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Short Communication

# Portal frame inertia and stiffness matrices by substructure synthesis

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**Abstract**

Generic expressions of mass and stiffness matrices of the portal frame are presented. These are derived by means of the substructure synthesis method. This method is exceptionally characterised by low-order eigenvalue problems and highly accurate eigensolutions.


## 1. Introduction

The symmetrical clamped portal frame is the basic framed or civil engineering structure. Thus, its dynamic characteristics have drawn attention since the early 20s (see Ref. [1] for an early survey); nevertheless, solid results were obtained later [2,3]. The determinant of eigenfunction coefficients and the dynamic stiffness methods [4] can be regarded as the first analytical procedures; several versions of the second method, however, were employed throughout the mid-century to obtain the frame's natural frequencies and modes: the receptance method [3], one based on the reciprocal theorem [5], another by Rieger and McCallion [6] and the force method [7]. Of course, the conventional finite element method (FEM) is the next method to consider and that has been used to solve the problem [8].

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**Nomenclature**

| | |
|---|---|
| $EI_b$ | beam flexural rigidity |
| $EI_c$ | columns flexural rigidity |
| $\mathbf{K}$ | stiffness matrix |
| $L_b$ | beam length |
| $L_c$ | columns length |
| $m_b$ | beam mass per unit length |
| $m_c$ | columns mass per unit length |
| $\mathbf{M}$ | inertia matrix |
| $\mathbf{q}$ | eigenvector |
| $\beta_n$ | Euler–Bernoulli beam characteristic betas |
| $\sigma_n$ | Euler–Bernoulli beam characteristic sigmas |
| $\omega_n$ | natural frequency |

The modern approaches to dynamic analysis of framed structures are based on the ideas of reducing the computational effort of the FEM [9–12] and enhancing the dynamic stiffness method which deals with highly irregular frequency functions [13,14]. For instance, it can be observed in one of these references that the FEM may not be the most indicated one for simple structures as the portal frame [10]; in fact, by means of the substructure synthesis method (SSM) [15], it is demonstrated in that work that the convergence characteristics of this method are superior to the ones associated with the FEM for the simple frame. Of course, faster convergence is synonymous with lower order system matrices or eigenvalue problems. Therefore, the structural dynamics question of how to obtain an accurate model with as few degrees of freedom as possible can be answered by means of the SSM in the one-portal frame case.

In this technical communication, generic expressions of those low-order SSM stiffness and mass matrices of the portal frame are presented. These matrices will permit easy and precise computation of natural frequencies and mode shapes of any symmetrical portal frame; furthermore, this accurate and low-order model can be utilised advantageously in additional analyses such as dynamic response, stability, active control and model updating.

## 2. SSM inertia and stiffness matrices

The analysed symmetrical portal frame is shown in Fig. 1. It is understood that both columns share the same flexural rigidity $EI_c$, mass per unit length $m_c$ and length $L_c$, which may differ from the corresponding properties of the beam: $EI_b$, $m_b$ and $L_b$; subscripts $c$ and $b$ stand for column and beam, respectively. Furthermore, the usual engineering assumption of slenderness is considered to neglect the effects of shear deformation, rotatory inertia and axial motion.

The application of the SSM to multiply supported structures, or to the one-portal frame for that matter, has been presented in a previous work [10]; in this short work only generic expressions, which are of course original, of the system matrices are presented along with the necessary definitions directly related to the matrices; thus, for the theoretical aspects of the SSM, readers are referred to that previous work and another [11].

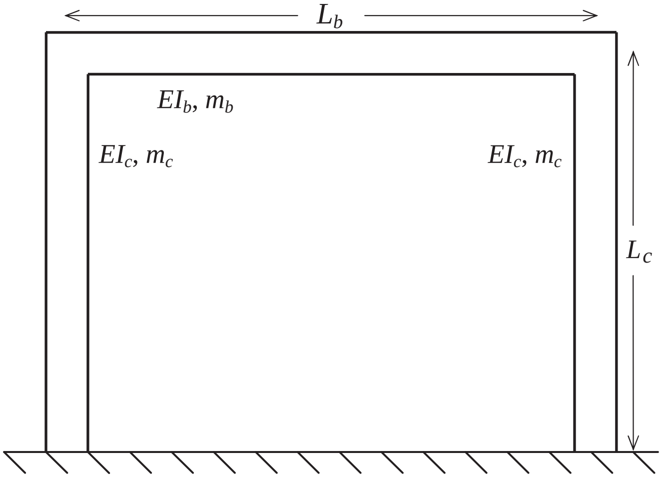


Fig. 1. Portal frame.

