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Journal of Sound and Vibration 283 (2005) 91–120
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Dynamical interaction of an elastic system and essentially nonlinear absorber

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Received 7 November 2003; accepted 30 March 2004

Abstract

The nonlinear two-degrees-of-freedom system under consideration consists of a linear oscillator with a relatively big mass which is an approximation of some continuous elastic system, and an essentially nonlinear oscillator with a relatively small mass which is an absorber of the main linear system vibrations. Free and forced vibrations of the system are investigated. Analysis of nonlinear normal vibration modes shows that a stable localized vibration mode, which provides the vibration regime appropriate for an absorption, exists in a large region of the system parameters. In this regime amplitudes of vibrations of the main elastic system are small; simultaneously vibrations of the absorber are significant. Frequency response of the system under external periodic force is obtained. The dynamical interaction of elastic string under impact impulse and the essentially nonlinear absorber is considered too. Absorption of a longitudinal traveling wave in the system is analyzed.

1. Introduction

Numerous scientific publications contain a description and analysis of different devices for the vibration absorption of machines and mechanisms due to the importance of these problems in engineering. It is known that often the effective vibro-absorption can be achieved by using linear absorbers with big mass, but this is impossible to realize in most concrete systems. So an analysis of

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doi:10.1016/j.jsv.2004.03.061

absorption by using nonlinear passive absorbers is interesting as a theory as well for engineering applications. Here only some of publications on the subject are selected. In particular, principal aspects of the nonlinear absorption theory are analyzed by Kolovski [1]. Haxton and Barr [2] considered the absorber in the form of a beam, which is attached to the system mass–spring. Shaw and Wiggins [3] analyzed a pendulum-type centrifugal vibration absorber to reduce torsion oscillations. They studied chaotic motions in the system too. Shaw et al. [4] considered forced resonances in the system containing the nonlinear absorber when the forced frequency was close to the half-sum of two natural frequencies. Natsiavas [5] offered to use the oscillator with a nonlinear spring to absorb forced oscillations of the Duffing system. In Ref. [6], the mass–spring nonlinear system was analyzed by using the perturbation method to reduce vibrations of some self-excited system. The linear and nonlinear absorber general theory is presented in the handbook [7]. Lee and Shaw [8] considered the absorption of torsion vibrations of the four-cylinder engine by using the pendulum-type centrifugal absorber. Haddow and Shaw [9] studied experimentally the rotating machinery with the centrifugal pendulum absorber. The pendulum was also used by Cuvalci and Ertas [10] as a vibration absorber to reduce the response of a flexible cantilever beam. In Ref. [11] the dynamics of a resonantly excited single-degree-of-freedom (dof) linear system coupled to an array of nonlinear autoparametric vibration pendulums is analyzed. Vakakis [12] considered a semi-infinite linear chain with an essentially nonlinear spring, which was attached to the chain to absorb the vibration energy. Impact systems can be used to absorb oscillations too [13,14]. Aoki and Watanabe [15] offered the impact absorber, which contains small mass hitting on the stop.

In this paper, the single dof essentially nonlinear oscillator is examined to use an absorption of oscillations of some elastic structure. In this case a part of the elastic oscillations energy is transferred to the oscillator. The elastic structure is approximated by the single-dof mass–spring model to study the principal capacity to absorb oscillations. Free oscillations of the two-dof system are studied in this paper by the methods of the nonlinear normal vibration modes (NNMs) theory [16–23]. Note that the NNMs approach is very effective if large amplitude oscillations are analyzed. The localized NNM in the two-dof nonlinear system under consideration is appropriate for the absorption, when the main linear system and absorber have small and large amplitudes, respectively. It is accepted here that the absorber mass is significantly smaller than the mass of the main linear subsystem. Such choice of the parameter is determined by the real engineering design conditions.

Free vibrations of the system under consideration are analyzed in Section 2 by using the nonlinear normal vibration theory. Analysis of stability of the localized and non-localized NNMs is presented in Section 3. Here two approaches are used. In the first place we used the Mathieu equation that is the simplest variant of the stability problem. Next, we used the algebraization by Ince to obtain more exact results. The regions in the system parameter space are selected where the non-localized mode is unstable, and the localized mode, appropriate to the absorption, is stable. In Section 4, forced resonances and its stability are considered. Finally, the absorption of elastic waves in string by using the essentially nonlinear absorber is considered in Section 5.

2. Nonlinear normal modes in a system containing the essentially nonlinear absorber

One considers a possibility to absorb vibrations of linear elastic structure. We use the single-dof oscillator as the absorber connected with the fixing point by spring with a cubic nonlinearity. To simplify an investigation we replace the elastic system by a single-dof linear oscillator. The transformation can be realized, for example, by using the standard Bubnov–Galerkin procedure. As a result we obtain the following two-dof nonlinear system (Fig. 1):

$$\begin{cases} \varepsilon m\ddot{x} + cx^3 + \gamma(x-y) = 0, \\ M\ddot{y} + \omega^2 y + \gamma(y-x) = 0. \end{cases} \tag{1}$$

Here x and y are displacements of the absorber and main elastic systems, respectively, and ω^2, γ and c are stiffness coefficients of the springs. By assumption of the Section 1, the mass of the absorber is significantly smaller than the corresponding parameter of the elastic system. Therefore, the formal small parameter ε is introduced.

In this system non-localized as well as localized vibration modes are possible. To analyze the vibration modes, methods of the NNMs theory are used [16–23]. Nonlinear NNMs are a generalization of the normal (principal) vibrations of linear systems. In this regime a finite-dimensional system behaves like a conservative one having a single dof, and all position coordinates can be analytically parametrized by any one of them.

One writes system (1) energy integral of the form

$$T + \Pi = h, \tag{2}$$

where $T \equiv \varepsilon m\dot{x}^2/2 + M\dot{y}^2/2$ and $\Pi \equiv cx^4/4 + \omega^2 y^2/2 + \gamma(x-y)^2/2$ are kinetic and potential energies, respectively, and h is a constant of the system total energy.

Following the NNMs approach [16,17] trajectories of the NNMs in system (1) configuration space are sought in the form $y(x)$. We use the following equation to eliminate t from Eqs. (1):

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} = \dot{x}\frac{\mathrm{d}(\cdot)}{\mathrm{d}x}, \qquad \frac{\mathrm{d}^2(\cdot)}{\mathrm{d}t^2} = \dot{x}^2\frac{\mathrm{d}^2(\cdot)}{\mathrm{d}x^2} + \ddot{x}\frac{\mathrm{d}(\cdot)}{\mathrm{d}x}.$$

Using the last relations and system (1) energy integral (2), we derive the following equation to obtain the trajectories:

$$M\left[2y''\frac{h - \left(cx^4/4 + \omega^2 y^2/2 + \gamma(x-y)^2/2\right)}{\varepsilon m + M(y')^2} + \frac{y'}{\varepsilon m}(-cx^3 - \gamma(x-y))\right] + \omega^2 y + \gamma(y-x) = 0. \tag{3}$$

Here prime means a derivation by x.

Eq. (3) has the singularity at the maximum isoenergetic surface $\Pi = h$ where $x = X_0, y = y(X_0)$, and all velocities are equal to zero. The NNM trajectory can be analytically continued up to the maximum equipotential surface by satisfying some additional boundary condition:

$$\left[M\frac{y'}{\varepsilon m}(-cx^3 - \gamma(x-y)) + \omega^2 y + \gamma(y-x)\right]\Bigg|_{\Pi=h} = 0, \tag{4}$$

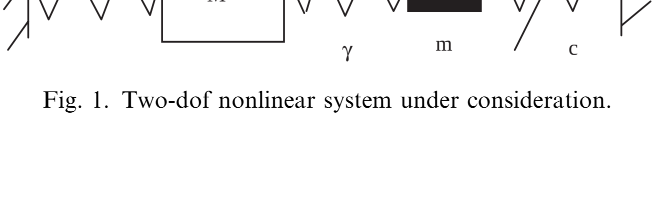

Fig. 1. Two-dof nonlinear system under consideration.

which is a condition of orthogonality of the NNMs trajectory to the maximum isoenergetic surface [16,17].

The zero approximation with respect to ε ($\varepsilon = 0$) gives us $y_0 = x + (c/\gamma)x^3$. This is *the non-localized vibration mode*. In this regime the vibration energy is distributed both in the linear oscillator, and in the essentially nonlinear absorber, that is, the vibration amplitudes of the subsystems are comparable. The corresponding limiting system, which can be obtained from Eqs. (1), is the following:

$$\begin{cases} cx^3 + \gamma(x - y) = 0, \\ M\ddot{y} + \omega^2 y + \gamma(y - x) = 0. \end{cases}$$

By using the classical procedure of the small parameter method we present the solution as the power series with respect to ε:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots. \tag{5}$$

Note that the power series method was proposed for a construction of the NNMs curvilinear trajectories in [16,20,23]. The first approximation equation by ε and corresponding boundary conditions are not presented here, but the equation can be easily obtained from relations (3) and (4). The solution of the first approximation by ε is obtained in the closed form:

$$y_1 = -\mu x \frac{12\sigma\left[H - \left(\sigma x^4/4 + \Omega x^2(1+\sigma x^2)^2/2 + \sigma^2 x^6/2\right)\right] + [\Omega(1+\sigma x^2) + \sigma x^2](1+3\sigma x^2)^2}{(1+3\sigma x^2)^3}, \tag{6}$$

where the following variables are introduced: $\mu = m/M$, $\sigma = c/\gamma$, $\Omega = \omega^2/\gamma$, $H = h/\gamma$.

Here

$$h = \frac{cX_0^4}{4} + \frac{\omega^2(y_0(X_0))^2}{2} + \frac{\gamma(X_0 - y_0(X_0))^2}{2},$$

$x = X_0$ is a value of the variable x at the maximal isoenergetic surface.

