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Short Communication

Application of Sherman–Morrison matrix inversion formula to damped vibration absorbers attached to multi-degree of freedom systems

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Abstract

Dynamic (tuned mass) vibration absorbers are among the most commonly used devices for vibration suppression. A new method is introduced based on the Sherman–Morrison matrix inversion formula for calculation of the optimal absorber parameter values for attachment to a damped multi-degree of freedom system. The new method can be used to minimize the motion of particular physical masses or modes of vibration, regardless of their location relative to the placement of the absorber subsystem. The new method can also minimize a linear summation of the degrees of freedom. Case studies illustrate the method's superior performance relative to classical analytical techniques for determining absorber parameters.

1. Introduction

The dynamic vibration absorber is one of the oldest vibration absorption devices. Its theory of operation was presented by Frahm [1] almost a hundred years ago. The dynamic vibration absorber he proposed was a mass–spring system, which is attached to the main system. When tuned properly this system was theoretically capable of setting the main system vibration to zero at a particular frequency. Later Ormondroyd and Den Hartog [2] proposed a vibration absorber

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with a viscous damper. In their analysis the main system was assumed to be an undamped system. The details of the derivation of the optimal parameters are given in Den Hartog's well-known book [3]. His elegant derivation makes use of the fixed points of the response curve of the main system. The amplitude of the response at certain frequencies is independent of the absorber damping for an undamped main system. The resulting vibration suppression is a broadband attenuation compared to Frahm's undamped vibration absorber.

After these two very important works in this field, there have been many studies, which consider damped main systems and multi-degree of freedom (mdof) main systems. However, since a damped main system invalidates the fixed points concept, most of the work with damped main systems has been numerical. There are quite a few studies done on damped vibration absorbers attached to a single-degree of freedom (sdof) damped main system. The study of Randall et al. [4] uses a numerical search method to find the optimal damping and stiffness values of an absorber attached to a damped main system. Thompson [5,6] attacked the same problem with a graphical frequency locus approach and found the expression for the tuning ratio and optimal damping ratio, which would minimize the main mass displacement and the throw of the absorber.

In a recent study of Nishihara and Asami [7], the possibility of calculating the optimal damping and tuning conditions without using the fixed points concept has been shown for an undamped sdof main system. Also, in another recent work of Asami et al. [8], it has been shown that an analytical series solution expression can be found for optimal damping and tuning ratios for a damped sdof main system.

Studies in the literature that consider a mdof main system are less in number compared to sdof main system studies. Lewis [9] showed that fixed points also exist for a damped absorber attached to a mdof undamped main system. He also found the optimal damping and tuning values for a two-dof main system. In a recent study by Ozer and Royston [10] it has been shown that the Den Hartog method can be extended to mdof undamped main systems. Vakakis and Paipetis [11] calculated the optimal parameters of a damped vibration absorber attached to an undamped mdof system making use of the A_q polynomials. Kitis et al. [12] employed a numerical optimization method for finding the optimal parameters of a damped vibration absorber attached to a damped mdof main system. The method is applied to a 22-dof main system in a case study.

In the present study, a semi-analytic method is proposed for determination of the parameters of a damped vibration absorber attached to a damped mdof main system. This new method makes use of the Sherman–Morrison (SM) [13] matrix inversion formula. The SM matrix inversion formula has been used in a variety of fields for different applications. These applications range from numerical partial differentiation calculations to circuit networks and from structures to least squares estimates in statistics models. Hager [14] reviewed several applications of the SM formula in different branches of science and engineering. In relation to vibration applications, there are a few studies that utilize the SM formula. Hong and Kim [15] used the SM formula to show the effect of the absorber material on response characteristics of acoustic–structural coupled systems.

2. Application of Shermann–Morison matrix inversion theorem in mechanical vibrations

The system to be analyzed is shown in Fig. 1. The *main* system, which is externally driven at the ith mass, has n-dof consisting of a series cascade of mass, spring and damper elements. A sdof dynamic vibration absorber is to be attached to the mth mass of the main system. An objective is to write the response of the *total* system in terms of the response of the main system and the parameters of the added system.

The equation of motion of the total system in Fig. 1 can be given as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{C_{add}}\dot{x}_\mathbf{a} + \mathbf{K_{add}}x_a = \mathbf{F}, \tag{1}$$

$$m_a\ddot{x}_a + c_a\dot{x}_a + k_a x_a + \mathbf{C}_{\mathbf{add}}^{\mathrm{T}}\dot{\mathbf{x}} + \mathbf{K}_{\mathbf{add}}^{\mathrm{T}}\mathbf{x} = 0, \tag{2}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & \\ & \cdots & & & & & \\ & & m_i & & & & \\ & & & \cdots & & & \\ & & & & m_m & & \\ & & & & & \cdots & \\ & & & & & & m_n \end{bmatrix}, \tag{3}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -k_i & k_i + k_{i+1} & -k_{i+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -k_m & k_m + k_{m+1} + k_a & -k_{m+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -k_n & k_n + k_{n+1} \end{bmatrix}, \tag{4}$$