### 2.1. Inertia matrix

The primary expression for the inertia matrix can be written as

$$\mathbf{M} = m_c \begin{pmatrix} \mathbf{M1}_{11} & & \\ \mathbf{M1}_{21} & \mathbf{M1}_{22} & \\ \mathbf{M1}_{31} & \mathbf{M1}_{32} & \mathbf{M1}_{33} \end{pmatrix} + m_b \begin{pmatrix} \mathbf{M2}_{11} & & \\ \mathbf{M2}_{21} & \mathbf{M2}_{22} & \\ \mathbf{M2}_{31} & \mathbf{M2}_{32} & \mathbf{M2}_{33} \end{pmatrix} + m_b L_b \begin{pmatrix} \mathbf{M3}_{11} & & \\ \mathbf{M3}_{21} & \mathbf{M3}_{22} & \\ \mathbf{M3}_{31} & \mathbf{M3}_{32} & \mathbf{M3}_{33} \end{pmatrix}. \tag{1}$$

The first term represents the columns inertia, the second one the beam elastic inertia and the third, the beam rigid-body inertia; the blanks indicate symmetry and the submatrices are defined as

$$\mathbf{M1}_{11} = \begin{pmatrix} f_{22} & & & & & & \\ f_{32} - f_{21} & f_{33} + (\eta_3^2 + 1) f_{11} & & & & & \\ 0 & f_{43} - f_{41} & f_{44} & & & & \\ f_{52} - f_{21} & (\eta_3\eta_5 + 1) f_{11} & f_{54} - f_{41} & f_{55} + (\eta_5^2 + 1) f_{11} & & & \\ 0 & f_{63} - f_{61} & 0 & f_{65} - f_{61} & f_{66} & & \\ f_{72} - f_{21} & (\eta_3\eta_7 + 1) f_{11} & f_{74} - f_{41} & (\eta_5\eta_7 + 1) f_{11} & f_{76} - f_{61} & f_{77} + (\eta_7^2 + 1) f_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2a}$$

$$\mathbf{M1}_{21} = \begin{pmatrix} 0 & \eta_3 f_{21} & 0 & \eta_5 f_{21} & 0 & \eta_7 f_{21} & \cdots \\ 0 & -\eta_3\alpha_3 f_{11} & 0 & -\eta_5\alpha_3 f_{11} & 0 & -\eta_7\alpha_3 f_{11} & \cdots \\ 0 & \eta_3 f_{41} & 0 & \eta_5 f_{41} & 0 & \eta_7 f_{41} & \cdots \\ 0 & -\eta_3\alpha_5 f_{11} & 0 & -\eta_5\alpha_5 f_{11} & 0 & -\eta_7\alpha_5 f_{11} & \cdots \\ 0 & \eta_3 f_{61} & 0 & \eta_5 f_{61} & 0 & \eta_7 f_{61} & \cdots \\ 0 & -\eta_3\alpha_7 f_{11} & 0 & -\eta_5\alpha_7 f_{11} & 0 & -\eta_7\alpha_7 f_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2b}$$

$$\mathbf{M1}_{31} = \begin{pmatrix} f_{21} & \gamma_3 f_{11} & f_{41} & \gamma_5 f_{11} & f_{61} & \gamma_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ f_{21} & \gamma_3 f_{11} & f_{41} & \gamma_5 f_{11} & f_{61} & \gamma_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ f_{21} & \gamma_3 f_{11} & f_{41} & \gamma_5 f_{11} & f_{61} & \gamma_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2c}$$

