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Closed-form solutions for the free longitudinal vibration of inhomogeneous rods

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Abstract

This paper aims at presenting a family of exact solutions for the longitudinal vibration of variable area rods. Area variations that give solutions in terms of the confluent hypergeometric function are being sought for and the governing differential equation is appropriately reduced to the confluent hypergeometric differential equation, using a generic transformation. The eigenfrequencies of rods with certain area variations, subjected to classical boundary conditions, are obtained and the parametric space of the solutions obtained is studied. These solutions are also highly useful in other topics of study such as torsional vibration of rods and wave propagation in ducts with variable cross-sectional areas, since the governing differential equations are very similar.

1. Introduction

The longitudinal vibration of rods is a topic of great interest with several numerical and analytical solutions, approximate and exact, available in the literature. Its importance can be understood from its application in the design of high-rise buildings and towers, high aspect-ratio aircraft wings, machine shafts, etc. Longitudinal and torsional vibrations are significant in these structures and their natural frequencies have to be considered while designing them. Exact solutions and numerical techniques for longitudinal vibration of homogeneous rods can be found in several books on the topic [1,2], but analytical studies on inhomogeneous rods are scarce.

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Inhomogeneity can arise due to variation in cross-sectional area or in density (and thus Young's modulus). Interest in variable area rods was instigated by Eisenberger's study [3] of vibration of rods with polynomial variation in the cross-sectional area. Matsuda et al. [4] studied the eigenfrequencies of a non-uniform bar by transforming the governing equation into a boundary integral equation. Bapat [5] derived exact solutions and studied the mode shapes for the vibration of rods with catenoidal and exponential area variations while Abrate [6] derived and studied the traveling wave solutions for rods with area variations of the form $A_0(1+\alpha x/L)^2$ and beams with area variations of the form $A_0(1+\alpha x)^4$. Kumar and Sujith [7] obtained exact solutions for rods with area variations of the form $(ax+b)^n$ and $A_0\sin^2(ax+b)$. Li [8] carried out a functional transformation of the governing differential equation and then obtained exact solutions for certain functional forms of an involved parameter.

In the present study, the governing differential equation is transformed by assuming a general form for the solution and also assuming a general transformation function for a change of variable. Then, the transformation function is subjected to constraints that would make the resulting differential equation of the confluent hypergeometric type. Then, applying more restrictions on the transformation function, some interesting area variations that give Kummer's hypergeometric function as solution are determined. After that, the obtained solutions are shown to be consistent with existing solutions. In the latter half of the paper, a study of the parametric space of the solutions obtained is carried out.

2. The equation of motion

The governing differential equation for the free longitudinal vibration of a finite, isotropic, variable area rod is given as [9]

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]=\rho A(x)\frac{\partial^2 u}{\partial t^2},\quad \text{on } l_1<x<l_2,\ t\geqslant 0. \tag{1}$$

The classical boundary conditions are:

$$\begin{aligned}&\text{(i)}\quad \text{Fixed}-\text{Fixed bar:}\quad u(l_1,t)=u(l_2,t)=0,\\ &\text{(ii)}\quad \text{Fixed}-\text{Free bar:}\quad u(l_1,t)=\partial u(l_2,t)/\partial x=0,\\ &\text{(iii)}\quad \text{Free}-\text{Free bar:}\quad \partial u(l_1,t)/\partial x=\partial u(l_2,t)/\partial x=0.\end{aligned} \tag{2}$$

Equating the external forces, acting on an infinitesimal element of a vibrating rod, to the inertial force of the element (see Fig. 1), one can derive these equations. In the above equations, $u(x,t)$ represents the longitudinal displacement of a rod section at a time instant t, $A(x)$ the cross-sectional area of the rod, E the Young's modulus of the material, ρ the density of the material and l_1 and l_2 the end coordinates of the rod.

Assuming the displacement function to be varying harmonically with time; i.e., a solution of the form $u(x,t)=W(x)e^{i\omega t}$ gives

$$\frac{d^2W}{dx^2}+\left(\frac{1}{A}\frac{dA}{dx}\right)\left[\frac{dW}{dx}\right]+\Omega^2 W=0, \tag{3}$$

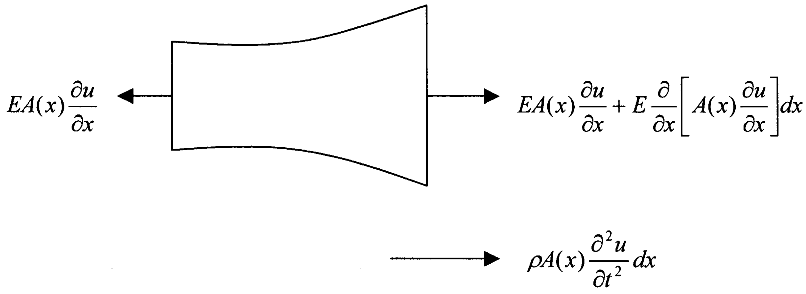

Fig. 1. An infinitesimal element of a non-uniform rod.

