hand side of Eq. (2) by inserting solution (9).

According to the method of averaging [20,21], the amplitude $a(t)$ and the phase $\theta(t)$ of the first-order approximate solution change very little during one period of motion $T = 2\pi/(3\Omega)$. Averaging Eq. (10) over the interval $[t_0, t_0 + T]$, during which $a(t)$ and $\theta(t)$ can be taken to be constants on the right-hand side of the equation yields

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{\varepsilon}{3\Omega} f_1(a, \theta),$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\varepsilon}{3\Omega a} f_2(a, \theta) \tag{11}$$

with

$$f_1(a,\theta) = \frac{2}{T}\left\{ \int_{t_0}^{t_1} [(c+d)(-3\Omega a \sin\varphi - \Omega H \sin\Omega t) - \sigma(a\cos\varphi + H\cos\Omega t) - \alpha(a\cos\varphi + H\cos\Omega t)^3]\sin\varphi\,\mathrm{d}t + \int_{t_1}^{t_0+T/2} [c(-3\Omega a\sin\varphi - \Omega H\sin\Omega t) - \sigma(a\cos\varphi + H\cos\Omega t) - k_0]\sin\varphi\,\mathrm{d}t \right\}$$

$$f_2(a,\theta) = \frac{2}{T}\left\{ \int_{t_0}^{t_1} [(c+d)(-3\Omega a \sin\varphi - \Omega H \sin\Omega t) - \sigma(a\cos\varphi + H\cos\Omega t) - \alpha(a\cos\varphi + H\cos\Omega t)^3]\cos\varphi\,\mathrm{d}t + \int_{t_1}^{t_0+T/2} [c(-3\Omega a\sin\varphi - \Omega H\sin\Omega t) - \sigma(a\cos\varphi + H\cos\Omega t) - k_0]\cos\varphi\,\mathrm{d}t \right\}$$

where t_0 and t_1 are the roots of the following two equations, respectively,

$$a\cos(3\Omega t_0 + \theta) + H\cos\Omega t_0 + y_s = 0,$$
$$a\cos(3\Omega t_1 + \theta) + H\cos\Omega t_1 - y_s = 0.$$

The steady-state solutions $a(t)$ and $\theta(t)$ for the super-harmonic resonance response can be obtained from Eq. (11) by letting $\mathrm{d}a/\mathrm{d}t = \mathrm{d}\theta/\mathrm{d}t = 0$, which are solved by a numerical method based on a root finding algorithm.

The stability of the steady-state response can be ascertained by evaluating the eigenvalues of the Jacobian matrix of the linearized component of the equation, which is obtained by adding small perturbations to the steady-state solutions and then substituting the perturbed solutions into Eq. (11).

4. Comparison of the approximate and numerical integration solutions

Based on the analytical procedure given in Sections 2 and 3, the first-order approximate solutions for the super-harmonic resonance response can be easily obtained for a given set of system parameters using the matching method and the method of averaging. It was found that the approximate analytical solutions and the results of numerical integration of Eq. (1) are in good agreement for the super-harmonic resonance response.

An illustrative example system is studied herein as defined by the system parameters $c = 0.1$, $d = 0.1$, $\alpha = 2.0$, $k = 5.0$, $\Omega = 1.0$, $y_s = 0.5$. Fig. 2 illustrates a comparison of the maximum amplitudes of the super-harmonic resonance response between the approximate solutions and the results of numerical integration in the region $\omega \in [8.8, 9.2]$, which corresponds to the external detuning in the region $\sigma \in [-0.2, 0.2]$. The values of the amplitudes obtained by the matching method and the method of averaging are indicated by circles and triangles, respectively, while the values of numerical simulations are given by the solid curve. Only small differences between the approximate and numerical integration solutions are found. Both of the first-order approximate solutions match well with the numerical integration solutions. The relative errors of the maximum amplitudes of the response between the approximate solutions and numerical solutions, as defined by $(y_{\text{num.}} - y_{\text{app.}})/y_{\text{num.}}$, are between 0.319% and 1.943% for the approximate solutions obtained using the matching method, and between -2.778% and -8.468% for those obtained using the averaging method. The first-order approximate solutions obtained using the matching method give slightly smaller values of the maximum amplitudes than those obtained from the numerical integrations of Eq. (1), while the approximate solutions obtained using the method of averaging give slightly larger values than the numerical integration solutions. The approximate solutions obtained using the matching method give more accurate results than those obtained using the method of averaging. The discrepancies are caused by the first-order truncation of the expansion solution. A more accurate approximation could be obtained if an