One now selects the *localized vibration mode*. When amplitudes of vibrations of the main elastic system are small, simultaneously vibrations of the absorber are significant. This regime can be analyzed if the next time transformation is imputed: $t = \sqrt{\varepsilon}\tau$. Then system (1) is written as

$$\begin{cases} m\ddot{x} + cx^3 + \gamma(x - y) = 0, \\ \dfrac{M}{\varepsilon}\ddot{y} + \omega^2 y + \gamma(y - x) = 0. \end{cases} \tag{7}$$

The corresponding limiting system (for $\varepsilon = 0$) has the form

$$\begin{cases} m\ddot{x} + cx^3 + \gamma(x - y) = 0, \\ M\ddot{y} = 0. \end{cases}$$

One has from here $y_0 = 0$. In this case the equation to obtain NNM trajectory $y(x)$ is the following:

$$M\left[2y'' \frac{h - \left(cx^4/4 + \omega^2 y^2/2 + \gamma(x-y)^2/2\right)}{m + (M/\varepsilon)(y')^2} + \frac{y'}{m}(-cx^3 - \gamma(x - y))\right] + \varepsilon\omega^2 y + \varepsilon\gamma(y - x) = 0. \tag{8}$$

By using the first approximation equation with respect to ε and the corresponding boundary conditions at the maximal isoenergetic surface, one obtains a solution of the form of the following power series by x:

$$y_1 = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots, \tag{9}$$

where

$$a_1 = -\mu \frac{\frac{X_0^2}{4H} + \frac{3X_0^4}{32H^2}}{1 + \frac{X_0^2}{4H} + \left(\frac{3}{4H} + \sigma\right)\frac{X_0^4}{8H}}, \quad a_3 = \frac{\mu + a_1}{12H}, \quad a_5 = \frac{1}{40H}\left[\frac{3}{4H}(\mu + a_1) + a_1\sigma\right], \ldots$$

3. Stability of localized and non-localized nonlinear normal modes

3.1. Analysis of stability by using the Mathieu equation

First, one considers a stability of *non-localized vibration mode*, which can be written up to terms of the order $O(\varepsilon)$ in the form

$$y \cong x + \frac{c}{\gamma} x^3. \tag{10}$$

By substituting approximation (10) as well derivatives with respect to time

$$\dot{y} = \left(1 + \frac{3c}{\gamma} x^2\right)\dot{x}, \quad \ddot{y} = \left(1 + \frac{3c}{\gamma} x^2\right)\ddot{x} + \frac{6cx}{\gamma}\dot{x}^2,$$

to the second equation of system (1), we can write the following equation to obtain the solution $x(t)$:

$$M\left[\left(1 + \frac{3c}{\gamma} x^2\right)\ddot{x} + \frac{6c}{\gamma} x\dot{x}^2\right] + \omega^2\left(x + \frac{c}{\gamma} x^3\right) + cx^3 = 0. \tag{11}$$

To get an approximate expression of the vibration mode frequency ω_0 we use the following harmonic approximation: $x_0 = A\cos\omega_0 t$; here A is the vibration amplitude. By substituting the approximation to Eq. (11) and extracting terms of $\cos\omega_0 t$, one obtains the next equation to obtain the frequency: $M\omega_0^2(1+\rho) = \omega^2(1+\rho) + \rho\gamma$. One has from here $\omega_0^2 = \omega^2/M + \rho\gamma/(M(1+\rho))$, where $\rho = \frac{3}{4}cA^2/\gamma$.

To investigate a stability of system (1) solutions we can write a system of the variational equations. Let $x = x_0 + u$, $y = y_0 + v$, where u and v are variations for the NNMs of system (1). Then we have

$$\begin{cases} \varepsilon m\ddot{u} + 3cx_0^2 u + \gamma(u - v) = 0, \\ M\ddot{v} + \omega^2 v + \gamma(v - u) = 0. \end{cases} \tag{12}$$

Note that we will investigate a stability of NNMs up to terms of the order $O(\varepsilon)$. In this case, eliminating the variable u from the first equation (12), $u = v/(1 + (3c/\gamma)x_0^2)$, from the second equation (12) we can obtain the following simplified variational equation:

$$M\ddot{v} + v\left[\omega^2 + \gamma\left(1 - \frac{1}{1 + 4\rho\cos^2\omega_0 t}\right)\right] = 0. \quad (13)$$

Then we use the next Fourier-series expansion:

$$\frac{1}{1 + 4\rho\cos^2\omega_0 t} = a_0 + a_2\cos 2\omega_0 t + \cdots, \quad (14)$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\,\mathrm{d}(\tau) = 2\frac{\omega_0}{2\pi}\int_0^{\omega_0}\frac{\mathrm{d}\tau}{1 + 4\rho\cos^2(\omega_0\tau)} = \frac{1}{\chi},$$

$$a_2 = \frac{2}{T}\int_{-T/2}^{-T/2} f(\tau)\cos(2\omega_0\tau)\,\mathrm{d}\tau = 2\frac{\omega_0}{\pi}\int_0^{\pi/\omega_0}\frac{\cos(2\omega_0\tau)\,\mathrm{d}\tau}{1 + 4\rho\cos^2(\omega_0\tau)} = -\frac{2(\chi - 1)}{\chi(\chi + 1)},$$
$$\ldots, \quad \chi = \sqrt{1 + 4\rho}.$$

Extracting two first harmonics of expansion (14) and using the transformation $\omega_0 t = \theta$, it is possible to rewrite the variational equation (13) as follows:

$$Mv''_{\theta\theta}\omega_0^2 + v[\omega^2 + \gamma(1 - a_0 - a_2\cos 2\theta)] = 0,$$

or

$$v''_{\theta\theta} + [\delta^* + 2\varepsilon^*\cos 2\theta]v = 0, \quad (15)$$

where

$$\delta^* = \frac{\omega^2}{M\omega_0^2} + \gamma\left(1 - \frac{1}{\chi}\right)\frac{1}{M\omega_0^2}, \quad \varepsilon^* = \frac{\gamma(\chi - 1)}{\chi(\chi + 1)}\frac{1}{M\omega_0^2}.$$

So an analysis of system (1) vibration modes is reduced to investigation of the well-known Mathieu equation in the form (15). The vibration amplitudes are not limited here. Regions of stability/instability of the Mathieu equation are known. The next analysis was made for fixed values of some parameters: $\varepsilon m = 0.1$, $M = 1$, $c = 1$, $\omega^2 = 1$. It is noteworthy that an influence of these parameters to results of the stability analysis is not principal. Fig. 2 shows lines corresponding to the considered solution, in the plane of the Mathieu equation parameters when the amplitude X_0 increases (for some fixed values of the parameter γ). Direction of the vibration amplitudes increase is showed by pointers. As we can see the non-localized vibration mode is situated in the region of instability (the region of instability is shaded), but for not very big vibration amplitudes the solution is near the boundary of stability/instability regions, that is, the instability develops slowly. There is a qualitative correspondence of the analysis with results of the direct numerical calculations.

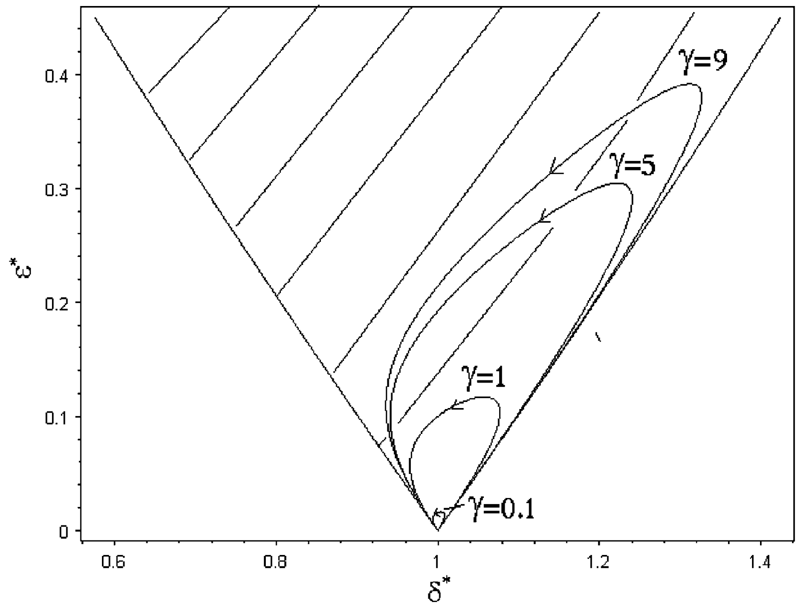

Fig. 2. Region of instability of the non-localized mode (for some fixed values of the parameter γ) in a plane of the Mathieu equation parameters when the amplitude X_0 increases. Direction of the vibration amplitudes increase is shown by pointers.

The asymptotic expansion for the positive exponent in the region of instability [24]

$$\mu = \frac{\varepsilon^*}{2}\sin(2\sigma) - \frac{3}{128}\varepsilon^{*^3}\sin(2\sigma) + \cdots.$$

Here σ is the parameter ($-\pi/2<\sigma<0$ in the instability region) which can be obtained from the following asymptotic expansion for δ^*:

$$\delta^* = 1 + \varepsilon^*\cos 2\sigma + \frac{\varepsilon^{*^2}}{8}(-2+\cos 4\sigma) + \cdots,$$

where the parameters ε^* and δ^* are presented in Eq. (15).

Table 1 contains some exponents which were calculated for some fixed values of the connection parameter γ.

Now one considers a stability of the *localized vibration mode* in the following approximate form:

$$y \cong 0. \tag{16}$$

By substituting the expression to the first equation of system (7) we can obtain the equation of motion in the form

$$m\ddot{x} + cx^3 + \gamma x = 0.$$

Repeating the transformation which was made in a case of the non-localized vibration mode, we can obtain the vibration frequency in harmonic approximation as $\omega_0^2 = \gamma + \rho/m$. In this case the variational equation is reduced to the Mathieu equation in the form (15) too.

Table 1
Values of positive exponents in the region of instability of the localized mode

$\gamma = 0.1$	$\mu_{max} = 0.008$ if $A = 0.441$
$\gamma = 1$	$\mu_{max} = 0.054$ if $A = 1.221$
$\gamma = 2$	$\mu_{max} = 0.079$ if $A = 1.595$
$\gamma = 5$	$\mu_{max} = 0.114$ if $A = 2.259$
$\gamma = 9$	$\mu_{max} = 0.131$ if $A = 2.856$

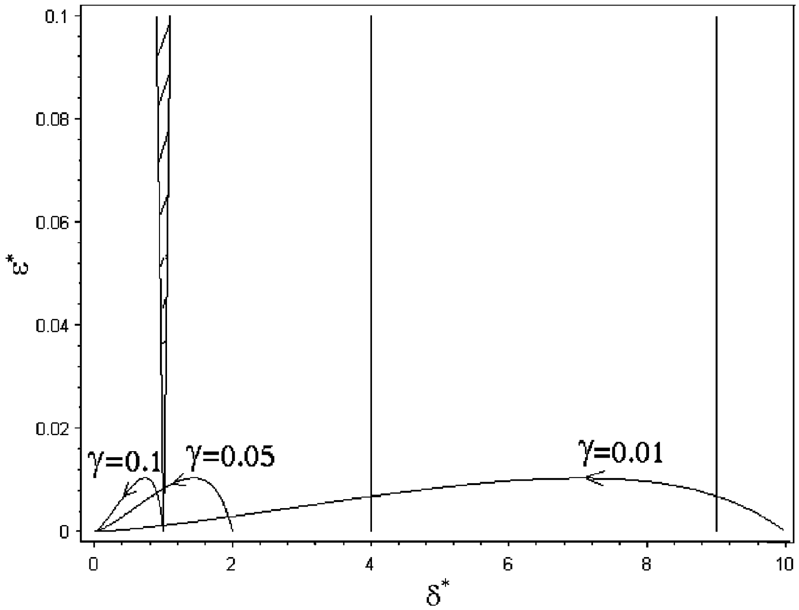

Fig. 3. Regions of instability of the localized mode (for some fixed values of the parameter γ) in a plane of the Mathieu equation parameters when the amplitude X_0 increases. Direction of the vibration amplitudes increase is shown by pointers.