where $W(x)$ represents the mode shape and the frequency $\Omega = \omega\sqrt{(\rho/E)}$, ω being the angular frequency. This equation has variable coefficients and so exact solutions cannot be found for a general area variation. However, we can find exact solutions for certain specified area variations. It is also possible to find area variations that give solutions in terms of special functions such as Bessel functions or hypergeometric functions.

3. Solution in terms of the confluent hypergeometric function

In this section, a general procedure is developed for obtaining solutions in terms of Kummer's hypergeometric function. Then assuming some simple transformation functions, area variations that give rise to the required solution are obtained. Assuming a functional variation for $W(x)$ of the form

$$W(x) = x^b e^{g(x)} f(x). \tag{4}$$

On substituting Eq. (4) into Eq. (3) and simplifying, Eq. (3) becomes

$$f''(x) + \left[2g'(x) + \frac{2b}{x} + \frac{A'}{A}\right] f'(x) + \eta(x) f(x) = 0, \tag{5}$$

where

$$\eta(x) = \frac{b(b-1)}{x^2} + \frac{2bg'(x)}{x} + g'^2(x) + g''(x) + \left(\frac{b}{x}\right)\frac{A'}{A} + g'(x)\frac{A'}{A} + \Omega^2. \tag{6}$$

Here, primes denote differentiation with respect to x. Now, assuming a general transformation function

$$s = h(x) \tag{7}$$

gives

$$f'(x) = h'(x)\,\frac{\mathrm{d}f(s)}{\mathrm{d}s} \tag{8}$$

and

$$f''(x) = h'^2(x)\,\frac{d^2f(s)}{ds^2} + h''(x)\,\frac{df(s)}{ds}. \tag{9}$$

In order to convert the standard differential equation

$$f''(x) + r(x)f'(x) + t(x)f(x) = 0 \tag{10}$$

to the confluent hypergeometric differential equation

$$s\,\frac{d^2f(s)}{ds^2} + (c-s)\,\frac{df(s)}{ds} - af(s) = 0 \tag{11}$$

the transformation function must satisfy the condition

$$h'^2(x) = \varphi(x)h(x), \tag{12}$$

where $\varphi(x)$ is any arbitrary function of x. Eq. (12) on integrating gives

$$h(x) = \frac{\left[\int(\sqrt{\varphi(x)})\,dx\right]^2}{4}. \tag{13}$$

Also, differentiating Eq. (12) gives

$$h''(x) = \frac{\varphi'(x)h(x)}{2h'(x)} + \frac{\varphi(x)}{2}. \tag{14}$$

Substituting Eqs. (8), (9), (12) and (14) into Eq. (5) gives

$$s\,\frac{d^2f(s)}{ds^2} + \left[\frac{1}{2} + \left\{\frac{\varphi'}{2\varphi} + 2g' + \frac{2b}{x} + \frac{A'}{A}\right\}\sqrt{\frac{s}{\varphi}}\right]\frac{df(s)}{ds} + \frac{\eta(x)}{\varphi(x)}f(s) = 0. \tag{15}$$

In order to obtain the confluent hypergeometric differential equation, the conditions to be satisfied are

$$\frac{1}{2} + \left\{\frac{\varphi'}{2\varphi} + 2g' + \frac{2b}{x} + \frac{A'}{A}\right\}\sqrt{\frac{s}{\varphi}} = c - s \tag{16}$$

and

$$\eta(x)/\varphi(x) = -a. \tag{17}$$

Now on assuming simple functional forms for $\varphi(x)$, a differential equation in $g(x)$ is obtained, by using Eqs. (16) and (17), whose solution can often be easily obtained by inspection. In this paper, two simple forms for $\varphi(x)$, which can be easily integrated, are considered for analysis.