$$\mathbf{M1}_{22} = \begin{pmatrix} f_{22} & & & & & & \\ f_{32} - \alpha_3 f_{21} & f_{33} + \alpha_3^2 f_{11} & & & & & \\ 0 & f_{43} - \alpha_3 f_{41} & f_{44} & & & & \\ f_{52} - \alpha_5 f_{21} & \alpha_3\alpha_5 f_{11} & f_{54} - \alpha_5 f_{41} & f_{55} + \alpha_5^2 f_{11} & & & \\ 0 & f_{63} - \alpha_3 f_{61} & 0 & f_{65} - \alpha_5 f_{61} & f_{66} & & \\ f_{72} - \alpha_7 f_{21} & \alpha_3\alpha_7 f_{11} & f_{74} - \alpha_7 f_{41} & \alpha_5\alpha_7 f_{11} & f_{76} - \alpha_7 f_{61} & f_{77} + \alpha_7^2 f_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2d}$$

$$\mathbf{M1}_{32} = \begin{pmatrix} f_{21} & -\alpha_3 f_{11} & f_{41} & -\alpha_5 f_{11} & f_{61} & -\alpha_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ f_{21} & -\alpha_3 f_{11} & f_{41} & -\alpha_5 f_{11} & f_{61} & -\alpha_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ f_{21} & -\alpha_3 f_{11} & f_{41} & -\alpha_5 f_{11} & f_{61} & -\alpha_7 f_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2e}$$

$$\mathbf{M1}_{33} = \begin{pmatrix} 2f_{11} & & & & & & \\ 0 & 0 & & & & & \\ 2f_{11} & 0 & 2f_{11} & & & & \\ 0 & 0 & 0 & 0 & & & \\ 2f_{11} & 0 & 2f_{11} & 0 & 2f_{11} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2f}$$

$$\mathbf{M2}_{11} = \begin{pmatrix} 0 & & & & & & \\ 0 & \eta_3^2 g_{11} & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & \eta_3\eta_5 g_{11} & 0 & \eta_5^2 g_{11} & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & \eta_3\eta_7 g_{11} & 0 & \eta_5\eta_7 g_{11} & 0 & \eta_7^2 g_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2g}$$

$$\mathbf{M2}_{21} = -\mathbf{M2}_{11}, \tag{2h}$$

$$\mathbf{M2}_{31} = \begin{pmatrix} 0 & \eta_3(g_{11}+g_{21}) & 0 & \eta_5(g_{11}+g_{21}) & 0 & \eta_7(g_{11}+g_{21}) & \cdots \\ 0 & -\eta_3 g_{11} & 0 & -\eta_5 g_{11} & 0 & -\eta_7 g_{11} & \cdots \\ 0 & \eta_3(g_{11}+g_{41}) & 0 & \eta_5(g_{11}+g_{41}) & 0 & \eta_7(g_{11}+g_{41}) & \cdots \\ 0 & -\eta_3 g_{11} & 0 & -\eta_5 g_{11} & 0 & -\eta_7 g_{11} & \cdots \\ 0 & \eta_3(g_{11}+g_{61}) & 0 & \eta_5(g_{11}+g_{61}) & 0 & \eta_7(g_{11}+g_{61}) & \cdots \\ 0 & -\eta_3 g_{11} & 0 & -\eta_5 g_{11} & 0 & -\eta_7 g_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2i}$$

$$\mathbf{M2}_{22} = \mathbf{M2}_{11}, \quad \mathbf{M2}_{32} = -\mathbf{M2}_{31} \tag{2j,k}$$

$$\mathbf{M2}_{33} = \begin{pmatrix} g_{11}+2g_{21}+g_{22} & & & & & & \\ g_{32}-g_{11}-g_{21} & g_{33}+g_{11} & & & & & \\ g_{11}+g_{21}+g_{41} & g_{43}-g_{11}-g_{41} & g_{11}+2g_{41}+g_{44} & & & & \\ g_{52}-g_{11}-g_{21} & g_{11} & g_{54}-g_{11}-g_{41} & g_{55}+g_{11} & & & \\ g_{11}+g_{21}+g_{61} & g_{63}-g_{11}-g_{61} & g_{11}+g_{41}+g_{61} & g_{65}-g_{11}-g_{61} & g_{11}+2g_{61}+g_{66} & & \\ g_{72}-g_{11}-g_{21} & g_{11} & g_{74}-g_{11}-g_{41} & g_{11} & g_{76}-g_{11}-g_{61} & g_{77}+g_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2l}$$