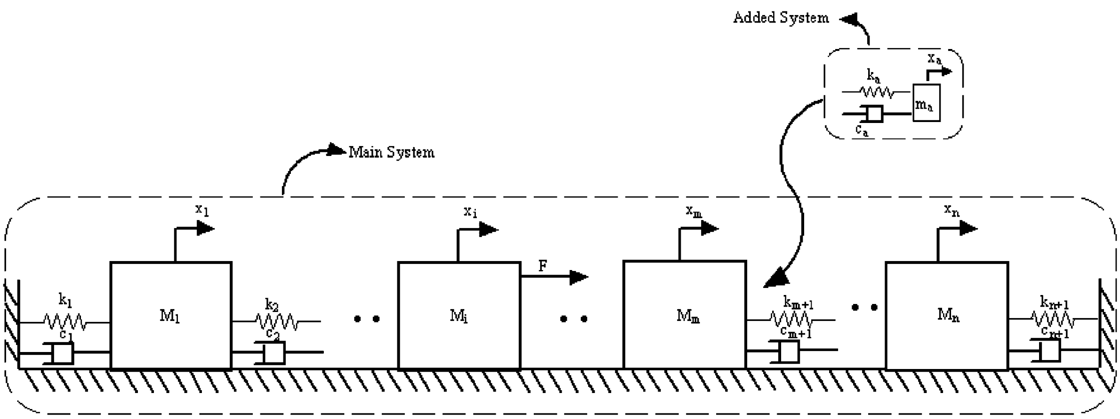


Fig. 1. The main system and the attached mass–spring–damper system.

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -c_i & c_i + c_{i+1} & -c_{i+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -c_m & c_m + c_{m+1} + c_a & -c_{m+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -c_n & c_n + c_{n+1} \end{bmatrix}, \tag{5}$$

$$\mathbf{F} = \begin{bmatrix} 0 \\ \cdots \\ F_i \\ \cdots \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ \cdots \\ x_i \\ \cdots \\ x_m \\ \cdots \\ x_n \end{bmatrix}, \quad \mathbf{C_{add}} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ -c_a \\ \cdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{K_{add}} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ -k_a \\ \cdots \\ 0 \end{bmatrix}. \tag{6−9}$$

These equations can be transformed to the frequency domain using the relation $\mathbf{X} = \mathbf{x}e^{i\omega t}$ such that Eq. (1) can be written as

$$(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})\mathbf{X} + (\mathbf{K_{add}} + i\omega\mathbf{C_{add}})X_a = \mathbf{F}, \tag{10}$$

where one can define

$$\mathbf{Z} \equiv \mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C}. \tag{11}$$

At this point the Z (dynamic stiffness) expression will be written in terms of the dynamic stiffness of the main system and remainder terms. The main system stiffness and the damping matrices can be written as (note that the mass matrix is the same as in Eq. (3))

$$\mathbf{K_{main}} = \begin{bmatrix} k_1 + k_2 & -k_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -k_i & k_i + k_{i+1} & -k_{i+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -k_m & k_m + k_{m+1} & -k_{m+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -k_n & k_n + k_{n+1} \end{bmatrix}, \tag{12}$$

$$\mathbf{C}_{\mathbf{main}} = \begin{bmatrix} c_1+c_2 & -c_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -c_i & c_i+c_{i+1} & -c_{i+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -c_m & c_m+c_{m+1} & -c_{m+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -c_n & c_n+c_{n+1} \end{bmatrix}. \tag{13}$$

Using Eqs. (3)–(5), (12) and (13) one can write the dynamic stiffness expression as

$$\mathbf{Z} = (\mathbf{K}_{\mathbf{main}} - \omega^2\mathbf{M}_{\mathbf{main}} + \mathrm{i}\omega\mathbf{C}_{\mathbf{main}}) + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ \sqrt{\mathrm{i}\omega c_a + k_a} \\ \cdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \ldots & 0 & \ldots & \sqrt{\mathrm{i}\omega c_a + k_a} & \ldots & 0 \end{bmatrix}. \tag{14}$$

Renaming some of the variables results in

$$\mathbf{Z}_{\mathbf{v}} \equiv \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ \sqrt{\mathrm{i}\omega c_a + k_a} \\ \cdots \\ 0 \end{bmatrix}, \quad \mathbf{Z}_{\mathbf{main}} = \mathbf{K}_{\mathbf{main}} + \mathrm{i}\omega\mathbf{C}_{\mathbf{main}} - \omega^2\mathbf{M}_{\mathbf{main}}. \tag{15,16}$$