3.1. Area variations given by $\varphi(x) = qx^{2m}$

Assuming

$$\varphi(x) = qx^{2m}. \tag{18}$$

Thus,

$$h(x) = \frac{qx^{2m+2}}{4(m+1)^2}. \tag{18a}$$

Substituting Eq. (18) into Eq. (16) and simplifying gives

$$\frac{A'}{A} + 2g' + \frac{(2b+2m+1)}{x} - \frac{2c(m+1)}{x} + \frac{qx^{2m+1}}{2m+2} = 0. \tag{19}$$

Substituting Eq. (18) into Eq. (17) and replacing A'/A using (19) gives

$$g''(x) - g'^2(x) - \Psi(x)g'(x) + \frac{\theta}{x^2} + \left[aq - \frac{bq}{2m+2}\right]x^{2m} + \Omega^2 = 0, \tag{20}$$

where

$$\Psi(x) = \frac{(2b+2m+1)}{x} - \frac{2c(m+1)}{x} + \frac{qx^{2m+1}}{2m+2}, \tag{21}$$

$$\theta = 2bmc + 2bc - 2bm - b^2 - 2b. \tag{22}$$

On inspecting Eq. (20), it is found that $g'(x)$ can only be a sum of powers of x. Substituting $g'(x) = x^i$ into Eq. (20) indicates that i can take values $0, -1,\ 2m+1$. Thus, $g'(x)$ can be assumed to be of the form

$$g'(x) = D + \frac{B}{x} + Cx^{2m+1}. \tag{23}$$

Substituting Eq. (23) into Eq. (20) gives

$$[\Omega^2 - D^2] + \frac{P}{x} + \frac{Q}{x^2} + Rx^{2m} + \left[-2DC - \frac{Dq}{2m+2}\right]x^{2m+1} + \left[-C^2 - \frac{Cq}{2m+2}\right]x^{4m+2} = 0, \tag{24}$$

where

$$P = 2Dc(m+1) - 2DB - (2b+2m+1)D, \tag{25}$$

$$Q = -B^2 - (2b+2m+2-2mc-2c)B - (b^2+2b+2bm-2bmc-2bc), \tag{26}$$

$$R = -2BC - (2b-2mc-2c)C - \frac{(B+b)q}{2m+2} + aq. \tag{27}$$

Only those values of m that keep Ω non-zero are relevant in this problem. The two values of m that satisfy this criterion are $m = 0$ and $-1/2$.

Case I: When $m = 0$. Substituting the above condition into Eq. (24) and equating the coefficients of like powers of x to zero gives

$$D = 0 \quad \text{and} \quad C = 0 \quad \text{or} \quad C = -q/2,$$
$$B + b = 0 \quad \text{or} \quad B + b = 2c - 2.$$

Taking any pair of conditions gives rise to one of the four pairs of linearly independent solutions. Here, for simplicity, the conditions $C = 0$ and $B + b = 0$ are taken. This gives

$$a = -\Omega^2/q. \tag{28}$$

On substituting the values into Eq. (19) and integrating, the suitable area variation is found to be

$$A = kx^{2c-1} \exp\left[\frac{-qx^2}{4}\right]. \tag{29}$$

Therefore, when a rod has a cross-sectional area variation of the form

$$A = kx^n \exp[bx^2] \tag{29a}$$

the solution of Eq. (2) is given as

$$W(x) = c_1 M\left(\frac{\Omega^2}{4b}, \frac{n+1}{2}, -bx^2\right) + c_2 U\left(\frac{\Omega^2}{4b}, \frac{n+1}{2}, -bx^2\right), \tag{30}$$

where $M(a, c, x)$ is the Kummer's hypergeometric function.

Case II: When $m = -1/2$. Once again substituting into Eq. (24) and equating the coefficients of like powers of x to zero gives

$$(D + C)(2b - c + 2B) + q(B + b - a) = 0, \tag{31}$$

$$(B + b) = 0 \quad \text{or} \quad B + b = c - 1, \tag{32}$$

$$(D + C)^2 + (D + C)q - \Omega^2 = 0. \tag{33}$$

The solutions given by either of the conditions (32) are equivalent. For the sake of convenience, the condition $B + b = 0$ is chosen.

$$D + C = -\frac{aq}{c}, \tag{34}$$

$$\left(\frac{a}{c}\right)^2 - \left(\frac{a}{c}\right) - \left(\frac{\Omega}{q}\right)^2 = 0. \tag{35}$$

Substituting these values into Eq. (19), the appropriate area variation is found to be

$$A = kx^n \exp\left[q\left(\frac{2a}{n} - 1\right)x\right]. \tag{36}$$

Thus it is found that an area variation of the form

$$A = kx^n \exp[bx] \tag{36a}$$

gives the solution of Eq. (2) as

$$W(x) = \exp\left(\frac{-b - v}{2}\, x\right)\left[c_1 M\left(\left(\frac{b}{v} + 1\right)\frac{n}{2}, n, vx\right) + c_2 U\left(\left(\frac{b}{v} + 1\right)\frac{n}{2}, n, vx\right)\right],$$

where

$$v = \sqrt{b^2 - 4\Omega^2}. \tag{37}$$

The general solutions obtained above can be shown to be consistent with existing special solutions for vibration of rods with uniform, exponential and polynomial area variations.