$$\mathbf{M3}_{11} = \begin{pmatrix} 0 & & & & & & \\ 0 & l_{33} - 2l_{31} + l_{11} & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & l_{53} - l_{51} - l_{31} + l_{11} & 0 & l_{55} - 2l_{51} + l_{11} & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & l_{73} - l_{71} - l_{31} + l_{11} & 0 & l_{75} - l_{71} - l_{51} + l_{11} & 0 & l_{77} - 2l_{71} + l_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2m}$$

$$\mathbf{M3}_{21} = \mathbf{0}, \tag{2n}$$

$$\mathbf{M3}_{31} = \begin{pmatrix} 0 & l_{31} - l_{11} & 0 & l_{51} - l_{11} & 0 & l_{71} - l_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & l_{31} - l_{11} & 0 & l_{51} - l_{11} & 0 & l_{71} - l_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & l_{31} - l_{11} & 0 & l_{51} - l_{11} & 0 & l_{71} - l_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2o}$$

$$\mathbf{M3}_{22} = \mathbf{0}, \quad \mathbf{M3}_{32} = \mathbf{0}, \tag{2p,q}$$

$$\mathbf{M3}_{33} = \begin{pmatrix} l_{11} & & & & & & \\ 0 & 0 & & & & & \\ l_{11} & 0 & l_{11} & & & & \\ 0 & 0 & 0 & 0 & & & \\ l_{11} & 0 & l_{11} & 0 & l_{11} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2r}$$

where

$$f_{2i-1\,2i-1} = \frac{L_c}{4\beta_{1i}^2\sigma_{1i}^2}, \quad f_{2i\,2i} = \frac{L_c}{\beta_{2i}^2\sigma_{6i}^2}, \tag{3a,b}$$

$$f_{2i-1\,2j} = f_{2j\,2i-1} = \frac{(-1)^{2i+j-1}2\beta_{2j}}{\beta_{1i}(\beta_{2j}^4 - \beta_{1i}^4)\sigma_{1i}\sigma_{6j}}(\beta_{2j}\sigma_{2j} - \beta_{1i}\sigma_{1i}), \tag{3c}$$

$$g_{2i-1\,2i-1} = \frac{L_b(\sigma_{4i} + \sigma_{5i})^2}{4\beta_{3i}^2}, \quad g_{2i\,2i} = \frac{L_b}{\beta_{3i}^2(\sigma_{4i} - \sigma_{5i})^2}, \tag{3d,e}$$

$$g_{2i-1\,2j} = g_{2j\,2i-1} = \begin{cases} \dfrac{1}{\beta_{3j}^4 - \beta_{3i}^4}\left(\beta_{3j}\dfrac{\sigma_{4i}^2 - \sigma_{5i}^2}{\sigma_{4j} - \sigma_{5j}} - \beta_{3i}(\sigma_{4i} + \sigma_{5i})\right), & \text{for } i \neq j, \\ \dfrac{\sigma_{4i} + \sigma_{5i}}{4\beta_{3i}^3}(\beta_{3i}L_b\sigma_{3i} + 1), & \text{for } i = j, \end{cases} \tag{3f}$$

$$l_{2i-1\,2j-1} = (\beta_{1i}\beta_{1j}\sigma_{1i}\sigma_{1j})^{-1}, \quad \alpha_{2i-1} = \frac{\beta_{11}\sigma_{11}}{\beta_{1i}\sigma_{1i}}, \tag{3g,h}$$

$$\eta_{2i-1} = \alpha_{2i-1} - 1, \quad \gamma_{2i-1} = \alpha_{2i-1} - 2, \tag{3i,j}$$

where in turn $\beta_{1i}, \beta_{2i}$ and $\beta_{3i}$ are the in-increasing-magnitude-order and high-precision roots of

$$\cos\,\beta_{1i}L_c \cosh\beta_{1i}L_c = -1, \quad \cos\,\beta_{2i}L_c \cosh\,\beta_{2i}L_c = 1, \quad \tan\,\beta_{3i}L_b = \tanh\,\beta_{3i}L_b \tag{4a−c}$$