The next analysis was conducted for the same fixed parameter values as for the non-localized vibration mode. Fig. 3 shows lines corresponding to the localized mode, in a plane of the Mathieu equation parameters when the amplitude X_0 increases (for some fixed values of the parameter γ). Here we can see that, in a case $\gamma = 0.01$, the curve meets three regions of instability. It is interesting that the regions are very narrow, that is, the localized mode is stable for almost all values of the system parameters.

Results of the NNMs stability analysis are approximate because only the harmonic approximation was used. But the results are basically confirmed by checking numerical calculations for small values of the parameter εm.

3.2. Algebraization of the variational equations

Another approach of the NNMs stability analysis is based on the so-called algebraization by Ince. In this case a new variable associated with the solution under consideration is chosen as an independent argument [25]. Then the variational equation is converted to the equation with singular points. This approach was used to investigate a problem of the NNMs stability and described in Refs. [16,17,26]. The algebraization gives us more exact results in the stability problem than if we reduce the problem to the Mathieu equation.

So, one considers a new a stability of the *non-localized vibration mode*. Motion along the mode is determined by Eq. (11), which was obtained by using the approximate representation of the NNM the form (10). The representation permits to write the energy integral (3) as

$$\frac{M}{2}\left(1+\frac{3c}{\gamma}x^2\right)^2\dot{x}^2+c\frac{x^4}{4}+\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{\gamma}{2}\left(\frac{c}{\gamma}x^3\right)^2=h. \tag{17}$$

Variational equation can be obtained with regard to relation (10) from the second equation of system (12):

$$M\ddot{v}+v\left[\omega^2+\gamma\left(1-\frac{1}{1+4\rho x^2}\right)\right]=0. \tag{18}$$

One introduces the following transformation of the independent variable: $t\rightarrow x$. Expressing the time derivatives in terms of the new independent variable x,

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}x}\dot{x},\qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\dot{x}^2+\frac{\mathrm{d}}{\mathrm{d}x}\ddot{x},$$

we can obtain the next equation instead of Eq. (12):

$$M(v''\dot{x}^2+v'\ddot{x})+v\left[\omega^2+\gamma\left(1-\frac{1}{1+\frac{3c}{\gamma}x^2}\right)\right]=0. \tag{19}$$

One has from the energy integral (17)

$$\dot{x}^2=\frac{2}{M}\frac{h-\left[c\frac{x^4}{4}+\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{c^2}{2\gamma}x^6\right]}{\left(1+\frac{3c}{\gamma}x^2\right)^2}$$

and from Eq. (11)

$$\ddot{x}=-\frac{\frac{6Mc}{\gamma}x\dot{x}^2+\omega^2\left(x+\frac{c}{\gamma}x^3\right)+cx^3}{M\left(1+\frac{3c}{\gamma}x^2\right)}.$$

Substituting the expressions into Eq. (19), after some transformations we can write the following:

$$\begin{aligned}
&2v''\left\{h-\left[c\frac{x^4}{4}+\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{c^2}{2\gamma}x^6\right]\right\}\left(1+\frac{3c}{\gamma}x^2\right)\\
&-v'\left(\frac{12c}{\gamma}x\left\{h-\left[c\frac{x^4}{4}+\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{c^2}{2\gamma}x^6\right]\right\}\right.\\
&\left.+\left(\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{c^2}{2\gamma}x^6\right)\left(1+\frac{3c}{\gamma}x^2\right)^2\right)\\
&+v\left(\omega^2\left(1+\frac{3c}{\gamma}x^2\right)+3cx^2\right)\left(1+\frac{3c}{\gamma}x^2\right)^2=0.
\end{aligned} \tag{20}$$

Singular points of Eq. (20) are situated on the maximal isoenergetic surface:

$$h-\left[c\frac{x^4}{4}+\frac{\omega^2}{2}\left(x+\frac{c}{\gamma}x^3\right)^2+\frac{c^2}{2\gamma}x^6\right]=0.$$

One denotes the points as $\pm X_0$. One introduces the next transformation in Eq. (20): $x^2-X_0^2=z$. Then we have

$$\begin{aligned}
&2[v''_{zz}4(z+X_0^2)+2v'_z](a_6(z+X_0^2)^2+(a_4+a_6X_0^2)(z+X_0^2)+a_2+a_4X_0^2+a_6X_0^4)\\
&\times\left(1+\frac{3c}{\gamma}(z+X_0^2)\right)z-v'_z2(z+X_0^2)\frac{12c}{\gamma}z\\
&\times(a_6(z+X_0^2)^2+(a_4+a_6X_0^2)(z+X_0^2)+a_2+a_4X_0^2+a_6X_0^4)\\
&-v\left(\omega^2\left(1+\frac{3c}{\gamma}(z+X_0^2)\right)+3c(z+X_0^2)\right)\left(1+\frac{3c}{\gamma}(z+X_0^2)\right)^2=0,
\end{aligned} \tag{21}$$

where $a_2=\omega^2/2$, $a_4=c/4+c\omega^2/\gamma$, and $a_6=c^2/(2\gamma)+c^2\omega^2/(2\gamma^2)$. Indices of the equation singular points are equal to $\alpha_1=0$ and $\alpha_2=\frac{1}{2}$. It is demonstrated in Ref. [25] that solutions corresponding to boundaries of stability/instability regions of equations with singular points, indices of which are equal to 0 and $\frac{1}{2}$ are the following:

$$v_1=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots, \tag{22}$$

$$v_2=\sqrt{X_0^2-x^2}(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+\cdots). \tag{23}$$

Substituting the series v_1 to the variational equation (21) and matching respective powers of x, we can obtain the next infinite recurrent system of linear algebraic equations to determine coefficients of expansion (22):

$$
\begin{aligned}
&x^0\colon 4ha_2 + \omega^2 a_0 = 0,\\
&x^1\colon 12ha_3 - 12\frac{ch}{\gamma}a_1 = 0,\\
&x^2\colon 3ca_0 - 12\frac{ch}{\gamma}a_2 - 3\omega^2 a_2 + 9\frac{\omega^2 c}{\gamma}a_0 + 24ha_4 = 0,\\
&x^3\colon 2ca_1 + 8\frac{c\omega^2}{\gamma}a_1 + 40ha_5 - 8\omega^2 a_3 = 0,\\
&x^4\colon 24\frac{hc}{\gamma}a_4 - 15\omega^2 a_2 + 18\frac{c^2}{\gamma}a_0 - 3\frac{\omega^2 c}{\gamma}a_2 + 27\frac{\omega^2 c^2}{\gamma^2}a_0 + 60ha_6 = 0,\\
&\quad\vdots
\end{aligned}
$$

It is clear that the system decomposes into two subsystems to determine coefficients with even and odd subscripts.

Analogously, substituting the solution v_2 to Eq. (21) and matching respective powers of x, one has the following recurrent equations:

$$
\begin{aligned}
&x^0\colon X_0^4\omega^2 b_0 - 2X_0^2 hb_0 + 4X_0^4 hb_2 = 0,\\
&x^1\colon 12X_0^4 hb_3 - 12\frac{X_0^4 hc}{\gamma}b_1 - 6X_0^2 hb_1 = 0,\\
&x^2\colon -12\frac{X_0^4 h}{\gamma}b_2 + 9\frac{X_0^4\omega^2 c}{\gamma}b_0 - 3X_0^4\omega^2 b_2 - 12X_0^2 hb_2 + 24X_0^4 hb_2 + 6\frac{X_0^2 hc}{\gamma}b_0 + 3X_0^4 cb_0 = 0,\\
&x^3\colon -40X_0^4 hb_5 - 12\frac{X_0^2 hc}{\gamma}b_1 - 38X_0^2 hb_3 - 8X_0^4\omega^2 b_3 + 4X_0^2\omega^2 b_1 + 4hb_1\\
&\qquad + 8\frac{X_0^4 c\omega^2}{\gamma}b_1 + 2X_0^4 cb_1 = 0,\\
&x^4\colon 60X_0^4 hb_6 + 24\frac{X_0^4 hc}{\gamma}b_4 - 66X_0^2 hb_4 - \frac{9}{2}X_0^2 cb_0 - 12\frac{X_0^2\omega^2 c}{\gamma}b_0 + 12hb_2 + 12X_0^2\omega^2 b_2\\
&\qquad - 3\frac{X_0^4\omega^2 c}{\gamma}b_2 - 15X_0^4\omega^2 b_4 - 12\frac{ch}{\gamma}b_0 + 6\frac{X_0^2 hc}{\gamma}b_2 + 18\frac{X_0^4 c^2}{\gamma}b_0 + 27\frac{X_0^4\omega^2 c^2}{\gamma^2}b_0 = 0,\\
&\quad\vdots
\end{aligned}
$$

which also decompose into two subsystems to determine coefficients with even and odd subscripts.

As a result four systems of the algebraic equations with respect to a_i, b_i , $i = \overline{1,\infty}$ are derived. These systems have non-trivial solutions if their determinants are equal to zero. These determinants were calculated until six order. Thus, four equations connecting the system parameters are derived. These equations give the instability region boundaries for the non-localized vibration mode. Three determinants have solutions only when the system parameters are

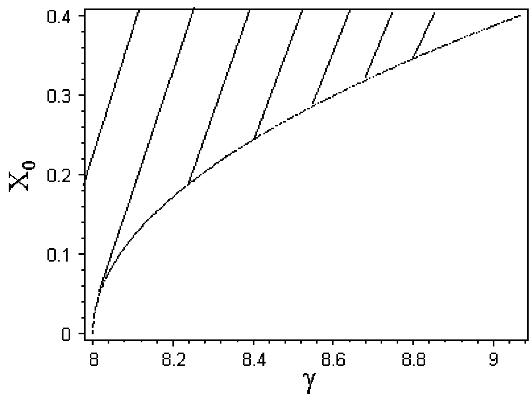

Fig. 4. Boundary of stability/instability regions of the non-localized mode in place of parameters X_0, γ, obtained by using the Ince algebraization.

equal to zero. The boundary of instability for the last determinant, which correspond to b_i, $i = 2k$, $k \in N$ for the parameters $\varepsilon m = 0.1$, $M = 1$, $c = 1$, $\omega^2 = 1$, is shown in Fig. 4. This boundary generally corresponds to the right one, which is obtained for the Mathieu equation (2). However, the boundary obtained by the algebraization for relatively big values of the parameter γ is situated above. We conclude that, at some values of the system parameters the non-localized vibration mode is situated in the domain of stability near its boundary and gets into the instability domain if the vibration amplitude increases.