Using Eqs. (14)–(16) one can write the relationship as

$$\boldsymbol{\alpha} = \mathbf{Z}^{-1} = [\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{v}}\mathbf{Z}_{\mathbf{v}}^{\mathrm{T}}]^{-1}. \tag{17}$$

At this point, to carry out the matrix inversion in Eq. (17), the SM matrix inversion formula [21] will be used. The SM matrix inversion theorem can be written as

$$[[\mathbf{A}] + \mathbf{u}\mathbf{v}^{\mathrm{T}}]^{-1} = [\mathbf{A}]^{-1} - \frac{[\mathbf{A}]^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}[\mathbf{A}]^{-1}}{1 + \mathbf{v}^{\mathrm{T}}[\mathbf{A}]^{-1}\mathbf{u}}, \tag{18}$$

where $\mathbf{A}$ is a $[n \times n]$ square full rank matrix and $\mathbf{u}$ and $\mathbf{v}$ are $[n \times 1]$ vectors. If one examines Eqs. (17) and (18), it can be seen that Eq. (17) is in the general form of the SM expression given in Eq. (18). Evaluation of the matrix inversion in Eq. (17) using Eq. (18) results in the expression

$$[\mathbf{Z}_{\mathbf{main}} + \mathbf{Z}_{\mathbf{v}}\mathbf{Z}_{\mathbf{v}}^{\mathrm{T}}]^{-1} = \boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{main}} - \frac{\boldsymbol{\alpha}_{\mathbf{main}}\mathbf{Z}_{\mathbf{v}}\mathbf{Z}_{\mathbf{v}}^{\mathrm{T}}\boldsymbol{\alpha}_{\mathbf{main}}}{1 + \mathbf{Z}_{\mathbf{v}}^{\mathrm{T}}\boldsymbol{\alpha}_{\mathbf{main}}\mathbf{Z}_{\mathbf{v}}}, \tag{19}$$

where α_{main} is the receptance matrix of the main system given in Fig. 1. The use of the SM formula has provided the receptance of the $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ matrices in Eqs. (3)–(5) in terms of the main system receptance and the added system's parameters. Further expansion of Eq. (19) using Eq. (15) results in the expression

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{main}} - \boldsymbol{\alpha}_{\mathbf{main}}^{\langle m \rangle}(\boldsymbol{\alpha}_{\mathbf{main}}^{\langle m \rangle})^{\mathrm{T}} \frac{k_a + \mathrm{i}\omega c_a}{1 + (\alpha_{\mathrm{main}})_{mm}(k_a + \mathrm{i}\omega c_a)}, \tag{20}$$

where $\alpha_{\mathrm{main}}^{\langle m \rangle}$ is the mth column in the main system's receptance matrix which can be expressed as

$$\boldsymbol{\alpha}_{\mathbf{main}}^{\langle m \rangle} = \begin{bmatrix} (\alpha_{\mathrm{main}})_{1m} \\ \cdots \\ (\alpha_{\mathrm{main}})_{im} \\ \cdots \\ (\alpha_{\mathrm{main}})_{mm} \\ \cdots \\ (\alpha_{\mathrm{main}})_{nm} \end{bmatrix}. \tag{21}$$

Considering the total system's second equation of motion (2), it is possible to transform this expression to the frequency domain and solving for X_a yields

$$X_a = -\frac{\mathrm{i}\omega\mathbf{C}_{\mathbf{add}}^{\mathrm{T}} + \mathbf{K}_{\mathbf{add}}^{\mathrm{T}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\mathbf{X}. \tag{22}$$

If Eq. (22) is substituted into Eq. (10) the following can be obtained:

$$\left[\mathbf{K} - \omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} - (\mathbf{K}_{\mathbf{add}} + \mathrm{i}\omega\mathbf{C}_{\mathbf{add}})\frac{\mathrm{i}\omega\mathbf{C}_{\mathbf{add}}^{\mathrm{T}} + \mathbf{K}_{\mathbf{add}}^{\mathrm{T}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\right]\mathbf{X} = \mathbf{F}. \tag{23}$$

The response of the total system is given by

$$\mathbf{X} = \boldsymbol{\alpha}_{\mathbf{tot}}\mathbf{F}, \tag{24}$$

where

$$\boldsymbol{\alpha}_{\mathbf{tot}} = \left[\mathbf{K} - \omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} - (\mathbf{K_{add}} + \mathrm{i}\omega\mathbf{C_{add}})\frac{\mathrm{i}\omega\mathbf{C}^{\mathrm{T}}_{\mathbf{add}} + \mathbf{K}^{\mathrm{T}}_{\mathbf{add}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\right]^{-1}. \tag{25}$$

