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Journal of Sound and Vibration 283 (2005) 1229–1234
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

On oscillations in a system with a piecewise smooth coefficient

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Received 15 September 2003; received in revised form 4 June 2004; accepted 19 July 2004
Available online 25 november 2004

Abstract

In this paper a straightforward perturbation procedure will be presented for a class of ordinary differential equations with piecewise smooth terms in the equations. From the constructed approximations of the solutions a map can be derived from which the existence and the stability of time-periodic solutions can be determined.

1. Introduction

Ordinary differential equations as model equation for piecewise smooth systems occur in a wide variety of applications. For dry friction problems the equation

$$m\ddot{x} + cx = F_0 \operatorname{sgn}(v_0 - \dot{x}) \tag{1}$$

is frequently used. In Eq. (1) m, c, F_0, and v_0 are constants and sgn is the signum function defined by $\operatorname{sgn}(u) = 1$ for $u > 0, 0$ for $u = 0$, and -1 for $u < 0$. For oscillation problems with a symmetrical restoring force of constant magnitude the equation

$$\ddot{x} + \operatorname{sgn}(x) = 0 \tag{2}$$

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doi:10.1016/j.jsv.2004.07.012

can be used. Eqs. (1) and (2), and many similar equations can for instance be found in Ref. [1]. Another example of an equation for a piecewise smooth system is

$$\ddot{x} + c\dot{x} + \mu x^{+} - \nu x^{-} = f(t, x, \dot{x}), \tag{3}$$

which has been introduced by Lazer and McKenna [2] as a simple model equation for the vertical oscillations of a long-span suspension bridge. In Eq. (3) c, μ, and ν are positive constants, $f(t, x, \dot{x})$ is an external force function, and $x^{+} = \max\{x, 0\}$ is the positive part of x, and $x^{-} = \max\{-x, 0\}$ its negative part. Recently Hogan [3] studied the equation

$$\ddot{x} + \varepsilon(|x| - 1)\dot{x} + x = 0, \tag{4}$$

where ε is a small positive constant, that is, $0 < \varepsilon \ll 1$. Eq. (4) is related to the equation

$$\ddot{x} + \varepsilon(|\dot{x}| - 1)\dot{x} + x = 0, \tag{5}$$

which has been studied in Ref. [1]. In fact by differentiating Eq. (5) with respect to t (and by rescaling x by a factor 2) an equation like Eq. (4) is obtained. In Ref. [1] the averaging technique and the harmonic balance method have been used to construct approximations of the periodic solution of Eq. (5) and in Ref. [3] the two-time-scales perturbation method has been applied to Eq. (4). To prove the existence of a unique, nontrivial periodic solution a Liénard theorem has been used in Ref. [3]. Usually, it is difficult to find (if available) the appropriate Liénard theorem for a given differential equation. In this paper it will be shown that for perturbed equations (like Eqs. (1)–(5)) a straightforward perturbation procedure can be used to construct approximations of the solutions. From these approximations a map can be defined. By using this map the existence and the stability of time-period solutions can be established. It should be remarked that the presented perturbation procedure is applicable to those equations (with piecewise smooth coefficients and a small parameter) for which on different (usually finite) time-intervals approximations of the solutions can be constructed. The so-constructed approximations on these different time intervals are then matched to obtain approximations of the solutions on sufficiently long time-intervals. A mathematical justification of the constructed results (that is, giving error estimates on time-scales) can be obtained by using standard mathematical techniques from perturbation theory as for instance described in Refs. [4–6]. As a prototype of problem Eq. (4) will be treated in detail.