Now the procedure of algebraization is used to analyze the stability of the *localized vibration mode*. We restrict ourselves by the following approximation $y \cong 0$. The energy integral along this vibration mode has the form

$$m\frac{\dot{x}^2}{2} + c\frac{x^4}{4} + \gamma\frac{x^2}{2} = h. \tag{24}$$

Let us exclude from system (7) first equation $\ddot{x} = -(\gamma x + cx^3)/m$ and from the energy integral (24)

$$\dot{x}^2 = 2\frac{h - (cx^4/4 + \gamma x^2/2)}{m}.$$

The results are substituted into the variational equation (19). We derive the following equation:

$$\frac{2M}{m}\left[h - \left(\frac{cx^4}{4} + \frac{\gamma x^2}{2}\right)\right]\left(1 + \frac{3c}{\gamma}x^2\right)v'' - \frac{M}{m}(\gamma x + cx^3)\left(1 + \frac{3c}{\gamma}x^2\right)v' + v\left(\omega^2\left(1 + \frac{3c}{\gamma}x^2\right) + 3cx^2\right)\left(1 + \frac{3c}{\gamma}x^2\right)^2 = 0. \tag{25}$$

Developing the same transformations as for the non-localized mode with respect to the singular points of Eq. (25) and its exponents, we obtain that the boundary solutions have the form of the power series (22) or (23).

Substituting these series into Eq. (20), we derive two systems of recurrent algebraic equations to obtain the series coefficients:

$$\begin{aligned}
&x^0: 4\frac{M}{m}ha_2 + \omega^2 a_0 = 0,\\
&x^1: -\frac{M}{m}\gamma a_1 + \omega^2 a_1 + 12\frac{M}{m}ha_3 = 0,\\
&x^2: 3ca_0 - 4\frac{M}{m}\gamma a_2 + 24\frac{M}{m}ha_4 + \omega^2 a_2 + 3\frac{\omega^2 c}{\gamma}a_0 + 12\frac{M}{m}\frac{ch}{\gamma}a_2 = 0,\\
&x^3: -4\frac{M}{m}ca_1 + \omega^2 a_3 + 40\frac{M}{m}ha_5 + 3\frac{\omega^2 c}{\gamma}a_1 + 36\frac{M}{m}\frac{hc}{\gamma}a_3 - 9\frac{M}{m}\gamma a_3 + 3ca_1 = 0,\\
&x^4: -16\frac{M}{m}\gamma a_4 + \omega^2 a_4 + 60\frac{M}{m}ha_6 - 15\frac{M}{m}ca_2 + 3ca_2 + 3\frac{\omega^2 c}{\gamma}a_2 + 72\frac{M}{m}\frac{hc}{\gamma}a_4 = 0,\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
x^0:\ & -2\frac{M}{m}X_0^2hb_0 + X_0^4\omega^2 b_0 + 4\frac{M}{m}X_0^4hb_2 = 0,\\
x^1:\ & -\frac{M}{m}X_0^4\gamma b_1 - 6\frac{M}{m}X_0^2hb_1 + X_0^4\omega^2 b_1 + 12\frac{M}{m}X_0^4hb_3 = 0,\\
x^2:\ & -4\frac{M}{m}X_0^4\gamma b_2 - 18\frac{M}{m}X_0^2hb_2 + X_0^4\omega^2 b_2 + 24\frac{M}{m}X_0^4hb_4 + 12\frac{M}{m}\frac{X_0^4hc}{\gamma}b_2\\
& -2X_0^2\omega^2 b_0 + 2\frac{M}{m}X_0^2\gamma b_0 - 6\frac{M}{m}\frac{X_0^2hc}{\gamma}b_0 + 3X_0^4cb_0 + 3\frac{X_0^4\omega^2 c}{\gamma}b_0 = 0,\\
x^3:\ & -9\frac{M}{m}X_0^4\gamma b_3 + 3X_0^4b_1c - 38\frac{M}{m}X_0^2hb_3 + X_0^4\omega^2 b_3 + 40\frac{M}{m}X_0^4hb_5 - 2X_0^2\omega^2 b_1\\
& -\frac{M}{m}hb_1 - 18\frac{M}{m}\frac{X_0^2hc}{\gamma}b_1 + 36\frac{M}{m}\frac{X_0^4hc}{\gamma}b_3 + 3\frac{X_0^4\omega^2 c}{\gamma}b_1 - 4\frac{M}{m}X_0^4cb_1 + 6\frac{M}{m}X_0^2\gamma b_1 = 0,\\
x^4:\ & -16\frac{M}{m}X_0^4\gamma b_4 - 66\frac{M}{m}X_0^2hb_4 + X_0^4\omega^2 b_4 + 60\frac{M}{m}X_0^4hb_6 + 14\frac{M}{m}X_0^2\gamma b_2\\
& +72\frac{M}{m}\frac{X_0^4hc}{\gamma}b_4 + \frac{15}{2}\frac{M}{m}X_0^2cb_0 - 6\frac{X_0^2\omega^2 c}{\gamma}b_0 - 6X_0^2cb_0 + \omega^2 b_0 + 12\frac{M}{m}hb_2\\
& -\frac{M}{m}\gamma b_0 - 54\frac{M}{m}\frac{X_0^2hc}{\gamma}b_2 - 15\frac{M}{m}X_0^4cb_2 - 2X_0^2\omega^2 b_2 + 3\frac{X_0^2\omega^2 c}{\gamma}b_2 + 3X_0^4cb_2 = 0,\\
& \vdots
\end{aligned}$$

Each of these systems is decomposed into two systems. One of these system corresponds to even powers of x and the other one to odd powers. The systems of linear equations have non-trivial solutions, if determinants are equal to zero. The determinants are calculated until six order. Thus, we obtain four equations to determine the boundary of instability, which depend on the system

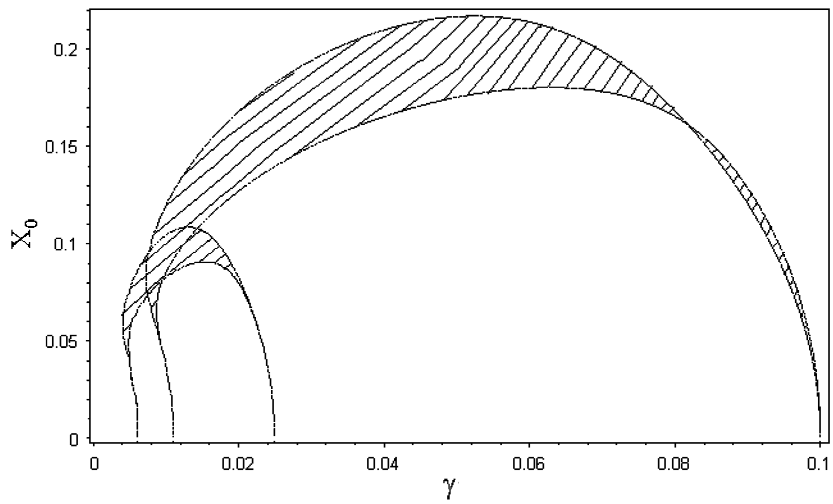

Fig. 5. Boundaries of the instability regions of the localized mode in place of parameters X_0 and γ, obtained by using the Ince algebraization.

parameters. The boundaries for localized mode with the following system parameters $\varepsilon m = 0.1$, $M = 1$, $c = 1$, $\omega^2 = 1$ are shown in Fig. 5. The regions of instability are shaded in this figure.

The numerical calculations generally confirm the analytical results which are obtained by the algebraization. In particular, the unstable non-localized vibration mode and stable localized mode are presented in Figs. 6 and 7, respectively. Here $\varepsilon m = 0.1$, $M = 1$, $c = 1$, $\omega^2 = 1$, $\gamma = 1.5$, $X_0 = 4$ (Fig. 6) and $\gamma = 1$, $X_0 = 1.5$ (Fig. 7).

Taking into account a weak dissipation in system (1) we show in Fig. 8 oscillations which are close to the stable localized normal mode if the vibration amplitudes are significant. Subsequently, the fast stability loss of the localized mode and a passage to the stable non-localized mode takes place, when the oscillation amplitude decreases and the trajectory hits the unstable region.

4. Forced vibrations of system containing the essentially nonlinear absorber

The forced oscillations of the two-dof system discussed in this paper are considered in this section. The system is similar to system (1) but the external periodic action is presented here:

$$\begin{cases} \varepsilon m\ddot{x} + cx^3 + \gamma(x - y) = 0, \\ M\ddot{y} + \omega^2 y + \gamma(y - x) = P_0 \cos kt, \end{cases} \tag{26}$$

where all coefficients mean the same as in system (1), $P_0 \cos kt$ is the time-periodic external force, and ε is the small parameter.

An aim of this section is to show, by using the standard approximate methods, that the forced resonance regimes are close to the localized and non-localized NNMs of the generating

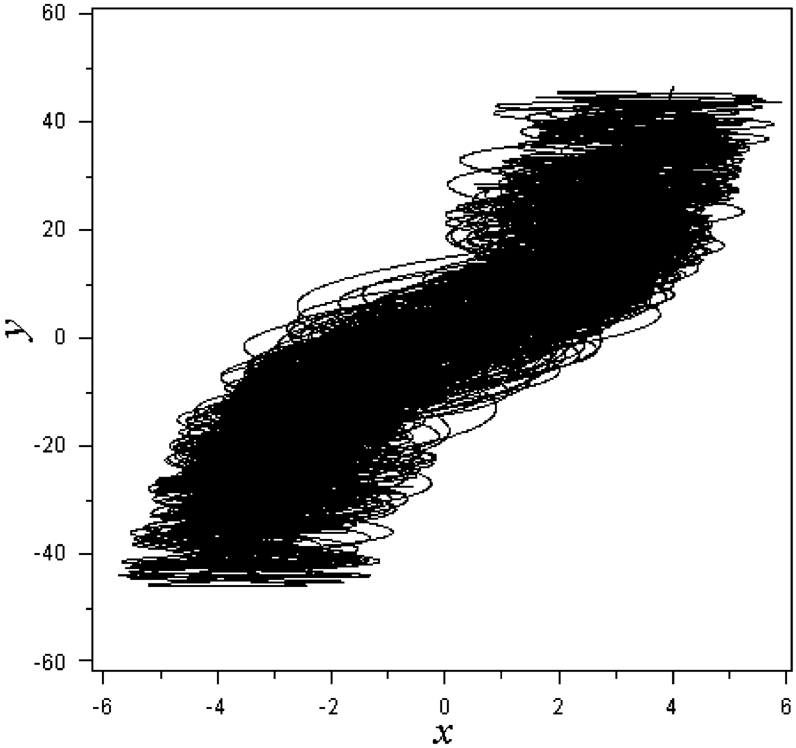

Fig. 6. Trajectory in a configuration place of solution close to the non-localized mode in the region of instability.

