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Short Communication

Improvable bounds on the largest eigenvalue of a completely positive finite element flexibility matrix

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1. Introduction

Under certain conditions of element configuration and order the finite element flexibility matrix derived for second-order problems is completely positive, $F_{ij} > 0$. Then the Perron–Frobenius theorem on the positivity of the eigenvector corresponding to the least eigenvalue, $\lambda_1(K)$, of stiffness matrix K, applies. For such matrices Gershgorin's theorem can be applied iteratively to produce ever tighter upper and lower bounds on $\lambda_1(K)$, and consequently on the spectral condition number [1] of the stiffness matrix K, a number that is essential for assessing the quality of any numerical computation involving K.

2. The Perron–Frobenius and Gershgorin's theorems

For completeness sake we restate these theorems here but without proof. See Ref. [2].

Perron–Frobenius theorem. *Let real matrix $F = F(n \times n)$ be positive and symmetrical, $F = F^{\mathrm{T}}$. Then the eigenvector v corresponding to the largest eigenvalue $\lambda_n(F)$ of matix F is positive, $v_i > 0$, $i = 1, 2, \ldots, n$.*

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Gershgorin's theorem. *Let s_i be the sum of the moduli of the elements F_{ij} along the ith row of real and symmetric matrix F excluding the diagonal element F_{ii}. Then each eigenvalue of F lies inside or on the boundary of at least one of the intervals $|\lambda - F_{ii}| = s_i$. In other words,*

$$\lambda_1(F) \geqslant \min_i \left(F_{ii} - \sum_{i \neq j} |F_{ij}| \right) \quad \text{and} \quad \lambda_n(F) \leqslant \max_i \left(F_{ii} + \sum_{i \neq j} |F_{ij}| \right), \quad i = 1, 2, \ldots, n. \tag{1}$$

The last theorem can be strengthened somewhat in case an eigenvector is known to be completely positive, but this is not essential to our following discussion.

3. Two-node string element

We look first at the simple boundary value problem

$$u'' + f(x) = 0 \quad 0 < x < 1, \quad u'(0) = u(1) = 0, \tag{2}$$

describing the deflection u of a loaded unit string held under unit tension. In case $f(x) > 0$, then $u'' < 0$, and the deflection curve of the string is concave, if positive u is down, as in Fig. 1.

The finite element stiffness and mass matrices for problem (2) are

$$k = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad m = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{3}$$

respectively, with h denoting the element size. Matrix k is symmetric and positive semidefinite, while matrix m is symmetric and positive definite.

Since $k_{ij} < 0$ if $i \neq j$ and $k_{ii} > 0$, the assembled global stiffness matrix K is also of this structure. For such matrices we have the following lemma.

Lemma. *Let symmetric tridiagonal matrix K be positive definite and of the general form*

$$K = \begin{bmatrix} + & - & & & \\ - & + & - & & \\ & - & + & - & \\ & & - & + & - \\ & & & - & + \end{bmatrix} = \begin{bmatrix} p & n & & & \\ n & p & n & & \\ & n & p & n & \\ & & n & p & n \\ & & & n & p \end{bmatrix} \tag{4}$$

in which p symbolizes a positive number, and n a negative. Then $F_{ij} = K_{ij}^{-1} > 0$.

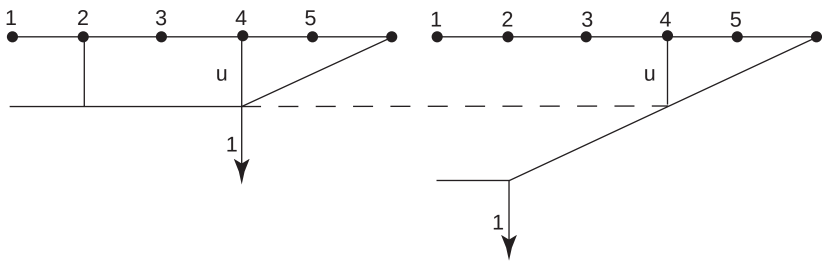

Fig. 1. A symmetrical taut string under one point force.