2. Construction of an approximation of the solution

In this section an approximation of the solution of Eq. (4) will be constructed. Since Eq. (4) is autonomous it can be assumed, without loss of generality, that $x(0) = 0$ and $\dot{x}(0) = A > 0$. So, the following problem has to be considered (as long as $x(t) \geqslant 0$ for $t > 0$):

$$\begin{aligned} &\ddot{x} + \varepsilon(x - 1)\dot{x} + x = 0, \quad t > 0, \\ &x(0) = 0, \quad \dot{x}(0) = A > 0, \end{aligned} \tag{6}$$

where $0 < \varepsilon \ll 1$, and where A is an ε-independent constant. To solve the initial-value problem for Eq. (6) approximately the following expansion in ε for $x(t)$, that is,

$$x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{7}$$

is substituted into Eq. (6). By taking together terms of equal powers in ε the following problems are then obtained:

$$\begin{aligned} O(1)\text{-problem:} \quad & \ddot{x}_0 + x_0 = 0, \quad t > 0, \\ & x_0(0) = 0, \quad \dot{x}(0) = A, \end{aligned} \tag{8}$$

$$\begin{aligned} O(\varepsilon)\text{-problem:} \quad & \ddot{x}_1 + x_1 = -(x_0 - 1)\dot{x}_0, \quad t > 0, \\ & x_1(0) = \dot{x}_1(0) = 0, \end{aligned} \tag{9}$$

and so on. The $O(1)$-problem (8) can readily be solved, yielding

$$x_0(t) = A\sin(t). \tag{10}$$

Substituting Eq. (10) into Eq. (9) the $O(\varepsilon)$-problem (9) then also can be solved, yielding

$$x_1(t) = \frac{-A^2}{3}\sin(t) + \frac{A^2}{6}\sin(2t) + \frac{A}{2}t\sin(t). \tag{11}$$

Now it should be observed that the approximation $\bar{x}(t) = x_0(t) + \varepsilon x_1(t)$ satisfies the original differential equation (6) and the initial values up to order ε^2, that is,

$$\begin{aligned} &\ddot{\bar{x}} + \varepsilon(\bar{x} - 1)\dot{\bar{x}} + \bar{x} \\ &\quad = \ddot{x}_0 + x_0 + \varepsilon(\ddot{x}_1 + x_1 + (x_0 - 1)\dot{x}_0) + \varepsilon^2((x_0 - 1)\dot{x}_1 + \dot{x}_0 x_1) + \varepsilon^3 x_1 \dot{x}_1 \\ &\quad = O(\varepsilon^2 t) + O(\varepsilon^3 t^2), \end{aligned} \tag{12}$$

and $\bar{x}(0) = 0, \dot{\bar{x}}(0) = A$.

By writing down the equivalent integral equations for the initial-value problems (6) and (12), by subtracting the so-obtained integral equations, by using the Lipschitz continuity of the nonlinearities, and by applying Gronwall's inequality it can be shown simply (see for instance Chapter 1 of Verhulst [4]) that (as long as $x(t) \geqslant 0$)

$$|x(t) - \bar{x}(t)| = O(\varepsilon^2 t^2). \tag{13}$$

Estimate (13) implies that (as long as $x(t) \geqslant 0$) $x(t) = \bar{x}(t) + O(\varepsilon^2) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$ for times t of order 1, where $x_0(t)$ and $x_1(t)$ are given by Eqs. (10) and (11), respectively. Now let $\tilde{t} > 0$ be the shortest time for which $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$ becomes zero, and assume that $\tilde{t}$ can be approximated by $\tilde{t}_0 + \varepsilon\tilde{t}_1 + \varepsilon^2\tilde{t}_2 + \cdots$. By substituting this approximation for $\tilde{t}$ into $x(\tilde{t}) = x_0(\tilde{t}) + \varepsilon x_1(\tilde{t}) + O(\varepsilon^2) = 0$ it easily follows that $\tilde{t}_0 = \pi$, and $\tilde{t}_1 = 0$, and so $\tilde{t} = \pi + O(\varepsilon^2)$. It then follows from $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$ that (for ε sufficiently small) $\dot{x}(\tilde{t}) < 0$. So, for $t > \tilde{t}$ the following initial value problem has to be solved (until x becomes zero again):