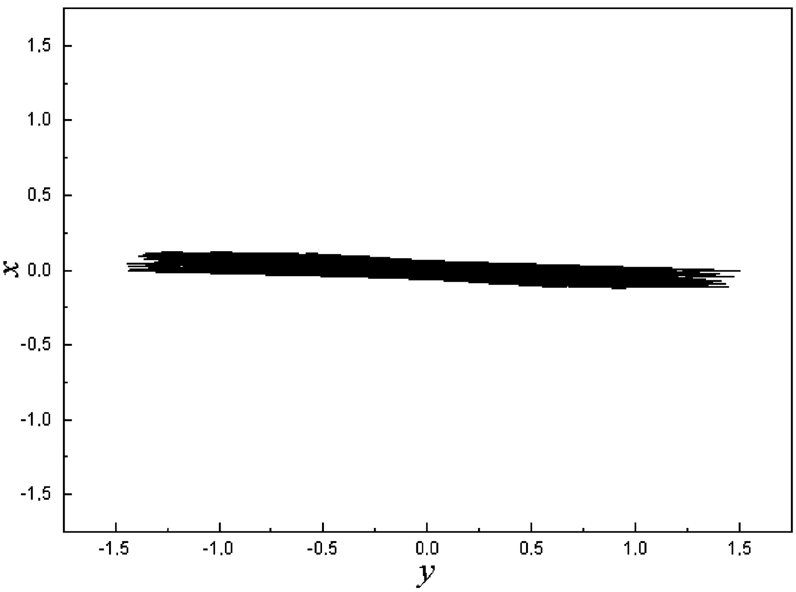

Fig. 7. Trajectory in a configuration place of solution close to the localized mode in the region of stability.

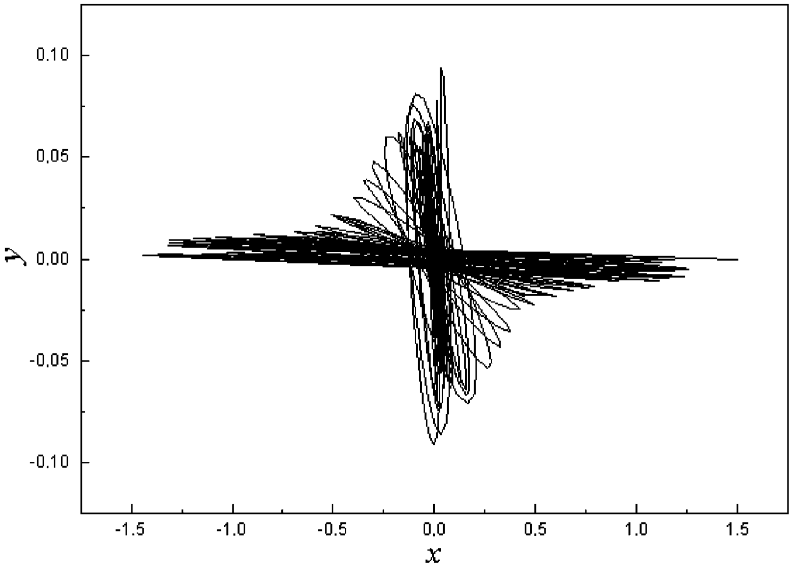

Fig. 8. Dynamics of system (1) with dissipation: transfer from the localized mode to non-localized one when the vibration amplitude decreases.

conservative system, analyzed in preceding sections. Frequency responses for approximate solutions of system (1) are obtained here. Skeletons of the frequency responses correspond to the localized and non-localized NNMs.

4.1. Damped forced oscillations

Let us introduce the viscous damping into system (26). As a result we obtain the following equations of motion:

$$\begin{cases} \varepsilon m\ddot{x} + \beta\dot{x} + cx^3 + \gamma(x-y) = 0, \\ M\ddot{y} + \beta\dot{y} + \omega^2 y + \gamma(y-x) = P_0\cos kt, \end{cases} \tag{27}$$

where β is a coefficient of viscous damping. Solutions of the system are presented in the following single-harmonic approximate form:

$$x = a\cos(kt + \varphi_1),$$

$$y = b\cos(kt + \varphi_2). \tag{28}$$

Substituting (28) into Eqs. (27) and using the simplest harmonic balance procedure, we obtain the system of nonlinear equations with respect to a, b, S_i, C_i, where $S_i = \sin\varphi_i$, $C_i = \cos\varphi_i$, $i = 1, 2$.

After some transformations one has the following equations:

$$\begin{cases} \varepsilon mak^2 S_1 - \beta akC_1 - \frac{3}{4}ca^3 S_1 - \gamma aS_1 + \gamma bS_2 = 0, \\ -\varepsilon mak^2 C_1 - \beta akS_1 + \frac{3}{4}ca^3 C_1 + \gamma aC_1 - \gamma bC_2 = 0, \\ Mbk^2 S_2 - \beta bkC_2 + \gamma aS_1 - \gamma bS_2 = 0, \\ -Mbk^2 C_2 - \beta bkS_2 - \gamma aC_1 + \gamma bC_2 - P_0 = 0, \\ S_1^2 + C_1^2 = 1, \\ S_2^2 + C_2^2 = 1. \end{cases} \tag{29}$$

Excluding from system (29) the variables S_1, C_1, S_2, and C_2, we obtain the next system of two equations:

$$\left(b^2(Mk^2 - \omega^2 - \gamma) + a^2\left(\frac{3}{4}ca^2 + \gamma - \varepsilon mk^2\right)\right)^2 + \beta^2 k^2 (a^2 + b^2)^2 = P_0 b^2,$$

$$\beta^2 k^2 a^2 + a^2\left(\frac{3}{4}ca^2 + \gamma - \varepsilon mk^2\right)^2 = \gamma^2 b^2. \tag{30}$$

Excluding b^2, we can obtain the nonlinear equation with respect to the amplitude a and the external frequency k. Fig. 9 shows the frequency response $a(k)$ with the following parameters, $M = 1$, $m = 0.1$, $\varepsilon = 1$, $c = 1$, $\gamma = 1$, $P_0 = 0.1$, $\omega^2 = 1$, $\beta = 0.005$. Fig. 10 shows the frequency response $b(k)$ for the parameters.

The additional calculations show that decreasing of the nonlinear coefficient c leads to the increase of oscillations amplitudes in the two main resonance domains. Thus, the choice of the essential nonlinear oscillator as the absorber is effective if the nonlinear stiffness is sufficiently large. We stress from the Figs. 9 and 10 that in the domain of the second main resonance the oscillations amplitudes of the main elastic system are considerably smaller than the oscillations

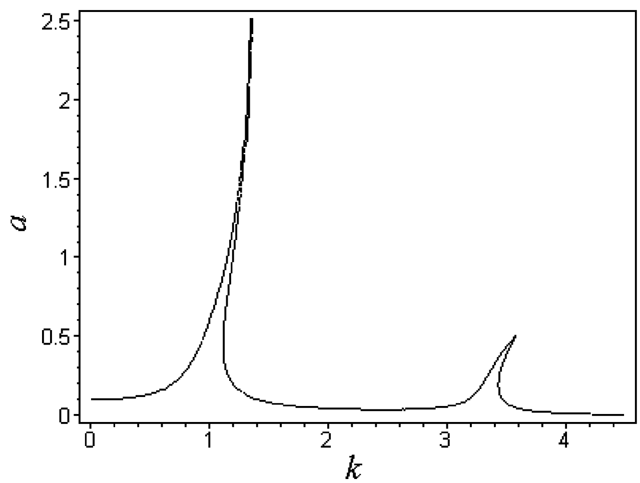

Fig. 9. Frequency response $a(k)$ in system (27): damped forced oscillations.

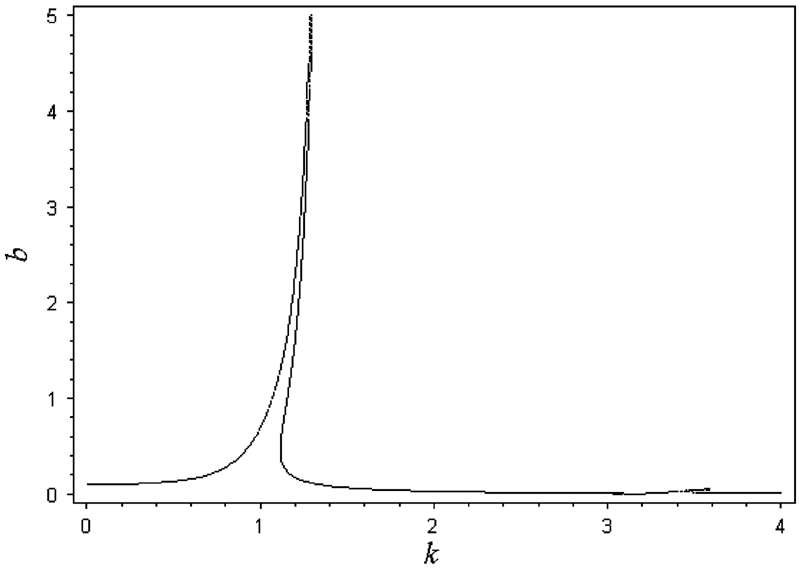

Fig. 10. Frequency response $b(k)$ in system (27): damped forced oscillations.

amplitudes of the absorber. In the frequency range of this second main resonance the absorption of elastic system oscillations takes place.

4.2. Biharmonic approximation of the forced oscillations

Let us increase the number of harmonics of the approximate solution to check the accuracy of the harmonic approximation. The solution is chosen for system (26) without dissipation of the form

$$x = a_1 \cos kt + a_3 \cos 3kt,$$

$$y = b_1 \cos kt + b_3 \cos 3kt. \quad (31)$$

Substituting approximation (31) into system (26) and equating amplitudes of the first and third harmonics of cosine, the following equations are derived:

$$\begin{cases} -ma_1k^2 + \frac{3}{4}a_1^3c + \frac{3}{4}a_1^2a_3c + \frac{3}{2}a_1a_3^2c + \gamma a_1 - \gamma b_1 = 0, \\ -9mk^2a_3 + \frac{1}{4}a_1^3c + \frac{3}{2}a_1^2a_3c + \frac{3}{4}a_3^3c + \gamma a_3 - \gamma b_3 = 0, \\ -Mb_1k^2 + \omega^2 b_1 + \gamma b_1 - \gamma a_1 - P_0 = 0, \\ -9Mb_3k^2 + \omega^2 b_3 + \gamma b_3 - \gamma a_3 = 0. \end{cases} \quad (32)$$

Eq. (32) allows for obtaining the frequency response for the different harmonics. Figs. 11 and 12 show the dependences of a_1 and a_3 on the frequency k with the parameters presented in the above sections, Figs. 13 and 14 show frequency responses $b_1(k)$ and $b_3(k)$, respectively. It is interesting that the checking numerical calculations performed by A. Koz'min confirm principally the frequency response obtained by using the biharmonic approximation.