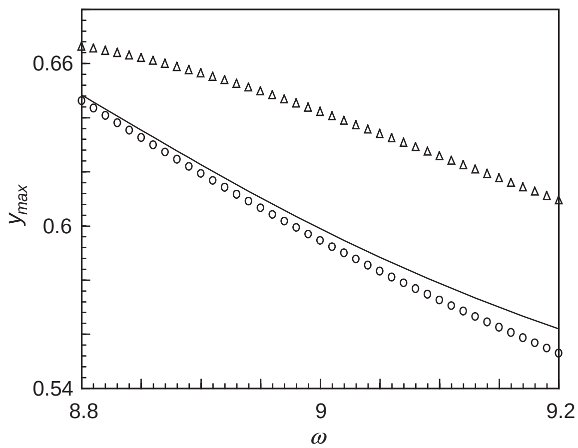

Fig. 2. Variation of the maximum amplitudes of large amplitude response with the frequency ω, where circles denote the approximate solutions obtained using the matching method, triangles represent the approximate solutions obtained using the method of averaging, and the solid curve indicates the results of numerical integration.

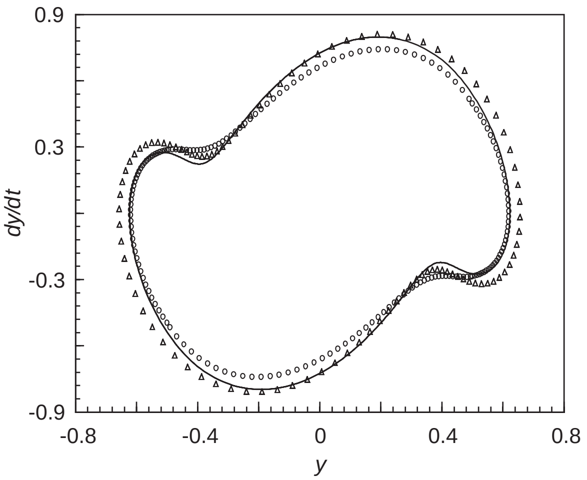

Fig. 3. Comparison of the phase portrait of the super-harmonic resonance response between the approximate and numerical integration solutions at $\omega = 8.9$, where circles and triangles denote the approximate solutions obtained using the matching method and the averaging method, respectively, and the solid curve represents the results of numerical simulations.

additional term of the second order is included in the approximate solution, but it seems unnecessary as the first-order approximations have given good representations of the super-harmonic resonance response.

Fig. 3 compares the phase portrait of the approximate solutions obtained using the matching method (indicated by circles) and using the method of averaging (indicated by triangles) with the numerical integration solution (solid curve) at $\omega = 8.9$. Small discrepancies between the first-order approximate and numerical integration solutions are found. Both of the first-order analytical solutions match well with the numerical integration solutions. The first-order approximate solutions can also give excellent representations of the super-harmonic resonance response for larger values of excitation amplitude.

5. Conclusion

An approximate periodic solution for the super-harmonic resonance response of a piecewise nonlinear oscillator has been analytically constructed using both the matching method and the averaging method, as no exact solution exists in closed-form. It was found that both of the first-order approximate solutions were an excellent representation of the exact solutions. The approximate solutions obtained using the matching method give more accurate results than those obtained using the averaging procedure, but involve more terms in the expressions of the first-order approximate solutions. Both the matching method and the averaging procedure can also be used to seek approximate solutions for the sub-harmonic resonance response, and could even be used to find approximate analytical solutions for the secondary resonance response of a piecewise nonlinear oscillator that is governed by a set of two nonlinear differential equations.

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