$$\begin{aligned} & \ddot{x} + \varepsilon(-x - 1)\dot{x} + x = 0, \quad t > \tilde{t}, \\ & x(\tilde{t}) = 0, \quad \dot{x}(\tilde{t}) = -A + \varepsilon A\left(\frac{2}{3}A - \frac{\pi}{2}\right) + O(\varepsilon^2), \end{aligned} \tag{14}$$

where $\tilde{t} = \pi + O(\varepsilon^2)$. To solve this initial value problem (14) approximately the function $x(t)$ is again approximated by expansion (7). This expansion is submitted into Eq. (14), and terms of

equal powers in ε are taken together, yielding as

$$\begin{aligned} O(1)\text{-problem:} \quad & \ddot{x}_0 + x_0 = 0, \quad t > \tilde{t}, \\ & x_0(\tilde{t}) = 0, \quad \dot{x}_0(\tilde{t}) = -A, \end{aligned} \tag{15}$$

$$\begin{aligned} O(\varepsilon)\text{-problem:} \quad & \ddot{x}_1 + x_1 = (x_0 + 1)\dot{x}_0, \quad t > \tilde{t}, \\ & x_1(\tilde{t}) = 0, \quad \dot{x}_1(\tilde{t}) = A\left(\tfrac{2}{3}A - \frac{\pi}{2}\right), \end{aligned} \tag{16}$$

and so on. Again the initial value problems (15), (16), and so on can be solved consecutively, yielding

$$x_0(t) = -A\sin(t - \tilde{t}), \tag{17}$$

$$x_1(t) = A\left(A - \frac{\pi}{2}\right)\sin(t - \tilde{t}) - \frac{A^2}{6}\sin(2(t - \tilde{t})) - \frac{A}{2}(t - \tilde{t})\sin(t - \tilde{t}), \tag{18}$$

and so on. Again it can be shown simply that the approximation $\bar{x}(t) = x_0(t) + \varepsilon x_1(t)$ (where x_0 and x_1 are given by Eqs. (17) and (18), respectively) satisfies the following estimate: $|x(t) - \bar{x}(t)| = O(\varepsilon^2)$ for times t of order 1 (as long as $x(t) \leqslant 0$). Now let $\hat{t} > \tilde{t}$ be the shortest time for which $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$ becomes zero again, and assume that $\hat{t}$ can be approximated by $\hat{t}_0 + \varepsilon\hat{t}_1 + \varepsilon^2\hat{t}_2 + \cdots$. By substituting this approximation for $\hat{t}$ into $x(\hat{t}) = x_0(\hat{t}) + \varepsilon x_1(\hat{t}) + O(\varepsilon^2) = 0$, where x_0 and x_1 are given by Eqs. (17)and (18), respectively, it easily follows that $\hat{t}_0 = 2\pi$, and $\hat{t}_1 = 0$, and so $\hat{t} = 2\pi + O(\varepsilon^2)$. Now $\dot{x}(\hat{t})$ can also be computed, yielding

$$\dot{x}(\hat{t}) = A + \varepsilon A(-\tfrac{4}{3}A + \pi) + O(\varepsilon^2). \tag{19}$$

For ε sufficiently small it follows from Eq. (19) that $\dot{x}(\hat{t}) > 0$. So, for $t > \hat{t}$ Eq. (6) has to be solved again subject to $x(\hat{t}) = 0$ and $\dot{x}(\hat{t})$ as given by Eq. (19). Since $A > 0$ is arbitrary it can be concluded that in fact all initial value problems for Eq. (4) have been solved already (up to $O(\varepsilon^2)$) for times of order 1. Comparing $\dot{x}(0) = A$ with $\dot{x}(\hat{t})$ as given by Eq. (19) it can also be concluded that for $A > \frac{3}{4}\pi$ (and ε sufficiently small and positive) the amplitude of the oscillation will decrease, and that for $A < \frac{3}{4}\pi$ the amplitude will increase. This suggests the existence of (at least) one non-trivial periodic solution (or closed orbit in the $(x, \dot{x})$ phase plane). In the next section it will be proven that there exists a unique, non-trivial periodic solution for Eq. (4).