Proof. We write out the matrix equation $KF = I$ as

$$\begin{bmatrix} p & n & & & \\ n & p & n & & \\ & n & p & n & \\ & & n & p & n \\ & & & n & p \end{bmatrix} F = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}. \tag{5}$$

Since matrix K is positive definite the Gauss elimination algorithm leaves all elimination pivots in this matrix positive, and we bring the matrix linear system by a succession of elementary operations to the upper triangular form

$$\begin{bmatrix} p & n & & & \\ & p & n & & \\ & & p & n & \\ & & & p & n \\ & & & & p \end{bmatrix} F = \begin{bmatrix} 1 & & & & \\ p & 1 & & & \\ p & p & 1 & & \\ p & p & p & 1 & \\ p & p & p & p & 1 \end{bmatrix}, \tag{6}$$

from which it easily results that $K^{-1} = F$ is completely positive, $F_{ij} > 0$. □

For the string global stiffness matrix

$$K^{-1} = \frac{1}{h} \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}^{-1}$$

$$= h \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = h \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = F, \tag{7}$$

written here for the specific $h = 1/n,\ n = 6$. Analytically,

$$F_{ij} = \frac{1}{n}(n + 1 - j), \quad j \geqslant i. \tag{8}$$

The jth column of F represents the discrete response of the string to a unit force at the jth node—it is the discrete Green's function for that point. See Fig. 1.

The flexibility matrix F is symmetric and positive definite since the stiffness matrix K is such. It is also *dense*—none of its entries is zero, *positive*—all its entries are larger than zero, and *bounded*—here $F_{ij} \leqslant 1$ for any n.

All this is physically plausible. It is in the nature of the string that a concentrated force applied at any interior point causes *all* points of the string to move, and all *in the direction* of the applied force, and this is manifested in F being dense and positive. The finite element deflection computed for the point-loaded string is theoretically exact for any number of nodes, and hence the maximum deflection, and with it $\max F_{ij}$, is fixed for any n.

A string cannot transmit rotation or torque (one cannot use a string or rope as a lever), and under the action of a concentrated load it abruptly changes slope. Consequently if the string is internally fixed, then the problem of computing its displacements separates into two disjoint problems between the two pairs of supports, or K becomes reducible. Otherwise K is irreducible.

Symmetry in F constitutes a discrete counterpart to the *Betti–Maxwell* reciprocal theorem of elasticity: *A unit force at node i causes the same deflection at node j as a unit force at j causes at i*. See Fig. 1.

An experimental F is constructed by measuring the nodal deflections resulting from a unit force applied sequentially at all nodes. Because F is symmetric, the deflection curve due to one force can be determined by measuring displacement u at a *fixed* point, where it may be measured most conveniently and accurately, while shifting the *force* from point to point.

To this extent the algebraic formulation of the string discretized by linear finite elements faithfully imitated the analytical model of the string; both sensibly duplicating nature. It must be borne in mind, however, that generally the analytical or physical properties of the algebraically described problem may correctly appear only in the limit of the discretization as the element size becomes ever smaller. In particular, while the positive definitness of the global stiffness matrix K is guaranteed by the variational nature of the finite element method, the positiveness of flexibility matrix F may generally materialize, even for problems with a positive Green's function, only for a fine mesh.

The previous lemma readily admits the following generalization.

Theorem. *Let the symmetric matrix K be positive definite and of the general form*

$$K = \begin{bmatrix} p & n & & & \\ n & p & n & & \\ & n & p & n & \\ & & n & p & n \\ & & & n & p \end{bmatrix} + \begin{bmatrix} & & z & z & z \\ & & & z & z \\ z & & & & z \\ z & z & & & \\ z & z & z & & \end{bmatrix}, \tag{9}$$

in which, generically, $p > 0$, $n < 0$, *and* $z \leqslant 0$. *Then* $F_{ij} = K_{ij}^{-1} > 0$.

4. Improvable Gershgorin bounds on $\lambda_1(K)$

For vector $e = (1, 1, ..., 1)$, vector Ke holds as components the row sums of K. For Matrix K such that $K_{ii} > 0$ and $K_{ij} \leqslant 0$, $i \neq j$, Gershgorin's theorem may be expressed as

$$\lambda_1(K) \geqslant \min_i (Ke)_i, \quad i = 1, 2, ..., n. \tag{10}$$

But since the row sum of the finite element global stiffness matrix K is mostly zero (K being the discrete counterpart to the second degree differential operator) Gershgorin's theorem applied directly to K reduces to the trivial bound $\lambda_1(K) \geqslant 0$, correct for any positive semidefinite matrix.

The great practical interest of the bound form of Eq. (10) is that Ke, or for that matter, Kx for any vector x, may be computed on the finite element level without the need to actually assemble the global stiffness matrix K.