In Eq. (29a), as b approaches 0, the area variation becomes polynomial in nature; similar to the area variation studied by Kumar and Sujith [7] (with $a = 1$ and $b = 0$). Therefore, substituting lim $b \to 0$ into Eq. (30) and using the following identities [10],

$$\lim_{\alpha\to\infty} M\left(\alpha, \beta, -\frac{z}{\alpha}\right) = \Gamma(\beta) z^{(1-\beta)/2} J_{\beta-1}(2\sqrt{z})$$

and

$$\lim_{\alpha\to\infty}\left\{\Gamma(1+\alpha-\beta) U\left(\alpha, \beta, -\frac{z}{\alpha}\right)\right\} = 2z^{(1-\beta)/2} Y_{\beta-1}(2\sqrt{z}) \tag{38}$$

gives

$$U_1(x) = \left(\frac{\Omega x}{2}\right)^{(1-n)/2}\left[C_1 J_{(n-1)/2}(\Omega x) + C_2 Y_{(n-1)/2}(\Omega x)\right] \tag{39}$$

which is the same as the solution of Kumar and Sujith [7].

Again, taking the limit as b approaches 0 in Eq. (37) and using the identities [11]

$$\begin{aligned} J_p(x) &= \frac{(x/2)^p}{\Gamma(p+1)} \mathrm{e}^{-\mathrm{i}x} M\left(p+\tfrac{1}{2}; 2p+1; 2\mathrm{i}x\right), \\ Y_p(x) &= \frac{(x/2)^p}{\Gamma(p+1)} \mathrm{e}^{-\mathrm{i}x} U\left(p+\tfrac{1}{2}; 2p+1; 2\mathrm{i}x\right). \end{aligned} \tag{40}$$

the Bessel functional forms as in Ref. [7] are once again obtained.

Also, substituting lim $b \to 0$ & lim $n \to 0$ in both Eqs. (30) and (37), the exponential solutions corresponding to that of the homogeneous rod can be readily obtained. Similarly, substituting lim $n \to 0$ into Eq. (37) easily gives the solution for the inhomogeneous rod with exponential area variation [12].

3.1.1. Natural frequencies of variable area rods—numerical examples and analysis

In this section, the variation of the natural frequencies of vibration, of rods with the above area variations, with respect to the involved parameters is analyzed. The natural frequencies are obtained for the classical boundary conditions given in Section 2. In all cases, $l_1 = 0.1$ units and $l_2 = 1.1$ units.

Consider a fixed–fixed rod with area variation $A = kx^n \exp(bx^2)$. Applying the appropriate boundary conditions, a pair of homogeneous, simultaneous equations is obtained:

$$c_1 M\left(\frac{\Omega^2}{4b}; \frac{n+1}{2}; -0.01b\right) + c_2 U\left(\frac{\Omega^2}{4b}; \frac{n+1}{2}; -0.01b\right) = 0, \tag{41}$$

$$c_1 M\left(\frac{\Omega^2}{4b}; \frac{n+1}{2}; -1.21b\right) + c_2 U\left(\frac{\Omega^2}{4b}; \frac{n+1}{2}; -1.21b\right) = 0. \tag{42}$$

These equations will have a non-trivial solution only when their determinant vanishes, thus giving the transcendental equation for the eigenvalues Ω as

$$M\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-0.01b\right)U\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-1.21b\right) - U\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-0.01b\right)M\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-1.21b\right) = 0. \quad (43)$$

The eigenvalues Ω that satisfy the above equation for different parametric values are given in Tables 1 and 2. A close look at these two tables will show that for a fixed value of b, the eigenvalues first decrease with increasing n and then increase. Moreover, the value of n at which the eigen frequency is minimum decreases with increase in b. Also, it can be seen that, unlike in the homogeneous case, the natural frequencies are no longer integral multiples of the fundamental frequency.

For a fixed–free rod with the above area variation and appropriate boundary conditions, the resultant equation is

$$\frac{n+1}{2}\,M\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-0.01b\right)U\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-1.21b\right) + U\left(\frac{\Omega^2}{4b};\frac{n+1}{2};-0.01b\right)M\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-1.21b\right) = 0. \quad (44)$$

The eigenvalues that satisfy the above transcendental equation are given in Table 3. In this case, however, the eigenvalues are found to be monotonically decreasing with increasing n for all values of b.