and $\sigma_{1i}, \sigma_{2i}, \sigma_{3i}, \sigma_{4i}, \sigma_{5i}$ and $\sigma_{6i}$ are defined by

$$\sigma_{1i} = \frac{\sinh\,\beta_{1i}L_c - \sin\,\beta_{1i}L_c}{\cosh\,\beta_{1i}L_c + \cos\,\beta_{1i}L_c}, \quad \sigma_{2i} = \frac{\sinh\,\beta_{2i}L_c + \sin\,\beta_{2i}L_c}{\cosh\,\beta_{2i}L_c - \cos\,\beta_{2i}L_c}, \tag{4d,e}$$

$$\sigma_{3i} = \cot\,\beta_{3i}L_b, \quad \sigma_{4i} = \operatorname{csch}\beta_{3i}L_b, \quad \sigma_{5i} = \csc\beta_{3i}L_b, \tag{4f−h}$$

$$\sigma_{6i} = \sinh\,\beta_{2i}L_c/4 + \sin\,\beta_{2i}L_c/4 - \sigma_{2i}(\cosh\beta_{2i}L_c/4 - \cos\,\beta_{2i}L_c/4). \tag{4i}$$

### 2.2. Stiffness matrix

The primary expression for the stiffness matrix is

$$\mathbf{K} = EI_c\begin{pmatrix} \mathbf{K1}_{11} & & \\ \mathbf{K1}_{21} & \mathbf{K1}_{22} & \\ \mathbf{K1}_{31} & \mathbf{K1}_{32} & \mathbf{K1}_{33} \end{pmatrix} + EI_b\begin{pmatrix} \mathbf{K2}_{11} & & \\ \mathbf{K2}_{21} & \mathbf{K2}_{22} & \\ \mathbf{K2}_{31} & \mathbf{K2}_{32} & \mathbf{K2}_{33} \end{pmatrix}. \tag{5}$$

The first term represents the columns stiffness and the second one the beam stiffness; the submatrices are defined as

$$\mathbf{K1}_{11} = \begin{pmatrix} h_{22} & & & & & & \\ h_{32} - h_{21} & h_{33} + (\eta_3^2 + 1)h_{11} & & & & & \\ 0 & h_{43} - h_{41} & h_{44} & & & & \\ h_{52} - h_{21} & (\eta_3\eta_5 + 1)h_{11} & h_{54} - h_{41} & h_{55} + (\eta_5^2 + 1)h_{11} & & & \\ 0 & h_{63} - h_{61} & 0 & h_{65} - h_{61} & h_{66} & & \\ h_{72} - h_{21} & (\eta_3\eta_7 + 1)h_{11} & h_{74} - h_{41} & (\eta_5\eta_7 + 1)h_{11} & h_{76} - h_{61} & h_{77} + (\eta_7^2 + 1)h_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6a}$$

$$\mathbf{K1}_{21} = \begin{pmatrix} 0 & \eta_3 h_{21} & 0 & \eta_5 h_{21} & 0 & \eta_7 h_{21} & \cdots \\ 0 & -\eta_3\alpha_3 h_{11} & 0 & -\eta_5\alpha_3 h_{11} & 0 & -\eta_7\alpha_3 h_{11} & \cdots \\ 0 & \eta_3 h_{41} & 0 & \eta_5 h_{41} & 0 & \eta_7 h_{41} & \cdots \\ 0 & -\eta_3\alpha_5 h_{11} & 0 & -\eta_5\alpha_5 h_{11} & 0 & -\eta_7\alpha_5 h_{11} & \cdots \\ 0 & \eta_3 h_{61} & 0 & \eta_5 h_{61} & 0 & \eta_7 h_{61} & \cdots \\ 0 & -\eta_3\alpha_7 h_{11} & 0 & -\eta_5\alpha_7 h_{11} & 0 & -\eta_7\alpha_7 h_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6b}$$

$$\mathbf{K1}_{31} = \begin{pmatrix} h_{21} & \gamma_3 h_{11} & h_{41} & \gamma_5 h_{11} & h_{61} & \gamma_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ h_{21} & \gamma_3 h_{11} & h_{41} & \gamma_5 h_{11} & h_{61} & \gamma_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ h_{21} & \gamma_3 h_{11} & h_{41} & \gamma_5 h_{11} & h_{61} & \gamma_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6c}$$