3. The existence of time-periodic solutions

To prove the existence of a time-periodic solution of Eq. (4) a map Q (actually the Poincaré-return map on the section $x = 0$, and $\dot{x} > 0$ in the $(x, \dot{x})$ phase-plane) will be considered. Based upon the error estimate (13) and upon Eq. (19) this map Q is defined by (for $n = 1, 2, 3, \ldots$):

$$A_n = A_{n-1} + \varepsilon A_{n-1}\left(-\tfrac{4}{3}A_{n-1} + \pi\right) + O(\varepsilon^2 n), \tag{20}$$

where A_n is the value of $\dot{x}(t)$ with t such that the solution $x(t)$ returned to the Poincaré section (that is, $x = 0$ and $\dot{x} > 0$) for the nth time. A map $Q : A \to Q(A) \Leftrightarrow A_n = Q(A_{n-1})$ has now be defined. By neglecting terms of $O(\varepsilon^2 n)$ in Eq. (20) a new map P is defined: $\tilde{A} \to P(\tilde{A}) \Leftrightarrow \tilde{A}_n = P(\tilde{A}_{n-1})$. The maps P and Q will now be studied, and it will be shown that

(i) if $|A_0 - \tilde{A}_0| = O(\varepsilon)$ for $\varepsilon \downarrow 0$, then $|A_n - \tilde{A}_n| = O(\varepsilon)$ for $n = O(1/\sqrt{\varepsilon})$, that is, for $n \sim 1/\sqrt{\varepsilon}$ and $\varepsilon \downarrow 0$, A_n and $\tilde{A}_n$ remain "ε-close";
(ii) the map P has a unique, hyperbolic fixed point $\tilde{A} = 3\pi/4$, which is asymptotically stable;
(iii) there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$ the map Q has a unique, hyperbolic fixed point $A = 3\pi/4 + O(\varepsilon)$ with the same stability property as the fixed point $A = 3\pi/4$ of the map P.

Proof. (i) From $|A_0 - \tilde{A}_0| = O(\varepsilon)$ for $\varepsilon \downarrow 0$ it follows that there exists a positive constant M_0 such that $|A_0 - \tilde{A}_0| = M_0\varepsilon$. Further, $|A_n - \tilde{A}_n| = |P(A_{n-1}) - P(A_{n-1}) + O(\varepsilon^2 n)| \leqslant |P(A_{n-1}) - P(\tilde{A}_{n-1})| + M_1\varepsilon^2 n$, where M_1 is a positive constant. We also have $|P(A_{n-1}) - P(\tilde{A}_{n-1})| \leqslant L|A_{n-1} - \tilde{A}_{n-1}|$, where L is a Lipschitz-constant with $L = 1 + \varepsilon M_2$ in which M_2 is a positive constant. And so, $|A_n - \tilde{A}_n| \leqslant (1 + \varepsilon M_2)|A_{n-1} - \tilde{A}_{n-1}| + M_1\varepsilon^2 n \leqslant \cdots \leqslant \varepsilon(M_0 + \varepsilon n^2 M_1)e^{\varepsilon n M_2}$. Thus for $n = O(1/\sqrt{\varepsilon})$ it follows that $|A_n - \tilde{A}_n| = O(\varepsilon)$.

(ii) The fixed points of the map P follow from $\tilde{A}_n = P(\tilde{A}_{n-1})$ for $n \to \infty$, or equivalently from $\tilde{A} = \tilde{A} + \varepsilon\tilde{A}(-\frac{4}{3}\tilde{A} + \pi) \Leftrightarrow \tilde{A}(-\frac{4}{3}\tilde{A} + \pi) = 0$. Therefore for $\tilde{A} > 0$ the unique fixed point is $\tilde{A} = \frac{3}{4}\pi$. If the linearized map of P around this fixed point has no eigenvalues of unit modulus, then this fixed point is hyperbolic. Let DP be this linearized map; then $DP = 1 - \varepsilon\pi$. Obviously the eigenvalue λ of DP is $1 - \varepsilon\pi$. Since $0 < \varepsilon \ll 1$ it follows that $|\lambda| < 1$, and so the fixed point is hyperbolic and stable.