At this point the SM theorem is to be applied to Eq. (25). Comparing Eq. (25) with the SM theorem expression (18) results in

$$\mathbf{A} = \mathbf{Z} = \mathbf{K} - \omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C}, \quad \mathbf{u} = -(\mathbf{K_{add}} + \mathrm{i}\omega\mathbf{C_{add}}), \tag{26,27}$$

$$\mathbf{v}^{\mathrm{T}} = \frac{\mathrm{i}\omega\mathbf{C}^{\mathrm{T}}_{\mathbf{add}} + \mathbf{K}^{\mathrm{T}}_{\mathbf{add}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}. \tag{28}$$

So, the evaluation of the receptance matrix of the total system using the SM theorem results in

$$\left[\mathbf{Z} - (\mathbf{K_{add}} + \mathrm{i}\omega\mathbf{C_{add}})\frac{\mathrm{i}\omega\mathbf{C}^{\mathrm{T}}_{\mathbf{add}} + \mathbf{K}^{\mathrm{T}}_{\mathbf{add}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\right]^{-1} = \boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}(\mathbf{K_{add}} + \mathrm{i}\omega\mathbf{C_{add}})\dfrac{\mathrm{i}\omega\mathbf{C}^{\mathrm{T}}_{\mathbf{add}} + \mathbf{K}^{\mathrm{T}}_{\mathbf{add}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\boldsymbol{\alpha}}{1 - \dfrac{\mathrm{i}\omega\mathbf{C}^{\mathrm{T}}_{\mathbf{add}} + \mathbf{K}^{\mathrm{T}}_{\mathbf{add}}}{k_a - \omega^2 m_a + \mathrm{i}\omega c_a}\boldsymbol{\alpha}(\mathbf{K_{add}} + \mathrm{i}\omega\mathbf{C_{add}})}. \tag{29}$$

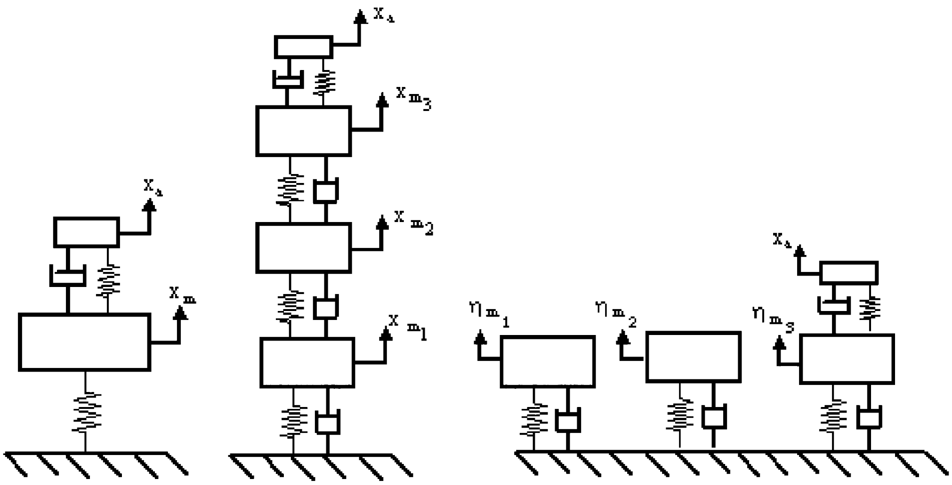
Using Eqs. (29), (8), (9) and (20) after several steps of simplifications one can obtain the displacement amplitude of the kth dof as

$$x_k = x_{\mathrm{main}_k} - \frac{x_{\mathrm{main}_m}\alpha_{\mathrm{main}_{km}}}{1/(-\omega^2 m_a) + 1/(k_a + \mathrm{i}\omega c_a) + \alpha_{\mathrm{main}_{mm}}}. \tag{30}$$

It can be seen from Eq. (30) that it is possible to write the displacement vector of the kth dof of the combined system in terms of the displacement of kth degree of the main system, the main system receptance values and the added system parameters.

3. Den Hartog's method for systems with a tuned damped vibration absorber

Before applying the previously discussed method to tuned vibration absorber systems, a widely used method for systems with vibration absorbers is discussed. Den Hartog's method was derived for a sdof main system with no damping. The added absorber system consists of a mass, stiffness and damper as shown in Fig. 2(a). Although the derivations given by Den Hartog [3] are for a sdof system, the principle can easily be extended to a mdof system, which can be represented as in Fig. 2(b). The mdof system must be transformed into the modal domain. This turns the n-dof main system in the physical domain into n-decoupled sdof systems in the modal domain as shown Fig. 2(c). At this point the target mode should be chosen and the tuned vibration absorber system should be attached to that modal coordinate. The tuned vibration absorber is effective in attenuating the resonance. So, it can be argued that the modal response of the system around the resonance is approximately equal to the response of the corresponding physical coordinate provided that the modes are sufficiently separated from each other in terms of resonant frequency.