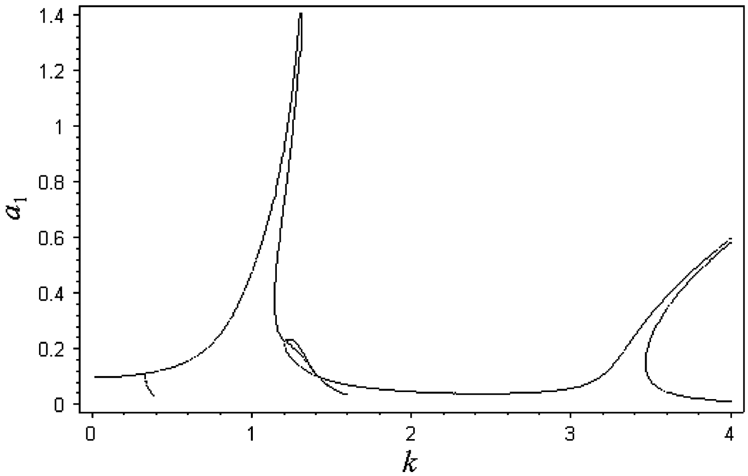

Fig. 11. Frequency response $a_1(k)$ in system (26): two-harmonic approximation of the forced oscillations.

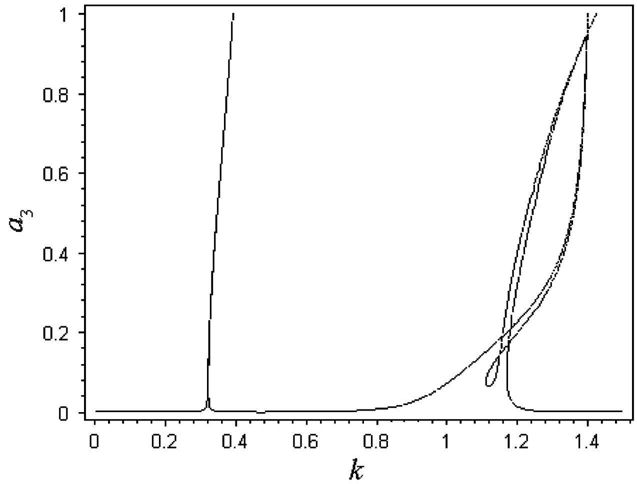

Fig. 12. Frequency response $a_3(k)$ in system (26): two-harmonic approximation of the forced oscillations.

4.3. The stability of forced oscillations

The system of variation equations coincides here with system (12) constructed for an analysis of the free vibrations stability.

By excluding the variation u from the first equation of system (12) and substituting the corresponding expression into the second equation of this system, we obtain the equation of the form (13). Expanding the function $1/(1 + 4\rho \cos^2 kt)$ into the Fourier series and preserving only two first harmonics, it can obtain the variational equation of the form (15) where it has to write the external frequency k instead of the ω_0.

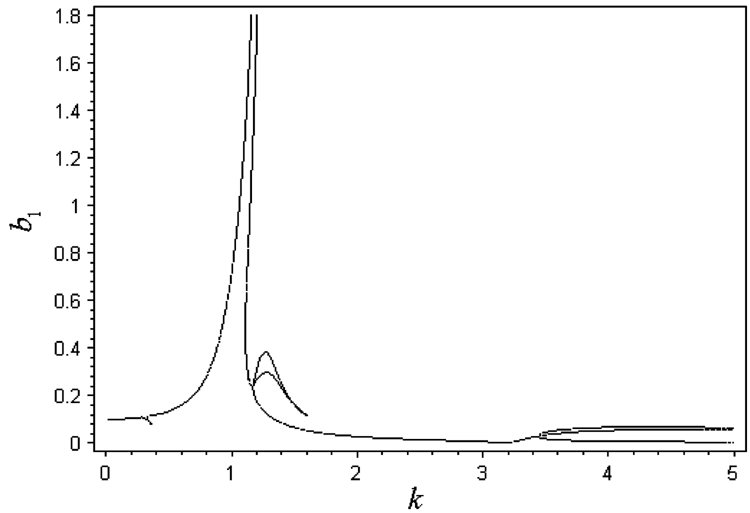

Fig. 13. Frequency response $b_1(k)$ in system (26): two-harmonic approximation of the forced oscillations.

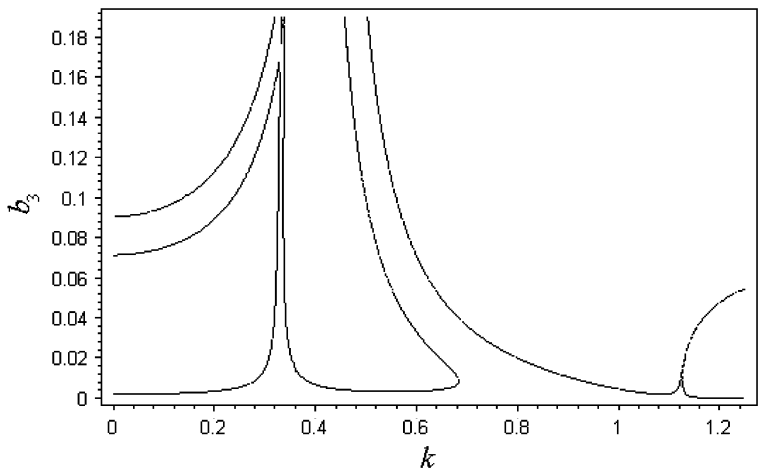

Fig. 14. Frequency response $b_3(kt)$ in system (26): two-harmonic approximation of the forced oscillations.

Here the restrictions on the oscillations amplitudes is not introduced. The future analysis is developed with parameters $\varepsilon m = 0.1$, $M = 1$, $c = 1$, $\omega^2 = 1$. It is important that these parameters change do not essentially influence the solution of the stability problem. Fig. 15 shows the forced vibration stability and instability (instability solutions are selected by markers), which are superimposed on the frequency response. We stress that the oscillations near the first resonance are unstable and the oscillations near the second resonance, which are provided by the absorption mode, are stable.

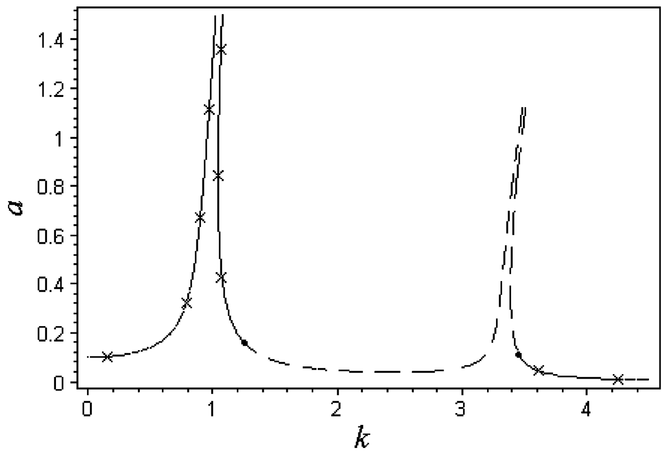

Fig. 15. Stability of the forced oscillations in system (26).

Fig. 16. The model "string-nonlinear oscillator" to investigate an absorption of impulse longitudinal waves.

The results of analytical stability study of the forced oscillations investigation are approximate but the checking numerical calculations basically confirm the results of the forced oscillations stability analysis.

5. Absorption of elastic waves by using the essentially nonlinear absorber

We consider in this section an interaction of the semi-infinite ideal elastic string and essentially nonlinear oscillator. It is assumed that a longitudinal traveling wave in the form of rectangular pulse induces longitudinal vibrations in the nonlinear oscillator. The nonlinear oscillator can be considered as an absorber if an essential part of elastic energy is concentrated in the nonlinear discrete unit with a relatively small mass. Note that the problem was studied in Ref. [27]. An analysis presented here is more exact and detailed because the approximate solution was obtained in Ref. [27] only for some limited case when the elastic constant of the coupling spring equalled to infinity, and the nonlinear oscillator mass equalled to zero.

Here an approximate analytical solution is constructed in a supposition that a duration of the pulse is small, that is the system is under an impact action.

The principal model is presented in Fig. 16. The equations describing this system can be written as

$$w_{tt}(t,x) - a^2 w_{xx}(t,x) = 0, \quad x<0, \tag{33}$$

$$m\ddot{v} + cv^3 + \gamma(v - w(t,0)) = 0, \tag{34}$$

$$\gamma(v - w(t,0)) = \rho a^2 \frac{\partial w}{\partial x}\bigg|_{x=0}, \tag{35}$$

where Eq. (33) describes a longitudinal motion in a string; Eq. (34) describes the single-dof oscillator vibrations and Eq. (35) describes a motion of the string extreme point. Here w denotes a displacement of the string elements, v is a displacement of the nonlinear oscillator, x and t are, respectively, space and time coordinates, γ is an elastic constant of the coupling spring, ρ is a density of the string, a is a sound velocity in the string and c is an elastic constant of the nonlinear oscillator "anchor" spring.

The solution for the region $x<0$ is searched in the D'Alambert form:

$$w(x,t) = F(x - at) + R(x + at), \tag{36}$$

where $F(x - at)$ is the external wave and $R(x + at)$ is the reflected wave. By substituting Eq. (36) into Eq. (33) and eliminating $R(x + at)$ by using the standard procedure [28], we have as a result the following system of equations:

$$\begin{cases} m\ddot{v} + cv^3 + \gamma(v - w(0)) = 0, \\ \rho a\dot{w}(0) + \gamma(w(0) - v) = -2F'(-at). \end{cases}$$

The external excitation is presented in a form of the rectangular pulse,

$$F'(x - at) = \begin{cases} 0, & at - x < 0, \\ -D/2, & 0 < at - x < \tau, \\ 0, & \tau < at - x. \end{cases} \tag{37}$$

Its approximates an effect of impact applied to the string at infinity. Here τ is small.

Introducing some renames, we formulate the problem as follows.

The next model is considered:

$$\begin{cases} \varepsilon m\ddot{v} + cv^3 + \gamma(v - u) = 0, \\ M\dot{u} + \gamma(u - v) = f(t), \end{cases} \tag{38}$$

where v is a variable which describes longitudinal displacements of the nonlinear oscillator; u is a variable which describes the wave front set in the extreme point of the string; $f(t)$ is an external action (the constant pulse), which acts during the small interval of time. It is assumed that the oscillator mass is essentially smaller than the string mass, so the *formal small parameter* ε in the first equation (38) is introduced. Here

$$f(t) = \begin{cases} D, & 0 \leqslant t \leqslant \tau, \\ 0, & t > \tau. \end{cases}$$

Initial values chosen are trivial.

Starting from the physical reasons we can select three regions and obtain analytical solutions for each of them: (1) a region of the fast increase of variables $u(t)$ and $v(t)$; (2) a region of damped oscillations; (3) a region of non-oscillating motion. Note that the next analysis and the numerical simulation confirm a correctness of the choice.