$$\mathbf{K1}_{22} = \begin{pmatrix} h_{22} & & & & & & \\ h_{32} - \alpha_3 h_{21} & h_{33} + \alpha_3^2 h_{11} & & & & & \\ 0 & h_{43} - \alpha_3 h_{41} & h_{44} & & & & \\ h_{52} - \alpha_5 h_{21} & \alpha_3\alpha_5 h_{11} & h_{54} - \alpha_5 h_{41} & h_{55} + \alpha_5^2 h_{11} & & & \\ 0 & h_{63} - \alpha_3 h_{61} & 0 & h_{65} - \alpha_5 h_{61} & h_{66} & & \\ h_{72} - \alpha_7 h_{21} & \alpha_3\alpha_7 h_{11} & h_{74} - \alpha_7 h_{41} & \alpha_5\alpha_7 h_{11} & h_{76} - \alpha_7 h_{61} & h_{77} + \alpha_7^2 h_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6d}$$

$$\mathbf{K1}_{32} = \begin{pmatrix} h_{21} & -\alpha_3 h_{11} & h_{41} & -\alpha_5 h_{11} & h_{61} & -\alpha_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ h_{21} & -\alpha_3 h_{11} & h_{41} & -\alpha_5 h_{11} & h_{61} & -\alpha_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ h_{21} & -\alpha_3 h_{11} & h_{41} & -\alpha_5 h_{11} & h_{61} & -\alpha_7 h_{11} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6e}$$

$$\mathbf{K1}_{33} = \begin{pmatrix} 2h_{11} & & & & & & \\ 0 & 0 & & & & & \\ 2h_{11} & 0 & 2h_{11} & & & & \\ 0 & 0 & 0 & 0 & & & \\ 2h_{11} & 0 & 2h_{11} & 0 & 2h_{11} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6f}$$

$$\mathbf{K2}_{11} = \begin{pmatrix} 0 & & & & & & \\ 0 & \eta_3^2 k_{11} & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & \eta_3\eta_5 k_{11} & 0 & \eta_5^2 k_{11} & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & \eta_3\eta_7 k_{11} & 0 & \eta_5\eta_7 k_{11} & 0 & \eta_7^2 k_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6g}$$

$$\mathbf{K2}_{21} = -\mathbf{K2}_{11}, \tag{6h}$$

$$\mathbf{K2}_{31} = \begin{pmatrix} 0 & \eta_3(k_{11}+k_{21}) & 0 & \eta_5(k_{11}+k_{21}) & 0 & \eta_7(k_{11}+k_{21}) & \cdots \\ 0 & -\eta_3 k_{11} & 0 & -\eta_5 k_{11} & 0 & -\eta_7 k_{11} & \cdots \\ 0 & \eta_3(k_{11}+k_{41}) & 0 & \eta_5(k_{11}+k_{41}) & 0 & \eta_7(k_{11}+k_{41}) & \cdots \\ 0 & -\eta_3 k_{11} & 0 & -\eta_5 k_{11} & 0 & -\eta_7 k_{11} & \cdots \\ 0 & \eta_3(k_{11}+k_{61}) & 0 & \eta_5(k_{11}+k_{61}) & 0 & \eta_7(k_{11}+k_{61}) & \cdots \\ 0 & -\eta_3 k_{11} & 0 & -\eta_5 k_{11} & 0 & -\eta_7 k_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6i}$$

$$\mathbf{K2}_{22} = \mathbf{K2}_{11}, \quad \mathbf{K2}_{32} = -\mathbf{K2}_{31}, \tag{6j,k}$$

$$\mathbf{K2}_{33} = \begin{pmatrix} k_{11}+2k_{21}+k_{22} & & & & & & \\ k_{32}-k_{11}-k_{21} & k_{33}+k_{11} & & & & & \\ k_{11}+k_{21}+k_{41} & k_{43}-k_{11}-k_{41} & k_{11}+2k_{41}+k_{44} & & & & \\ k_{52}-k_{11}-k_{21} & k_{11} & k_{54}-k_{11}-k_{41} & k_{55}+k_{11} & & & \\ k_{11}+k_{21}+k_{61} & k_{63}-k_{11}-k_{61} & k_{11}+k_{41}+k_{61} & k_{65}-k_{11}-k_{61} & k_{11}+2k_{61}+k_{66} & & \\ k_{72}-k_{11}-k_{21} & k_{11} & k_{74}-k_{11}-k_{41} & k_{11} & k_{76}-k_{11}-k_{61} & k_{77}+k_{11} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6l}$$