A better bound on $\lambda_1(K)$ may be obtained by modifying K by the similarity transformation PKP^{-1} designed to render K diagonally dominant while leaving all eigenvalues unchanged. Matters are considerably simplified if for large n we restrict matrix P to a diagonal form, denoted by D, and being such that $D_{ij} = 0$, $i \neq j$.

Let stiffness matrix K be, as before, such that $K_{ii} > 0$ and $K_{ij} \leqslant 0$, then the flexibility matrix $F = K^{-1}$ is positive and eigenvector v corresponding to $\lambda_n(F)$ is positive. But $\lambda_1(K) = 1/\lambda_n(F)$, and v is also the eigenvector of K corresponding to its lowest eigenvalue. Under these circumstances, the optimal D is such that $D^{-1}KDe = \lambda e$, or $KDe = \lambda De$, where λ is a scalar constant, and where vector $e = (1, 1, 1, ..., 1)$. Because $v_i > 0$ we may set $D_{ii} = v_i$ so that $De = v$, and $KDe = \lambda De$ becomes $Kv = \lambda v$, implying that $\lambda = \lambda_1(K)$.

In conclusion, if x is a good approximation to the eigenvector corresponding to the lowest eigenvalue of matrix K, a vector that may be obtained from any iterative method designed to produce such good eigenvector approximations, then

$$\lambda_1(K) \geqslant \min_i \frac{(Kx)_i}{x_i}, \quad i = 1, 2, ..., n, \tag{11}$$

with equality happening for $x = v$.

For example, if

$$K = \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 1.8 \\ 1 \end{bmatrix}, \tag{12}$$

then

$$\lambda_1(K) \geqslant \min_i \begin{bmatrix} 0.2/2 \\ 0.6/1.8 \\ 0.2/1 \end{bmatrix} = 0.1 \tag{13}$$

as compared to $\lambda_1(K) = 0.198$. Taking $x = (2.25, 1.8, 1)$ we obtain the better result

$$\lambda_1(K) \geqslant \min_i \begin{bmatrix} 0.45/2.25 \\ 0.35/1.8 \\ 0.2/1 \end{bmatrix} = \min_i \begin{bmatrix} 0.2 \\ 0.19444 \\ 0.2 \end{bmatrix} = 0.19444 \tag{14}$$

as compared to $\lambda_1(K) = 0.198$.

5. The infinity norm of flexibility matrix F

The infinity norm of matrix $F = F(n \times n)$ is defined as

$$\|F\|_\infty = \max_i \sum_j |F_{ij}|. \tag{15}$$

If flexibility matrix F is positive, then $\|F\|_\infty = \max_i (Fe)_i$, where i is the row index. It results then from Gershgorin's theorem that

$$\lambda_n(F) \leqslant \|F\|_\infty \quad \text{or} \quad \lambda_1(K) \geqslant \frac{1}{\|F\|_\infty}, \tag{16}$$

since $F = K^{-1}$ and $\lambda_1(K) \cdot \lambda_n(F) = 1$.

The following theorem indicates how to iteratively tighten the bounds on $\|F\|_\infty$, and hence bounds on $\lambda_1(K)$, using only matrix K.

Theorem. *Let flexibility matrix F be positive, x an arbitrary vector, and $r = Kx - e$. Then*

$$\frac{\|x\|_\infty}{1 + \|r\|_\infty} \leqslant \|F\|_\infty \leqslant \frac{\|x\|_\infty}{1 - \|r\|_\infty} \tag{17}$$

provided that $\|r\|_\infty < 1$.

Proof. Write $x = F(r + e)$ to have $\|x\| \leqslant \|F\| \, \|r + e\|$. Consequently, $\|x\| \leqslant \|F\|(1 + \|r\|)$ from which the left-hand side of inequality (17) readily follows. Write $x - Fe = Fr$ to have $\|Fr\| = \|x - Fe\| = \|Fe - x\|$. Hence, $\|F\| \, \|r\| \geqslant \|Fe - x\|$. But $\|Fe - x\| \geqslant \|Fe\| - \|x\|$ so that $\|F\| \, \|r\| \geqslant \|Fe\| - \|x\|$. The right-hand side of inequality (17) follows then immediately from the fact that $\|Fe\|_\infty = \|F\|_\infty$. □