5.1. Construction of the approximate analytical solution

5.1.1

One considers the first region: the fast increase of variables $u(t)$ and $v(t)$ ($D \neq 0,\ 0 \leqslant t \leqslant \tau$). Here $f(t)$ induces an increase by t of the displacement $u(t)$; therefore the terms $f(t)$ and $M\dot{u}$ are selected in the second equation of system (38). Besides, an increase of u induces an increase of the variable v; therefore the terms $m\ddot{v}$ and $\gamma(v-u)$ are selected in the first equation of system (38). To obtain the corresponding limiting system we assume that τ is small; the variables $u(t)$ and $v(t)$ are small too. We introduce the formal small parameters μ_1 and μ_2 to system (38) by using $t \to \mu_1 t$, $u \to \mu_2 u$, $v \to \mu_2 v$. As a result we can obtain the following system:

$$\begin{cases} \dfrac{\varepsilon\mu_2}{\mu_1^2} m\ddot{v} + \mu_2^3 c v^3 + \gamma\mu_2(v-u) = 0, \\ \dfrac{M}{\mu_1}\mu_2\dot{u} + \gamma\mu_2(u-v) = D. \end{cases}$$

To select the above discussed principal terms in the system it is necessary to assume $\varepsilon/\mu_1^2 = 1$, $\mu_2/\mu_1 = 1$. We obtain from here that $\mu_1 = \sqrt{\varepsilon}$ and $\mu_2 = \sqrt{\varepsilon}$. Denoting by $\sqrt{\varepsilon} = \mu$, one can write

$$\begin{cases} m\ddot{v} + \mu^2 c v^3 + \gamma(v-u) = 0, \\ M\dot{u} + \mu\gamma(u-v) = D. \end{cases} \tag{39}$$

The limiting system of the zero approximation by μ can be written in the form

$$\begin{cases} m\ddot{v}_0 + \gamma(v_0 - u_0) = 0, \\ M\dot{u}_0 = D. \end{cases}$$

Taking into account the trivial initial values we obtain from here that $u_0 = (D/M)t$.

One now writes an equation to determine v_0: $\ddot{v}_0 + \omega^2 v_0 = (\gamma/m)(D/M)t$ (here $\omega^2 = \gamma/m$) that gives us (taking into account the trivial initial values) that $v_0 = D/M(t - \sin\omega t/\omega)$.

The obtained solution (u_0, v_0) can be easily made more precise by the small parameter method. In this case in the first approximation by μ the term $\mu\gamma(u-v)$ will be considered in the second equation of system (39) too.

5.1.2

One considers the next region: a continuation of the variables increase and damped oscillations ($D = 0,\ \tau \leqslant t \leqslant \tau_1$). Conditions of joining of the first and second regions solutions gives us initial conditions for the second region in a point $t = \tau$. The initial conditions are the following:

$$u_{\mathrm{II}}(\tau) = \frac{D}{M}\tau, \quad v_{\mathrm{II}}(\tau) = \frac{D}{M}\left(\tau - \frac{\sin\omega\tau}{\omega}\right), \quad \dot{v}_{\mathrm{II}}(\tau) = \frac{D}{M}(1 - \cos\omega\tau). \tag{40}$$

Assuming anew that the variables u, v are not large, we introduce the small parameter μ: $u \to \mu u$, $v \to \mu v$. Then the corresponding limiting system can be written as

$$\begin{cases} m\ddot{v} + \mu c v^3 + \gamma\,(v - u) = 0, \\ M\ddot{u} + \gamma(u - v) = 0. \end{cases} \tag{41}$$

One expresses the variable u from the first equation of system (41), $u = (m/\gamma)\ddot{v} + v + \mu(c/\gamma)v^3$, and it is substituted into the second equation. As a result one obtains the next differential equation of the third order by τ:

$$\dddot{v} - 2\alpha\ddot{v} + \omega^2\dot{v} = -\mu\frac{c}{m}[3v^2\dot{v} - 2\alpha v^3],$$

where $\alpha = -\gamma/2M$, $\omega = \sqrt{\gamma/m}$.

A solution of the equation can be expanded in powers of the small parameter μ:

$$v = v_0 + \mu v_1 + \cdots.$$

One has the following equation of the zero approximation by μ:

$$\dddot{v} - 2\alpha\ddot{v}_0 + \omega^2\dot{v}_0 = 0. \tag{42}$$

We can write the characteristic equation for Eq. (42) as

$$k^3 - 2\alpha k^2 + \omega^2 k = 0, \tag{43}$$

from here $k_1 = 0$, $k_{2,3} = \alpha \pm \mathrm{i}\Omega$, where $\Omega = \sqrt{\alpha^2 - \omega^2}$. One now writes a solution of Eq. (42) of the form $v_0 = a_0 + \mathrm{e}^{\alpha t}(B_{01}\sin\Omega t + B_{02}\cos\Omega t)$, and it is possible to determine the coefficients a_0, B_{01}, and B_{02} from the initial conditions (40).

Equation of the first approximation by μ is the following:

$$\dddot{v}_1 - 2\alpha\ddot{v}_1 + \omega^2\dot{v}_1 = -\mu\frac{c}{m}[3v_0^2\dot{v}_0 - 2\alpha v_0^3]. \tag{44}$$

Here the initial conditions for v_1 and its derivatives are trivial.

To solve system (41) (or the corresponding Eqs. (42) and (44)) in two approximations by the small parameter it is not advisable to utilize the multi-scale method [29], which here leads to cumbersome calculations. Here we use the method, which is some generalization of the harmonic balance method.

The solution $v(t)$ is presented of the form

$$v = \tilde{v}_0 + \mu v_1 + \cdots, \tag{45}$$

where $\tilde{v}_0 = a_0\mathrm{e}^{\bar{\omega}_0 t} + \mathrm{e}^{\bar{\omega}_1 t}(B_{01}\sin\bar{\omega}_2 t + B_{02}\cos\bar{\omega}_2 t)$.

One introduces the following:

$$\bar{\omega}_0 = \mu\omega_{01} + \cdots, \quad \bar{\omega}_1 = \alpha + \mu\omega_{11} + \cdots, \quad \bar{\omega}_2 = \Omega + \mu\omega_{21} + \cdots. \tag{46}$$

One substitutes the zero approximation solution v_0 into Eq. (44). To remove the secular terms from the equation the following procedure is proposed. Substituting series (45) into Eq. (44) and linearizing the equation by v_1, we can write the following:

$$\begin{aligned}
&\dddot{v}_1 - 2\alpha\ddot{v}_1 + \omega^2\dot{v}_1 \\
&\quad = -\omega^2\{a_0\bar{\omega}_0 - \mathrm{e}^{\bar{\omega}_1 t}[(\omega_{11}B_{01} - \omega_{21}B_{02})\sin\bar{\omega}_2 t + (\omega_{11}B_{02} + \omega_{21}B_{01})\cos\bar{\omega}_2 t]\} \\
&\qquad + 2\alpha\mathrm{e}^{\bar{\omega}_1 t}\{2(\alpha\omega_{11}B_{01} - (\alpha\omega_{21} + \Omega\omega_{11})B_{02} - \Omega\omega_{21}B_{01})\sin\bar{\omega}_2 t \\
&\qquad + 2(\alpha\omega_{11}B_{02} + (\alpha\omega_{21} + \Omega\omega_{11})B_{01} - \Omega\omega_{21}B_{02})\cos\bar{\omega}_2 t\} \\
&\qquad - \mathrm{e}^{\bar{\omega}_1 t}\{3(\alpha^2\omega_{11}B_{01} - (\alpha^2\omega_{21} + 2\alpha\Omega\omega_{11})B_{02} - (\Omega^2\omega_{11} + 2\alpha\Omega\omega_{21})B_{01} \\
&\qquad + \Omega^2\omega_{21}B_{02})\sin\bar{\omega}_2 t + 3(\alpha^2\omega_{11}B_{02} + (\alpha^2\omega_{21} + 2\alpha\Omega\omega_{11})B_{01} \\
&\qquad - (\Omega^2\omega_{11} + 2\alpha\Omega\omega_{21})B_{02} - \Omega^2\omega_{21}B_{01})\cos\bar{\omega}_2 t\} - \frac{c}{m}[3v_0^2\dot{v}_0 - 2\alpha v_0^3].
\end{aligned}$$

Then, setting $\mu = 0$, we can obtain the next differential equation with respect to v_1:

$$\dddot{v}_1 - 2\alpha\ddot{v}_1 + \omega^2\dot{v}_1 = P_0 + \mathrm{e}^{\alpha t}[P_s\sin\Omega t + P_c\cos\Omega t] - \frac{c}{m}[3v_0^2\dot{v}_0 - 2\alpha v_0^3], \tag{47}$$

where

$$P_0 = -a_0\omega_{01}\omega^2, \quad P_s = 2\Omega((\alpha B_{02} + \Omega B_{01})\omega_{11} + (\alpha B_{01} - \Omega B_{02})\omega_{21}),$$

$$P_c = -2\Omega((\alpha B_{01} - \Omega B_{02})\omega_{11} - (\alpha B_{02} + \Omega B_{01})\omega_{21}).$$

One calculates separately the last summand in Eq. (47):

$$\begin{aligned}
3v_0^2\dot{v}_0 - 2\alpha v_0^3 = &-2\alpha a_0^3 + 3a_0^2\mathrm{e}^{\alpha t}[N_1\sin\Omega t + N_2\cos\Omega t] \\
&+ 3a_0\mathrm{e}^{2\alpha t}[N_3\sin 2\Omega t + N_4\cos 2\Omega t] \\
&+ \mathrm{e}^{3\alpha t}[N_5\sin\Omega t + N_6\cos\Omega t + N_7\sin 3\Omega t + N_8\cos 3\Omega t],
\end{aligned}$$

where $N_1 = -(\alpha B_{01} + \Omega B_{02})$, $N_2 = -(\alpha B_{02} - \Omega B_{01})$, $N_3 = \Omega(B_{01}^2 - B_{02}^2)$, $N_4 = 2\alpha B_{01}B_{02}$,

$$N_5 = \tfrac{3}{4}(B_{01}^2 + B_{02}^2)(\alpha B_{01} - \Omega B_{02}), \quad N_6 = \tfrac{3}{4}(B_{01}^2 + B_{02}^2)(\alpha B_{02} + \Omega B_{01}),$$

$$N_7 = \tfrac{1}{4}(\alpha B_{01}(3B_{02}^2 - B_{01}^2) - 3\Omega B_{02}(B_{02}^2 - 3B_{01}^2)),$$

$$N_8 = \tfrac{1}{4}(\alpha B_{02}(B_{02}^2 - 3B_{01}^2) + 3\Omega B_{01}(3B_{02}^2 - B_{01}^2)).$$

Then Eq. (47) takes the form

$$\begin{aligned}
\dddot{v}_1 - 2\alpha\ddot{v}_1 + \omega^2\dot{v}_1 = P_0 &+ \mathrm{e}^{\alpha t}[P_s\sin\Omega t + P_c\cos\Omega t] \\
&- \frac{c}{m}\{-2\alpha a_0^3 + 3a_0^2\mathrm{e}^{\alpha t}[N_1\sin\Omega t + N_2\cos\Omega t] \\
&+ 3a_0\mathrm{e}^{2\alpha t}[N_3\sin 2\Omega t + N_4\cos 2\Omega t] \\
&+ \mathrm{e}^{3\alpha t}[N_5\sin\Omega t + N_6\cos\Omega t + N_7\sin 3\Omega t + N_8\cos 3\Omega t]\}.
\end{aligned}$$

To take off the secular terms in Eq. (44) solution, we equate the constant term and coefficients at $e^{\alpha t}\sin\Omega t$, $e^{\alpha t}\cos\Omega t$ to zero. One obtains the following equalities:

$$\begin{cases} -a_0\omega^2\omega_{01} + 2\alpha\dfrac{c}{m}a_0^3 = 0, \\ (\alpha B_{02} + \Omega B_{01})\omega_{11} + (\alpha B_{01} - \Omega B_{02})\omega_{21} = \dfrac{3a_0^2 cN_1}{2\Omega m}, \\ (\alpha B_{01} - \Omega B_{02})\omega_{11} - (\alpha B_{02} + \Omega B_{01})\omega_{21} = -\dfrac{3a_0^2 cN_2}{2\Omega m}. \end{cases}$$

Thus, the system of linear algebraic equations with respect to the first approximations of the characteristic values ω_{01}, ω_{11}, ω_{21} is obtained. One has a solution of the system:

$$\omega_{01} = -\frac{ca_0^2}{M}, \quad \omega_{11} = -\frac{3a_0^2 c}{2m\Omega\Delta}(N_2(\alpha B_{01} - \Omega B_{02}) - N_1(\alpha B_{02} + \Omega B_{01})),$$

$$\omega_{21} = \frac{3a_0^2 c}{2m\Omega\Delta}(N_1(\alpha B_{01} - \Omega B_{02}) + N_2(\alpha B_{02} + \Omega B_{01})),$$

where $\Delta = (\alpha B_{01} - \Omega B_{02})^2 + (\alpha B_{02} + \Omega B_{01})^2$.