where

$$h_{2i-1\,2i-1} = \beta_{1i}^4 f_{2i-1\,2i-1}, \quad h_{2i\,2i} = \beta_{2i}^4 f_{2i\,2i}, \tag{7a,b}$$

$$h_{2i-1\,2j} = h_{2j\,2i-1} = \beta_{1i}^4 f_{2i-1\,2j}, \quad k_{2i-12i-1} = \beta_{3i}^4 g_{2i-1\,2i-1}, \quad k_{2i\,2i} = \beta_{3i}^4 g_{2i\,2i}, \tag{7c−e}$$

$$k_{2i-1\,2j} = k_{2j\,2i-1} = \begin{cases} \dfrac{\beta_{3i}\beta_{3j}}{\beta_{3j}^4 - \beta_{3i}^4}\left(\beta_{3i}^3 \dfrac{\sigma_{4i}^2 - \sigma_{5i}^2}{\sigma_{4j} - \sigma_{5j}} - \beta_{3j}^3(\sigma_{4i} + \sigma_{5i})\right) & \text{for } i \neq j, \\ \dfrac{\beta_{3i}(\sigma_{4i} + \sigma_{5i})}{4}(\beta_{3i} L_b \sigma_{3i} - 3) & \text{for } i = j. \end{cases} \tag{7f}$$

## 3. Remarks on these results

The implication of this work is that there now are available generic system matrices expressions that will allow the attainment of *very accurate* natural frequencies and mode shapes of the portal frame through *very-low-order* eingenvalue problems. This is possible by means of the SSM [10] because of two reasons: (1) the structure is divided into its natural and intuitive substructures or *superelements* and (2) the selection of admissible functions that are dynamically related to the vibration problem at hand and that make up a *quasicomparison* function. It has been shown that in this structural case the FEM demands larger order eigenvalue problems to satisfy the same specified accuracy, with the associated numerical and computational-cost problems; granted, the system matrices are simpler by this more widespread method.

Note that the order of the submatrices in Eqs. (2) and (6) is just 5 for 5-digit convergence of the first three natural frequencies in actual cases [10]. This computational fact has been corroborated with a *steel* structure with dimensions different from the ones in Ref. [10], where a *concrete* frame was considered.

Further, the expressions in Eqs. (3) and (7) are that simple because of the simplest-expression integrals involving beam eigenfunctions and derivatives that have been previously obtained [16,17]. The inherent computational superiority of the SSM is enhanced by these integrals because these reduce the number of computer operations (e.g. with badly behaved hyperbolic functions) and eliminate the need of numerical integration. Also, note that many of the simplifying zeros in the submatrices in Eqs. (2) and (6) are a result of the *orthogonality* of beam eigenfunctions [8].

Regarding obtaining the (approximate) mode shapes or eigenfunctions of the frame, which might not have been clear in Ref. [10], these are obtained, as well as the natural frequencies, through the solution of the usual vibrational eigenvalue problem

$$\mathbf{Kq} = \omega_n^2 \mathbf{Mq} \tag{8}$$

by inserting the eigenvectors $\mathbf{q}$ into Eq. (25) of Ref. [11], by introducing the resulting vectors $\mathbf{q}_d$ in conjunction with the vectors of the admissible functions (Eqs. (29) of Ref. [11]) into Eqs. (10) of Ref. [11] and ultimately by *synthesising*.

Finally, this SSM has been applied to an $n$-story single-bay frame [11], which means that system matrices expressions can also be developed for that case or, in principle, for the $n$-story $m$-bay general case.

## 4. Conclusions

Generic expressions of substructure-synthesis mass and stiffness matrices of the one-portal frame have been presented, which will allow easy and accurate computation of its natural frequencies and mode shapes through the solution of low-order eigenproblems.

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