6. Three-node string element

The element matrices for a three-node quadratic element of size $2h$ are

$$k = \frac{1}{6h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \quad \text{and} \quad m = \frac{h}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \tag{18}$$

with the element stiffness matrix k seen to have the positive off-diagonal entry $k_{13} = 1$. Hence, the assembled global matrix K is also with positive off diagonal entries, and it is not in the form of Eq. (9). Yet, $K^{-1} = F$ is still positive, and we may apply to it Eq. (17) . For example, if

$$K = \begin{bmatrix} 7 & -8 & 1 & \\ -8 & 16 & -8 & \\ 1 & -8 & 14 & -8 \\ & & -8 & 16 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 1.65 \\ 1.5 \\ 1.2 \\ 0.65 \end{bmatrix}, \quad \text{then} \quad r = Kx - e = \begin{bmatrix} -0.25 \\ 0.20 \\ 0.25 \\ -0.20 \end{bmatrix} \tag{19}$$

and

$$1.32 \leqslant \|F\|_\infty| \leqslant 2.2 \quad \text{so that} \quad 0.4545 \leqslant \lambda_1(K) \tag{20}$$

as compared to the directly computed $\lambda_1(K) = 0.7279$.

7. Three-node triangular membrane element

The bounding possibilities for the extremal eigenvalues of the global finite element stiffness and flexibility matrices K and F, described in the previous sections for the string, is of practical interest as they happen also in higher dimensions.

The linear membrane element stiffness matrix of a triangle of sides L_1, L_2, L_3 is

$$k = \frac{1}{8A}\left(L_1^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} + L_2^2 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} + L_3^2 \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right), \tag{21}$$

in which A denotes the area of the triangle. The same element stiffness matrix may be written in terms of the three vertex angles α_1, α_2, α_3 as

$$k = \frac{1}{4A} \begin{bmatrix} L_1^2 & -L_1L_2 \cos \alpha_3 & -L_1L_3 \cos \alpha_2 \\ -L_1L_2 \cos \alpha_3 & L_2^2 & -L_2L_3 \cos \alpha_1 \\ -L_1L_3 \cos \alpha_2 & -L_2L_3 \cos \alpha_1 & L_3^2 \end{bmatrix}, \tag{22}$$

with all off-diagonal entries of k being negative if the triangle is acute, that is, if $\alpha_i < 90^\circ$ or $\cos \alpha_i > 0$. The global stiffness matrix K assembled from such elements is of the form of Eq. (9). Consequently, its inverse, flexibility matrix F, is positive and the bounding procedure of Eqs. (11) and (17) may be applied to both.

As an example consider the equilateral triangular membrane fixed at its rim and discretized by the triangular finite elements of Eq. (22), as in Fig. 2. We take the vector $x = (1.1, 2.1, 1.9, 0.9, 2.1, 2.9, 1.9, 2.1, 1.9, 1.1)$ as an approximation to the eigenvector corresponding to the least eigenvalue of K, and obtain from Eq. (11) that $\lambda_1(K) \geqslant 1.78$ as compared to the computed $\lambda_1(K) = 2$.

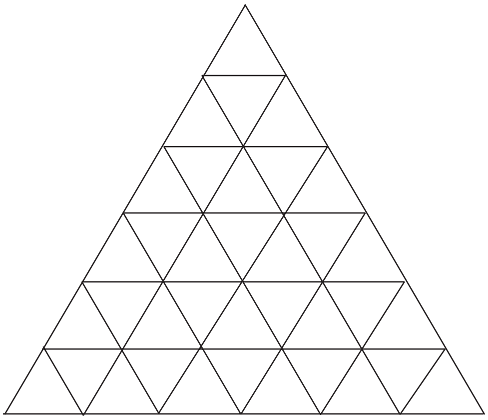

Fig. 2. An equilateral triangular membrane discretized by first-order elements.

8. Four-node rectangular membrane element

The stiffness matrix of the four-node rectangular element of length a and height b is

$$k = \frac{b}{6a}\begin{bmatrix} 2 & -2 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & -2 & 2 \end{bmatrix} + \frac{a}{6b}\begin{bmatrix} 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & -2 \\ -2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 \end{bmatrix}, \tag{23}$$

or in particular for a square element, when $a = b$,

$$k = \frac{1}{6}\begin{bmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -2 & -1 & -1 & 4 \end{bmatrix}, \tag{24}$$

which is with negative off-diagonal entries. Actually, element matrix k, and consequently global matrix K, is with negative off-diagonal entries as long as $\sqrt{2}/2 < a/b < \sqrt{2}$. To be able to confidently apply Eq. (11) to K one would want to keep the element aspect ratio within these bounds—not too stretched and not too compressed.

We have numerically observed that for a rectangular membrane of aspect ratio greater than 5.26, discretized by a mesh of 8×8 rectangular finite elements, the eigenvector corresponding to the least eigenvalue is no longer positive.

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