As a result the general solution of Eq. (44) can be written in the form

$$\begin{aligned} v_1 = a_1 &+ e^{\alpha t}(B_{11}\sin\Omega t + B_{12}\cos\Omega t) + e^{2\alpha t}[G_1\sin 2\Omega t + H_1\cos 2\Omega t] \\ &+ e^{3\alpha t}[G_2\sin\Omega t + H_2\cos\Omega t] + e^{3\alpha t}[G_3\sin 3\Omega t + H_3\cos 3\Omega t]. \end{aligned}$$

Here a_1, B_{11}, B_{12} are arbitrary constants which can be calculated by using the trivial initial values

$$G_1 = -\frac{3a_0 c}{2m\Delta_1}[\alpha(\alpha^2 - 7\Omega^2)N_3 + \Omega(5\alpha^2 - 3\Omega^2)N_4],$$

$$H_1 = -\frac{3a_0 c}{2m\Delta_1}[\alpha(\alpha^2 - 7\Omega^2)N_4 - \Omega(5\alpha^2 - 3\Omega^2)N_3], \quad \Delta_1 = \alpha^2(\alpha^2 - 7\Omega^2)^2 + \Omega^2(5\alpha^2 - 3\Omega^2)^2,$$

$$G_2 = -\frac{c}{4\alpha m\Delta_2}[(3\alpha^2 - \Omega^2)N_5 + 4\alpha\Omega N_6], \quad H_2 = -\frac{c}{4\alpha m\Delta_2}[(3\alpha^2 - \Omega^2)N_6 - 4\alpha\Omega N_5],$$

$$\Delta_2 = (3\alpha^2 - \Omega^2)^2 + 16\alpha^2\Omega^2,$$

$$G_3 = -\frac{c}{12m\Delta_3}[\alpha(\alpha^2 - 5\Omega^2)N_7 + 2\Omega(2\alpha^2 - \Omega^2)N_8],$$

$$H_3 = -\frac{c}{12m\Delta_3}[\alpha(\alpha^2 - 5\Omega^2)N_8 - 2\Omega(2\alpha^2 - \Omega^2)N_7], \quad \Delta_3 = \alpha^2(\alpha^2 - 5\Omega^2)^2 + 4\Omega^2(2\alpha^2 - \Omega^2)^2.$$

As a result, the asymptotic expansions of system (41) solution are obtained in the region of the damped vibrations:

$$v_{\mathrm{II}} = \tilde{v}_0 + \mu v_1 + \cdots,$$

$$u_{\mathrm{II}} = \left(\frac{m}{\gamma}\ddot{\tilde{v}}_0 + \tilde{v}_0\right) + \mu\left(\frac{m}{\gamma}\ddot{v}_1 + v_1 + \frac{c}{\gamma}v_0^3\right) + \cdots.$$

5.1.3

One considers now the third region where oscillations are absent. We denote an initial time of motion at the region by t_1. The value can be determined from the initial conditions for the region.

Here motion is slow, so the inertial term $\varepsilon m\ddot{v}$ can be neglected. Then the limiting system has the next form

$$\begin{cases} cv^3 + \gamma(v - u) = 0, \\ M\dot{u} + \gamma(u - v) = 0. \end{cases} \tag{48}$$

One writes a solution of the system $u = v + (c/\gamma)v^3$, $M/(2cv^2) - (3M/\gamma)\ln|v| = t + \varphi$, where φ is an arbitrary constant. One introduces initial conditions for the third region as

$$u_{\mathrm{II}}(t_1) = u_{\mathrm{III}}(t_1), \quad v_{\mathrm{II}}(t_1) = v_{\mathrm{III}}(t_1).$$

It is possible to determine φ and t_1 from the conditions. The obtained solution can be made more precise in the first approximation by ε.

5.2. Comparison of the analytical and numerical solutions

Numerical calculations which were made for different values of the system parameters show a good accuracy of the analytical solution for a sufficiently small time interval of the external pulse. Fig. 17 shows a comparison of the checking numerical calculations and the asymptotics constructed here for different values of the connection parameter γ and the time interval τ if the next parameter values are fixed at $c = 1$, $D = 1$, $M = 1$, and $\varepsilon m = 0.1$. In this figure the analytical solution is presented by points, and the numerical one by a solid line. Fig. 18 presents a dependence of the maximal kinetic energy on the parameter γ. The dependence shows that the nonlinear oscillator with a relatively small mass, connected to the linear elastic system, can keep up to 20% of the traveling elastic wave energy.

6. Conclusions

In this paper the two-dof system consisting of the linear oscillator with a relatively big mass which is an approximation of some continuous elastic system, and the nonlinear oscillator with a relatively small mass which is an absorber of the main linear system vibrations, is analyzed by using the nonlinear normal modes theory and methods of perturbations. Analysis of free and forced vibrations shows that there are large regions of the system parameters favorable for

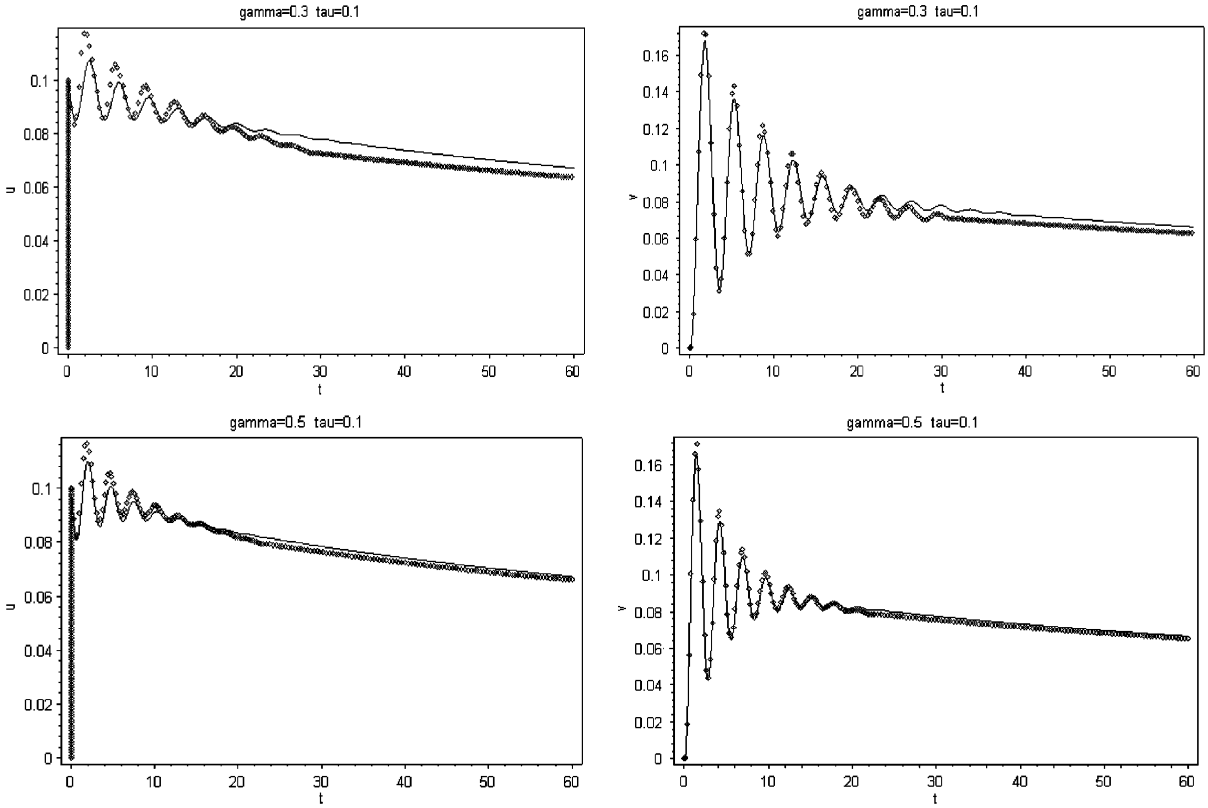

Fig. 17. Comparison of the checking numerical calculations and the asymptotics for system (38) for different values of the connection parameter γ and the impulse time interval τ if the next parameter values are fixed at: $c = 1$, $D = 1$, $M = 1$, $m = 0.1$. The analytical solution is presented by points, and the numerical one by solid line.

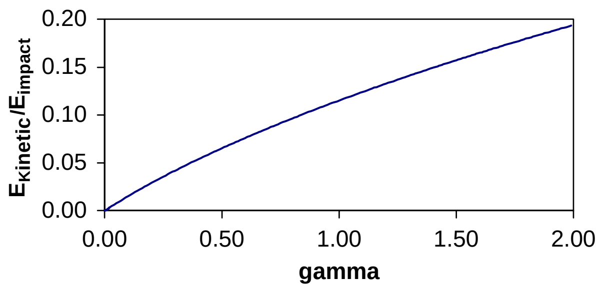

Fig. 18. Dependence of the maximal kinetic energy of the nonlinear oscillator and the parameter γ.

extinguishing elastic vibrations where the non-localized mode is unstable, and the localized mode is stable. In a case when the localized mode, appropriate for an absorption, is realized, the main elastic system and absorber have small and significant amplitudes, respectively.

Acknowledgements

The authors are very grateful to Dr. K.V. Avramov (NTU ''Kharkov Polytechnical Institute''), Prof. L.I. Manevich and Dr. O.V. Gendelman (Chemical Physics Institute, RAS, Moscow) for the useful discussions of the obtained results.

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