Table 1
Fundamental frequency for fixed–fixed rods with $A = kx^n \exp(bx^2)$

n	b							
	−3	−2	−1	−0.5	0.5	1	2	3
−2.5	5.286	4.914	4.608	4.486	4.316	4.270	4.260	4.353
−2.0	4.885	4.546	4.280	4.180	4.058	4.038	4.082	4.226
−1.5	4.471	4.170	3.949	3.875	3.810	3.820	3.923	4.122
−0.5	3.622	3.420	3.317	3.309	3.384	3.464	3.701	4.015
0.0	3.206	3.068	3.043	3.077	3.235	3.356	3.663	4.034
0.5	2.817	2.759	2.825	2.906	3.154	3.314	3.689	4.115
1.0	2.479	2.518	2.686	2.816	3.152	3.349	3.787	4.260
1.5	2.221	2.369	2.642	2.818	3.232	3.462	3.951	4.465
2.0	2.068	2.327	2.695	2.910	3.386	3.642	4.173	4.720
2.5	2.030	2.386	2.829	3.074	3.598	3.873	4.438	5.012
3.0	2.094	2.525	3.023	3.290	3.850	4.140	4.731	5.328

Table 2
First overtone frequency for fixed–fixed rods with $A = kx^n \exp(bx^2)$

n	b							
	−3	−2	−1	−0.5	0.5	1	2	3
−2.5	7.939	7.700	7.518	7.445	7.345	7.310	7.302	7.350
−2.0	7.580	7.367	7.209	7.151	7.081	7.069	7.090	7.173
−1.5	7.234	7.046	6.917	6.876	6.839	6.845	6.903	7.023
−0.5	6.612	6.484	6.422	6.417	6.458	6.501	6.636	6.835
0.0	6.355	6.262	6.238	6.252	6.331	6.397	6.573	6.811
0.5	6.147	6.090	6.107	6.142	6.263	6.349	6.564	6.839
1.0	5.992	5.976	6.034	6.090	6.252	6.357	6.612	6.921
1.5	5.897	5.924	6.024	6.100	6.302	6.433	6.715	7.055
2.0	5.864	5.934	6.075	6.171	6.404	6.552	6.871	7.238
2.5	5.892	6.003	6.184	6.298	6.572	6.730	7.074	7.464
3.0	5.976	6.126	6.344	6.475	6.773	6.936	7.317	7.727

Table 3
Fundamental frequency for fixed–free rods with $A = kx^n \exp(bx^2)$

n	b							
	−3	−2	−1	−0.5	0.5	1	2	3
−2.5	4.884	4.310	3.738	3.455	2.904	2.638	2.133	1.674
−2.0	4.494	3.946	3.399	3.129	2.603	2.350	1.874	1.447
−1.5	4.089	3.570	3.051	2.795	2.298	2.061	1.617	1.226
−0.5	3.233	2.782	2.330	2.109	1.684	1.485	1.120	0.812
0.0	2.791	2.377	1.966	1.765	1.384	1.207	0.891	0.630
0.5	2.350	1.978	1.610	1.432	1.100	0.948	0.682	0.470
1.0	1.924	1.594	1.274	1.121	0.840	0.715	0.501	0.338
1.5	1.525	1.241	0.970	0.844	0.616	0.518	0.354	0.233
2.0	1.169	0.931	0.710	0.610	0.435	0.359	0.241	0.156
2.5	0.864	0.672	0.500	0.424	0.295	0.242	0.159	0.101
3.0	0.617	0.468	0.341	0.285	0.194	0.158	0.101	0.064

For a free–free rod of the same area variation, the appropriate equation is

$$\begin{aligned} &M\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-0.01b\right)U\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-1.21b\right) \\ &\quad - U\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-0.01b\right)M\left(\frac{\Omega^2}{4b}+1;\frac{n+3}{2};-1.21b\right)=0. \end{aligned} \tag{45}$$

The natural frequencies for this case are given in Table 4. In this case, similar patterns as in the case of fixed–fixed rods are observed.

Table 4
Fundamental frequency for free–free rods with $A = kx^n \exp(bx^2)$

n	b							
	−3	−2	−1	−0.5	0.5	1	2	3
−2.5	5.012	4.438	3.873	3.600	3.074	2.829	2.386	2.030
−2.0	4.720	4.173	3.642	3.386	2.910	2.695	2.327	2.068
−1.5	4.465	3.951	3.461	3.232	2.819	2.642	2.369	2.221
−0.5	4.115	3.689	3.314	3.154	2.906	2.825	2.759	2.817
0.0	4.034	3.663	3.356	3.235	3.077	3.043	3.068	3.206
0.5	4.015	3.700	3.464	3.384	3.309	3.317	3.420	3.622
1.0	4.048	3.792	3.624	3.581	3.581	3.624	3.792	4.048
1.5	4.122	3.923	3.820	3.810	3.875	3.949	4.170	4.471
2.0	4.226	4.082	4.038	4.047	4.179	4.279	4.546	4.885
2.5	4.353	4.260	4.270	4.316	4.489	4.608	4.914	5.286
3.0	4.495	4.450	4.509	4.578	4.792	4.933	5.273	5.675

Now, consider a fixed–fixed rod with area variation $A = kx^n e^{bx}$. Applying the appropriate boundary conditions, another pair of homogeneous simultaneous equations is obtained:

$$c_1 M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) + c_2 U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) = 0, \tag{46}$$

$$c_1 M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right) + c_2 U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right) = 0, \tag{47}$$

where $v = \sqrt{b^2 - 4\Omega^2}$.