(a) (b) (c)

Fig. 2. (a) The sdof Den Hartog tuned vibration absorber system; (b) the mdof system with the vibration absorber; (c) the mdof system with vibration absorber in modal domain.

By applying Den Hartog's principle to mdof systems one can write the necessary equations for calculation of stiffness and damping values for the vibration absorber as

$$k_a = \left(\frac{1}{1+\mu}\right)^2 m_a \Omega_r, \quad c_a = 2 m_a \Omega_r \sqrt{\frac{3\mu}{8(1+\mu)^3}}, \tag{31,32}$$

where

$$\mu = \frac{m_a}{M_r}, \quad \Omega_r = \sqrt{\frac{K_r}{M_r}}. \tag{33a,b}$$

In the above equations M_r and K_r are the rth × rth elements of the modal mass and modal stiffness of the main system and "r" is the target mode.

There are two drawbacks to this method: (1) there is the assumption of an undamped main system, and (2) it is necessary to transform the whole system into the modal domain and assume that around a specific resonant frequency the physical system response is dominated by one mode. The first assumption is never valid and the second assumption holds only if the modes are well-separated in terms of their resonant frequency and lightly damped.

4. Application of Sherman–Morrison method to tuned dynamic vibration absorbers

Consider again the system of Fig. 1. The main system has n-dof and a mass, spring and damper system is to be attached to the mth dof to reduce the vibrations of the kth dof around a given frequency. In the sections to come, the methods for finding the stiffness and damper parameters of the tuned vibration absorber through the SM method, Den Hartog's method and a hybrid method will be discussed. It will also be shown that using the newly developed method one can minimize the weighted linear summation of the physical coordinates in the vicinity of a particular resonant frequency.

4.1. Determination of tuned vibration absorber parameters to suppress the response of a physical coordinate using the Sherman–Morrison method

It will be shown that it is possible to set the displacement x_k to zero by adjusting the added system's parameters. At this point assume that the mass of the added system is known and it is necessary to find values for k_a and c_a, which will result in zero displacement amplitude at the kth physical dof. The required k_a and c_a value can be found from Eq. (30) by setting x_k to zero at a specific frequency ω and solving for k_a and c_a, which will result in

$$k_a = \mathrm{Re}\left(\left(\frac{1}{\omega^2 m_a} + \frac{\alpha_{\mathrm{main}_{km}}\alpha_{\mathrm{main}_{im}}}{\alpha_{\mathrm{main}_{ik}}} - \alpha_{\mathrm{main}_{mm}}\right)^{-1}\right), \tag{34}$$

$$c_a = \mathrm{Im}\left(\left(\frac{1}{\omega^2 m_a} + \frac{\alpha_{\mathrm{main}_{km}}\alpha_{\mathrm{main}_{im}}}{\alpha_{\mathrm{main}_{ik}}} - \alpha_{\mathrm{main}_{mm}}\right)^{-1}\right)\frac{1}{\omega}. \tag{35}$$

There are two important observations that can be drawn from the above expressions. One of them is that the response amplitude and phase of the coordinate k can be set to anything and mathematical values of k_a and c_a necessary to achieve that amplitude and phase can be calculated (the values calculated may or may not be physically achievable).

The other observation is that Eqs. (34) and (35) can simplify to Frahm's vibration absorber values for a sdof system. For a sdof system the subscripts 'm', 'k' and 'i' are all equal to each other when the force is applied on the main mass. This results in the following expressions for required k_a and c_a values:

$$k_a = \mathrm{Re}(\omega^2 m_a) = \omega^2 m_a, \quad c_a = \mathrm{Im}(\omega^2 m_a)\frac{1}{\omega} = 0. \tag{36,37}$$

If the response is desired to be zero at the natural frequency of the main system, Eq. (36) can be written as

$$\frac{k_a}{m_a} = \frac{K}{M}, \tag{38}$$

where K and M are the sdof main system's stiffness and damping, respectively. The results in Eqs. (37) and (38) are the values for Frahm's absorber design.

4.2. Determination of tuned vibration absorber parameters that minimize the weighted linear summation of the physical coordinates

Most of the time the equations of motion are obtained using some modal discretization methods such as Rayleigh–Ritz or Galerkin. In order to relate these equations to displacements of physical coordinates one needs to consider weighted linear sums of the coordinates that are used in the discretization method. It is also possible to minimize the force applied to a particular element in the model (spring elements or damper elements). Since the force on these elements can be represented by the weighted linear summation of the physical coordinates (relative displacement or relative velocity between the physical coordinates), using the method described in this section one can minimize the forces applied a particular element in the model.

If the weighted linear sum of the coordinates is to be minimized then the expression to be minimized is

$$J = w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n, \tag{39}$$

or alternatively it may be written as

$$J = \sum_{r=1}^{n} w_r x_r, \tag{40}$$

where w_r is in the most general sense a complex constant.