(iii) It follows from (ii) that $I - DP$ is invertible. Therefore from (i), the implicit function theorem, and the linearization theorem of Hartman–Grobman, it follows (see also Chapter 1 in Ref. [5] or [6]), that the map Q continues to have a unique fixed point, which is ε-close to $3\pi/4$. Furthermore, the eigenvalue of DQ (DQ is the linearized map of Q around its fixed point) depends continuously on ε, and is (up to $O(\varepsilon^{3/2})$) equal to the eigenvalue of DP. Therefore for sufficiently small ε the two fixed points have the same stability properties. □

So far it can be concluded that a unique, asymptotically stable, non-trivial periodic solution exists for Eq. (4). The period T of this unique, nontrivial periodic solution can be approximated by $2\pi + O(\varepsilon^2)$, and the periodic solution itself can be approximated by

$$\frac{3\pi}{4}\sin(t) + \varepsilon\left(-\frac{3\pi^2}{16}\sin(t) + \frac{3\pi^2}{32}\sin(2t) + \frac{3\pi}{8}t\sin(t)\right) + O(\varepsilon^2)$$

for $0 \leqslant t \leqslant \tilde{t}$, and by

$$\frac{3\pi}{4}\sin(t) + \varepsilon\left(-\frac{9\pi^2}{16}\sin(t) - \frac{3\pi^2}{32}\sin(2t) + \frac{3\pi}{8}t\sin(t)\right) + O(\varepsilon^2)$$

for $\tilde{t} \leqslant t \leqslant T$, where $\tilde{t}$ is the time as defined before and $\tilde{t}$ can be approximated by $\pi + O(\varepsilon^2)$. Higher order approximations of the period T and of the periodic solution can be obtained by solving the $O(\varepsilon^n)$-problems with $n \geqslant 2$ as defined in Section 2 of this paper.

4. Conclusions and remarks

In this paper it has been shown how in a straightforward way approximations of the solutions can be obtained for a class of ordinary differential equations with piecewise smooth terms and

with a small parameter in the equations. By using these approximations a map can be constructed from which the existence and the stability of time-periodic solutions can be determined. As example the equation

$$\ddot{x} + \varepsilon(|x| - 1)\dot{x} + x = 0$$

has been studied in detail. Not only approximations of the solutions of this equation have been constructed but also order ε^2 accurate approximations of the unique and stable periodic solution have been presented. The perturbation procedure as presented in this paper can be applied to a large class of ordinary differential equations with piecewise smooth terms and with a small parameter in the equations. In a forthcoming paper the equation

$$\ddot{x} + \mu x^+ - \nu x^- = \varepsilon(\dot{x} - \dot{x}^3) \tag{21}$$

will be studied. In this equation μ, ν, and ε are positive constants with $0 < \varepsilon \ll 1$, and $x^+ = \max\{x, 0\}$, and $x^- = \max\{-x, 0\}$. This equation serves as a simple model equation to describe the wind-induced vertical vibrations of a long-span suspension bridge. When $\mu = \nu$ the well-known Rayleigh equation is obtained, which is related to the Van der Pol equation. The presented perturbation procedure can also be applied successfully to Eq. (21), and it can be shown that also Eq. (21) has a unique and stable periodic solution.

In this paper a straightforward perturbation expansion for the solutions has been used. As a consequence so-called secular terms can or will occur in the approximations. When only approximations are needed on timescales of order 1 then these secular terms will not cause problems. However, when for instance approximations are needed on timescales of order ε^{-1} then these secular terms will lead to inconsistencies. To avoid these incorrect results other perturbation methods (such as the multiple timescales perturbation method, or the averaging method, or etc.) instead of the method of straightforward expansions have to be used to construct asymptotically valid results.

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