On making the determinant of these homogeneous equations vanish, the transcendental equation for the natural frequencies is obtained as:

$$\begin{aligned} &M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right) \\ &\quad - U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right) = 0. \end{aligned} \tag{48}$$

Solution of the above equation for the natural frequencies Ω for different values of the parameters is given in Tables 5–7.

A study of the Tables 5–7 illustrates that the eigenvalues first decrease with increase in n for a given b and then increase. In addition, the value of n corresponding to minimum eigenfrequency decreases with increase in b. Also, in all cases, higher mode frequencies are not integral multiples of the fundamental frequency.

For a fixed–free rod with the above area variation and appropriate boundary conditions, the resultant equation is

$$F(v) M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) + G(v) U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right) = 0,$$

Table 5
Fundamental frequency for fixed–fixed rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	5.238	4.923	4.637	4.388	4.181	4.024	3.921
−1.5	4.551	4.267	4.023	3.828	3.690	3.614	3.605
−0.5	3.835	3.606	3.436	3.331	3.300	3.345	3.460
0.5	3.149	3.025	2.979	3.016	3.134	3.323	3.572
1.5	2.645	2.686	2.814	3.016	3.280	3.593	3.941
2.5	2.509	2.729	3.008	3.331	3.689	4.071	4.473

Table 6
First overtone frequency for fixed–fixed rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	8.014	7.780	7.570	7.387	7.234	7.113	7.023
−1.5	7.335	7.143	6.980	6.850	6.754	6.694	6.670
−0.5	6.720	6.586	6.488	6.428	6.406	6.423	6.478
0.5	6.245	6.187	6.170	6.193	6.256	6.359	6.498
1.5	5.982	6.012	6.083	6.193	6.342	6.525	6.740
2.5	5.965	6.083	6.237	6.428	6.650	6.900	7.176

Table 7
Second overtone frequency for fixed–fixed rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	10.858	10.675	10.511	10.368	10.248	10.150	10.077
−1.5	10.249	10.105	9.985	9.888	9.816	9.769	9.748
−0.5	9.743	9.649	9.581	9.539	9.523	9.532	9.568
0.5	9.386	9.349	9.339	9.355	9.398	9.468	9.563
1.5	9.205	9.228	9.278	9.355	9.458	9.586	9.738
2.5	9.211	9.295	9.404	9.539	9.697	9.879	10.080

where

$$F(v) = \left[nU\left(\left(\frac{b}{v} + 1 \right) \frac{n}{2} + 1; n + 1; 1.1v \right) + U\left(\left(\frac{b}{v} + 1 \right) \frac{n}{2}; n; 1.1v \right) \right],$$
$$G(v) = \left[M\left(\left(\frac{b}{v} + 1 \right) \frac{n}{2} + 1; n + 1; 1.1v \right) - M\left(\left(\frac{b}{v} + 1 \right) \frac{n}{2}; n; 1.1v \right) \right]. \tag{49}$$

Table 8
Fundamental frequency for fixed–free rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	4.415	3.992	3.578	3.176	2.789	2.417	2.065
−1.5	3.742	3.328	2.928	2.543	2.177	1.832	—
−0.5	3.020	2.625	2.248	1.893	1.563	1.263	—
0.5	2.256	1.898	1.564	1.261	—	—	—
1.5	—	1.209	0.949	0.725	—	—	—
2.5	—	—	—	0.356	—	—	—

Table 9
First overtone frequency for fixed–free rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	6.960	6.644	6.348	6.073	5.824	5.604	5.416
−1.5	6.239	5.954	5.689	5.458	5.259	5.094	4.973
−0.5	5.541	5.301	5.093	4.922	4.794	4.714	4.687
0.5	4.942	4.769	4.639	4.558	4.530	4.559	4.648
1.5	4.534	4.453	4.425	4.455	4.544	4.691	4.891
2.5	4.387	4.413	4.498	4.639	4.832	5.072	5.351