The objective is to find the k_a and c_a value that set the J expression to zero at a particular frequency ω. Using Eqs. (30) and (43) one can derive the expression which would result in $J = 0$:

$$k_a = \mathrm{Re}\left(\left(\frac{1}{\omega^2 m_a} + \frac{\alpha_{\mathrm{main}_{im}}\left(\sum_{r=1}^{n} w_r \alpha_{\mathrm{main}_{rm}}\right)}{\left(\sum_{r=1}^{n} w_r \alpha_{\mathrm{main}_{ir}}\right)} - \alpha_{\mathrm{main}_{mm}}\right)^{-1}\right), \tag{41}$$

$$c_a = \mathrm{Im}\left(\left(\frac{1}{\omega^2 m_a} + \frac{\alpha_{\mathrm{main}_{im}}\left(\sum_{r=1}^{n} w_r \alpha_{\mathrm{main}_{rm}}\right)}{\left(\sum_{r=1}^{n} w_r \alpha_{\mathrm{main}_{ir}}\right)} - \alpha_{\mathrm{main}_{mm}}\right)^{-1}\right)\frac{1}{\omega}. \tag{42}$$

5. Case studies

Example case studies are provided to demonstrate the effectiveness of the model. A simple four-dof system will be used as shown in Fig. 3. The absorber is to be attached to the mth dof, the

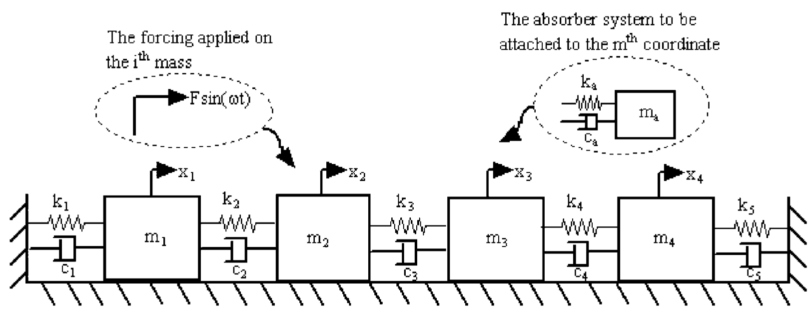


Fig. 3. The four-dof main system and the absorber system used in the case studies.

Table 1
The data table for case studies

	Index				
	1	2	3	4	5
Mass (kg)	1	1	1	1	—
Stiffness (N/m)	1000	2000	50	2000	1000
Damping (Ns/m)	1.5	3	0.075	3	1.5
Natural frequency (rad/s)	20.94	22.36	67.54	67.82	
Mass of absorber (kg)			0.2		
Force (N)			2		

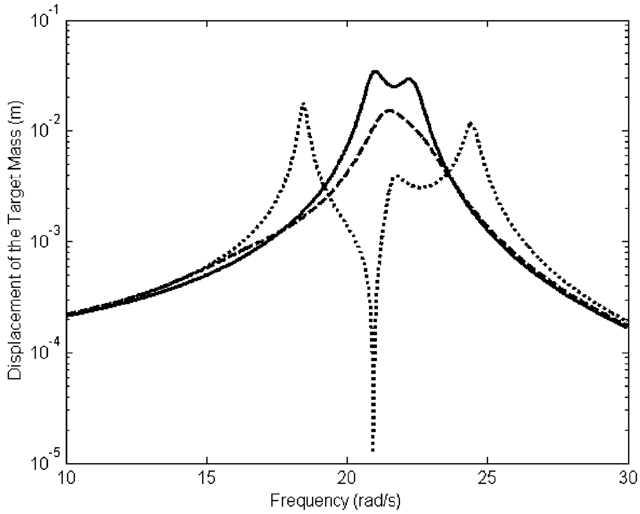


Fig. 4. The response of the target mass: ——, with no absorber; – – –, with absorber using Den Hartog's method ($k_a = 60.89$ N/m, $c_a = 1.75$ Ns/m); - - - -, with absorber using the SM method ($k_a = 87.7$ N/m, $c_a \approx 0$ Ns/m).

forcing is to be applied to the ith dof and the kth dof is the targeted for vibration minimization (Table 1).