The eigenvalues for the above equation are given in Tables 8 and 9. In this case, the fundamental mode is found to decrease with increasing n and disappears altogether for certain values of the parameters n and b. A similar pattern of disappearing modes was observed by Kumar and Sujith [7]. For a free–free rod of the same case, the appropriate equation is

$$J(v)N(v) - K(v)L(v) = 0,$$

where

$$\begin{aligned}
J(v) &= M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}+1; n+1; 0.1v\right) - M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right),\\
K(v) &= nU\left(\left(\frac{b}{v}+1\right)\frac{n}{2}+1; n+1; 0.1v\right) + U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 0.1v\right),\\
L(v) &= M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}+1; n+1; 1.1v\right) - M\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right),\\
N(v) &= nU\left(\left(\frac{b}{v}+1\right)\frac{n}{2}+1; n+1; 1.1v\right) + U\left(\left(\frac{b}{v}+1\right)\frac{n}{2}; n; 1.1v\right).
\end{aligned} \tag{50}$$

Table 10
Fundamental frequency for free–free rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	4.473	4.071	3.689	3.331	3.008	2.729	2.510
−1.5	3.941	3.593	3.281	3.016	2.814	2.686	2.644
−0.5	3.572	3.323	3.134	3.016	2.979	3.025	3.149
0.5	3.461	3.344	3.300	3.331	3.436	3.606	3.835
1.5	3.604	3.614	3.690	3.828	4.023	4.266	4.551
2.5	3.921	4.022	4.180	4.388	4.637	4.923	5.239

Table 11
First overtone frequency for fixed–free rods with $A = kx^n e^{bx}$

n	b						
	−3	−2	−1	0	1	2	3
−2.5	7.176	6.900	6.650	6.428	6.237	6.081	5.965
−1.5	6.740	6.525	6.342	6.193	6.081	6.012	5.982
−0.5	6.498	6.358	6.256	6.193	6.17	6.188	6.245
0.5	6.478	6.422	6.406	6.428	6.488	6.586	6.720
1.5	6.668	6.694	6.754	6.85	6.981	7.143	7.336
2.5	7.022	7.112	7.238	7.387	7.570	7.779	8.014

The eigenvalues for the above example are given in Tables 10 and 11. In this case, too, patterns, similar to that of corresponding fixed–fixed rods, are observed.

The governing differential equation was also numerically integrated, using a Fourth-Order Runge–Kutta Scheme with adaptive step sizing, for all cases considered above, and the eigenfrequencies obtained were verified.

3.2. Area variations specified by $\varphi(x) = qx^{2p-2}\exp(mx^p)$

In this section, another easily integrable form of $\varphi(x)$ is chosen as follows:

$$\varphi(x) = qx^{2p-2}\exp(mx^p). \tag{51}$$

This gives

$$h(x) = \frac{q}{(mp)^2}\exp(mx^p). \tag{51a}$$

Substituting Eq. (51) into Eq. (16) gives

$$\frac{A'(x)}{A(x)} + 2g'(x) + \frac{2b+p-1}{x} + (1-c)mpx^{p-1} + \frac{q}{mp}\,x^{p-1}\,\exp(mx^p) = 0. \tag{52}$$

Substituting Eq. (51) into Eq. (17) and using A'/A from (52) gives

$$g'' - g'^2 - \Phi g' - \frac{b(b+p)}{x^2} + b(c-1)mpx^{p-2} - \frac{bq}{mp}\,x^{p-2}\,\exp(mx^p) + aqx^{2p-2}\,\exp(mx^p) + \Omega^2 = 0,$$

where

$$\Phi = \frac{2b+p-1}{x} + \frac{q}{mp}\,x^{p-1}\,\exp(mx^p) - (c-1)mpx^{p-1}. \tag{53}$$

Upon inspection, it is found that $g'(x)$ can, once again, only contain terms that are powers of x. Substitution $g'(x) = x^l$ into Eq. (53) indicates that the only values that l can take are -1 and $p-1$.

Thus, assuming a functional form for $g'(x)$ as

$$g'(x) = \frac{B}{x} + Dx^{p-1} \tag{54}$$

and substituting into Eq. (53) gives

$$\frac{P}{x^2} + Qx^{p-2} + [-D^2 + D(c-1)mp]x^{2p-2} - \left[\frac{(B+b)q}{mp}\right]x^{p-2}\,\exp(mx^p) + \left[\left(a - \frac{D}{mp}\right)q\right]x^{2p-2}\,\exp(mx^p) + \Omega^2 = 0,$$