5.1. Case study 1: determination of tuned vibration absorber parameters for vibration suppression of the response of a physical coordinate using Sherman–Morrison method

In this part of the study, the necessary tuned vibration parameters will be calculated to minimize the vibration of a particular mass around a target mode. In this case study the target mass is chosen to be mass 1, the target mode is the first mode and the force is applied to the third mass. The absorber is attached to the first mass. In order to find the absorber parameters, Eqs. (34) and (35) should be evaluated at the natural frequency of the first mode (20.94 rad/s). The two

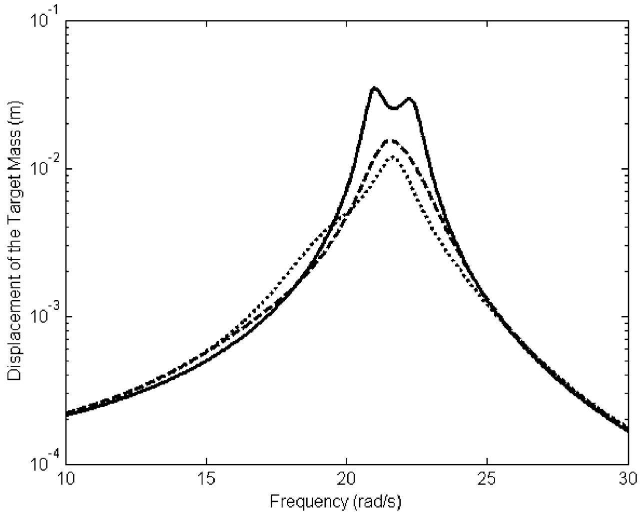


Fig. 5. The response of the target mass: ——, with no absorber; – – –, with absorber using Den Hartog's method ($k_a = 60.89\,\mathrm{N/m}$, $c_a = 1.75\,\mathrm{Ns/m}$); - - - -, with absorber using the SM method with Den Hartog's damping value ($k_a = 87.7\,\mathrm{N/m}$, $c_a = 1.75\,\mathrm{Ns/m}$).

values found will be the vibration absorber parameters that result in zero vibration of the first mass (target mass) at the first resonant frequency.

It can be observed from Fig. 4 that the SM method is very effective at the resonant frequency of the first mode. However, it acts like a narrow-band absorber. Generally, in applications it is required to have more broadband attenuation. It should be noted that to enforce the vibration amplitude to zero, the damping value is almost zero when the SM method is employed. At this point, one can use the Den Hartog's damping value along with the SM method's stiffness value. As shown in Fig. 5, this hybrid method decreases the peaks at the two sides of the resonance peak and it performs better than the Den Hartog method.

The system given in this example has two close modes and there is main system damping. Recall, Den Hartog method's two main assumptions were an undamped main system and distant modes. This system does not satisfy these two conditions and that is why the stiffness and damping values calculated are not optimal in the sense of Den Hartog (the two invariant points do not have the same amplitude). While employing the SM method the damping value used is the damping value of the Den Hartog method. So, one way to come up with better estimates of the damping value might be a numerical search for the damping value while the stiffness estimate remains the same. This numerical search basically scans all the frequencies around the first mode for different values of damping and chooses the damping value, which gives the smallest resonant value. Since the value for stiffness is fixed the numerical burden is not as intense as a two-dimensional search. The result of this method is given in Fig. 6.

The plot in Fig. 6 shows that only a minor performance improvement is possible by changing the damping value for the Den Hartog case. The reason for this is that using the Den Hartog's method the stiffness estimate is far from being optimal and different damping values do not

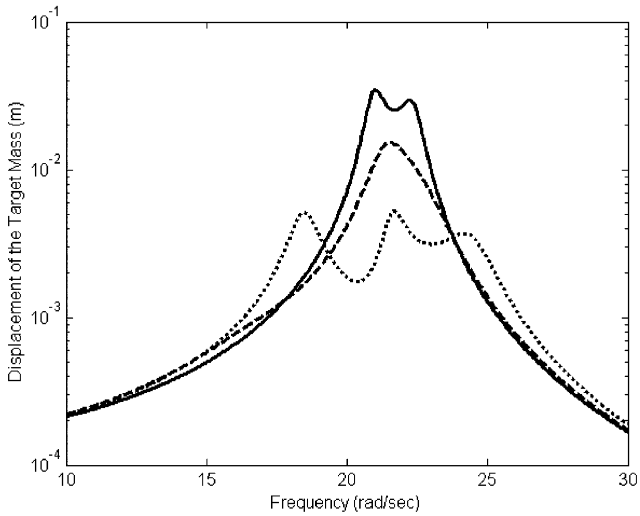


Fig. 6. The response of the target mass: ——, with no absorber; – – –, with absorber using Den Hartog's method with numerically found damping value ($k_a = 60.89$ N/m, $c_a = 1.43$ Ns/m); - - - -, with absorber using the SM method with numerically calculated damping value ($k_a = 87.7$ N/m, $c_a = 0.384$ Ns/m).

increase the performance appreciably. However, the performance improvement in the SM method is much more because its stiffness estimate is calculated taking the main system damping into consideration. Another reason why the SM stiffness estimate is better than the Den Hartog stiffness estimate is due to closely spaced modes. Recall, the Den Hartog method transfers the system into the modal domain and tries to minimize the corresponding mode. That approach may give unsatisfactory results when more than one mode is effective in determining the physical coordinates response, which is the case in this problem. However, the SM approach deals with the problem in the physical domain and the close spacing of the modes is not a disadvantage for the calculation of the stiffness estimate.