where

$$P = -(B+b)(B+a+p),$$
$$Q = (B+b)(-2D+mcp-mp). \tag{55}$$

The only possible values of p that maintain Ω non-zero are $p=1$ and 2, as obtained by examining the above equation. Again, substitution of $p=2$ into the above equation and equating the coefficients of like powers of x to zero gives $\Omega = 0$. Thus, the only possibility left is $p=1$. Substituting this value into Eq. (55) and equating the coefficients of like powers of x to zero gives

$$B + b = 0 \quad \text{and} \quad a = D/m,$$
$$D^2 - Dm(c-1) - \Omega^2 = 0.$$

Substituting the above values into Eq. (52) and integrating gives

$$A = k\,\exp((c-2a-1)mx)\,\exp\left(-\frac{q}{m^2}\,\exp(mx)\right). \tag{56}$$

Thus, an area variation of the form

$$A = k\,\exp(bx)\,\exp(n\,\exp(mx)) \tag{56a}$$

gives the solution of Eq. (2) as

$$W(x) = \mathrm{e}^{(v-b)x/2}\left[c_1 M\left(\frac{v-b}{2m}; 1+\frac{v}{2m}; -n\mathrm{e}^{mx}\right) + c_2 U\left(\frac{v-b}{2m}; 1+\frac{v}{2m}; -n\mathrm{e}^{mx}\right)\right],$$
$$\text{where } v = \sqrt{b^2 - 4\Omega^2}. \tag{57}$$

4. Discussion

Since the paper mainly deals with a mathematical technique for obtaining special solutions, the ideas developed here can also be used to examine other mathematically identical problems. Other problems governed by a similar differential equation

$$\frac{\partial}{\partial u_1}\left[X(u_1)\frac{\partial y}{\partial u_1}\right] = Y(u_1)\frac{\partial^2 y}{\partial t^2} \tag{58}$$

are

	$X(u_1)$	$Y(u_1)$	y	u_1
(a) Torsional vibration of rods	$J(x)$	$\frac{\rho J(x)}{G}$	$\theta(x)$	x
(b) Vibration of strings	1	$\lambda m(x)$	$u(x)$	x
(c) Vibration of membranes	1	$\lambda\rho(r)$	$u(r)$	r
(d) Wave propagation in ducts	$A(x)$	$\frac{A(x)}{c^2}$	$p(x)$	x

Thus, the solutions developed in this study could be used for other problems as well [13].

5. Conclusions

In this study, a general analytical technique that helps in obtaining cross-sectional area variations, which gives specific, closed-form solutions for the longitudinal vibration of rods, has been derived. The technique involves assuming a general functional form for the displacement and a generic transformation function and then placing restrictions on the transformation function to obtain solutions of a particular kind. Some area variations that give the solution to the problem in terms of Kummer's hypergeometric function have been obtained. Expressions have been obtained to calculate the natural frequencies of rods, subjected to classical boundary conditions, and the parametric space of the solutions obtained has been studied. The variation of the fundamental and first overtone frequencies with the parameter values has been listed.

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References

[1] P.M. Morse, K.U. Ingard, *Theoretical Acoustics*, Princeton University Press, Princeton, NJ, 1987.
[2] L. Meirovitch, *Elements of Vibration Analysis*, McGraw-Hill, New York, 1975.
[3] M. Eisenberger, Exact longitudinal vibration frequencies of a variable cross-section rod, *Applied Acoustics* 34 (1991) 123–130.
[4] H. Matsuda, T. Sakiyama, C. Morita, M. Kawakami, Longitudinal impulsive response analysis of variable cross-section bars, *Journal of Sound and Vibration* 181 (1995) 541–551.
[5] C.N. Bapat, Vibration of rods with uniformly tapered sections, *Journal of Sound and Vibration* 185 (1995) 185–189.
[6] S. Abrate, Vibration of non-uniform rods and beams, *Journal of Sound and Vibration* 185 (1995) 703–716.
[7] B.M. Kumar, R.I. Sujith, Exact solutions for the longitudinal vibration of non-uniform rods, *Journal of Sound and Vibration* 207 (5) (1997) 721–729.
[8] Q.S. Li, Exact solutions for free longitudinal vibration of non-uniform rods, *Journal of Sound and Vibration* 234 (1) (2000) 1–19.
[9] L. Meirovitch, *Analytical Methods in Vibrations*, MacMillan, New York, 1967.
[10] M. Abramovitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970.
[11] L.C. Andrews, *Special Functions of Mathematics for Engineers*, Oxford University Press, Oxford, 1998.
[12] J.B. Seaborn, *Hypergeometric Functions and their Applications*, Springer, New York, 1991.
[13] C.O. Horgan, A.M. Chan, Vibration of inhomogeneous strings, rods and membranes, *Journal of Sound and Vibration* 225 (3) (1999) 503–513.