5.2. *Case study 2: determination of tuned vibration absorber parameters that minimize the weighted linear summation of the coordinates*

In this case study a weighted linear sum of the dof will be minimized. The same data and the system of case study 1 are used. The performance index that is to be minimized is given as

$$J = 0.15x_1 + 0.35x_2 + 0.35x_3 + 0.15x_4. \tag{43}$$

As in the previous case the vibration absorber is to be attached to the first mass, the force is to be applied on the third mass and the performance index is to be minimized around the first mode. The responses with and without the absorbers are shown in Figs. 7 and 8. It can be seen from Fig. 8 that the SM method gives more superior results than the Den Hartog method. The damping values used for the case study in Fig. 7 are calculated using the Den Hartog method. The damping values are also calculated using the numerical method and the case study with numerically calculated damping values is shown in Fig. 8.

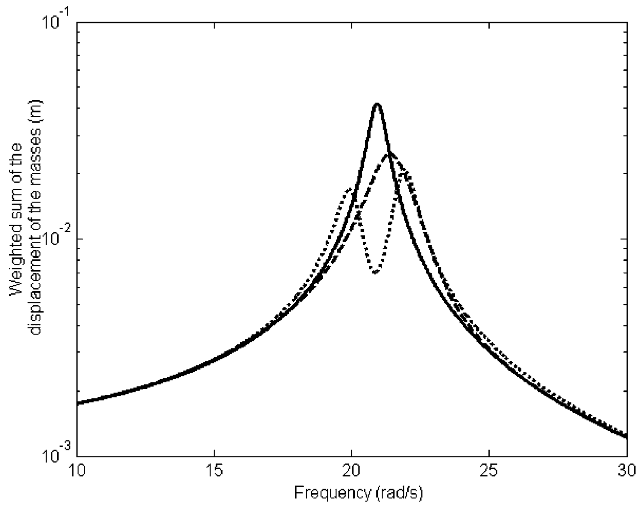


Fig. 7. The response of the target mass: ——, with no absorber; – – –, with absorber using Den Hartog's method ($k_a = 60.89\,\text{N/m}$, $c_a = 1.75\,\text{Ns/m}$); - - - -, with absorber using the SM method with Den Hartog's damping value ($k_a = 173.15\,\text{N/m}$, $c_a = 1.75\,\text{Ns/m}$).

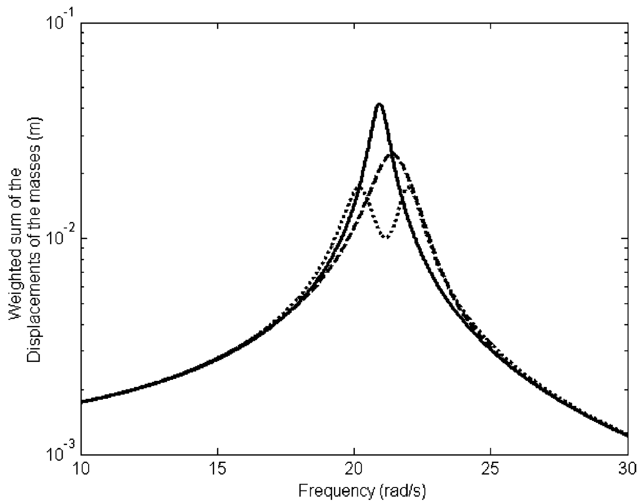


Fig. 8. The response of the target mass: ——, with no absorber; – – –, with absorber using Den Hartog's method with numerically found damping value ($k_a = 60.89\,\text{N/m}$, $c_a = 4.47\,\text{Ns/m}$); - - - -, with absorber using the SM method with numerically calculated damping value ($k_a = 173.15\,\text{N/m}$, $c_a = 6.049\,\text{Ns/m}$).

One can see that the SM method gives superior results compared to Den Hartog's method and the use of a numerical method to calculate the damping value gives slightly better results for the two methods. If one compares the SM plots of Figs. 7 and 8 it can be seen that the use of the numerical method in the damping value calculation equates the amplitudes of the two peaks compared to the uneven peaks of Fig. 7.

6. Conclusions

A new approach has been introduced for writing the response expression of a system where a sdof mass–spring–damper system is attached to a mdof main system. It has been shown that the necessary parameter values of the sdof system can be calculated for a desired vibration amplitude and phase of a particular dof of the main system. This new expression is utilized to find the parameters of a damped vibration absorber attached to a mdof damped system. The new method can also be used for minimization of the linear combination of the dof (which also can be used to minimize the force on a particular element in the model such as a spring or a damper). In all cases the estimates calculated using the introduced method gave better results than the estimates calculated using a modal based Den Hartog method.

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