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Journal of Sound and Vibration 283 (2005) 1250–1256
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Largest Lyapunov exponent for second-order linear systems under combined harmonic and random parametric excitations

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Received 5 January 2004; accepted 25 July 2004
Available online 23 November 2004

Abstract

The principal resonance of a second-order linear stochastic oscillator to combined harmonic and random parametric excitations is investigated. The method of multiple scales is used to determine the equations of modulation of amplitude and phase. The effects of damping, detuning, bandwidth, and magnitudes of random excitation are analyzed. The method of path integration is used to obtain the steady state probability density function of the system, and then the largest Lyapunov exponent is calculated. The almost-sure stability or instability of the stochastic system depends on the sign of the largest Lyapunov exponent. The theoretical analyses are verified by numerical results.

1. Introduction

Loadings imposed on the structures are quite often random forces, such as those arising from earthquakes, wind and ocean waves, which can be described satisfactorily only in probabilistic terms. The response of the structure is governed by the stochastic differential equations, in which the parameters or coefficients are stochastic processes. Investigations of stability under parametric random excitation have become increasingly important.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.07.025

According to the multiplicative ergodic theorem of Oseledec [1], the almost certain stability of the trivial solution of a system can be determined by the largest Lyapunov $\lambda = \lambda_{\max}$, i.e., when $\lambda < 0$ the trivial solution is almost certainly stable and when $\lambda > 0$ the trivial one is unstable. There are some studies on the calculation of the largest Lyapunov exponent under stochastic excitations [2–8]. However, the result is quite limited under combined harmonic and stochastic excitations [9]. In this paper, the principal resonance of a second-order linear stochastic system to combined harmonic and random parametric excitation is investigated. The method of multiple scales is used to determine the equations of modulation of amplitude and phase. The effects of damping, detuning, bandwidth, and magnitudes of random excitation are analyzed. The method of path integration is used to obtain the steady-state probability density function of the system, and then the largest Lyapunov exponent is calculated. The almost-sure stability or instability of the stochastic system depends on the sign of the largest Lyapunov exponent.

Consider the following second-order system parametric excited by combined harmonic and random excitations:

$$\ddot{u} + \varepsilon\beta\dot{u} + \omega_0^2 u + \varepsilon u(k \cos \Omega_1 t + \xi(t)) = 0, \tag{1}$$

where dots indicate differentiation with respect to the time t, $\varepsilon \ll 1$ is a small parameter, β and ω_0 are damping coefficient and natural frequency, respectively, and $\xi(t)$ is a stochastic process which is governed by the following equation advanced by Wedig [4]:

$$\xi(t) = h \cos(\Omega_2 t + \bar{\gamma} W(t)),$$

where $W(t)$ is a standard Wiener process. According to Wedig [4], in the case when $h = \bar{\gamma}/\sqrt{2} \to \infty$, $\xi(t)$ may represent a wide-band noise, and in the case when $\bar{\gamma} \to 0$ $\xi(t)$ may represent a narrow-band random noise.

For $h = 0$, the parametric excitations are only the deterministic harmonic ones; in this case system (1) goes over to the well-known Mathieu equation, and there are many well-established theories [10,11] for the stability of the trivial solution of system (1). For $k = 0$, the parametric excitations are only the random ones, in this case the invariant measures and largest Lyapunov exponent of system (1) have been evaluated by Wedig [4] using numerical simulation and perturbation method, Dimentberg [5] and Huang and Zhu [6] using stochastic averaging method, and the authors of this paper [7] using the multiple scales method. The moment Lyapunov exponents of system (1) have been studied by Xie [8] using the regular perturbation method in the case when $k = 0$, recently. However, the largest Lyapunov exponent of system (1) under combined harmonic and random parametric excitations has not been evaluated in the case for $k \neq 0$, $h \neq 0$.

2. Multiple scales method

The method of multiple scales [10,11], which has been widely used in the analysis of deterministic systems, has been extended to the analysis of nonlinear stochastic systems in recent years [7,12–14]. In this paper, the multiple scales method is used to investigate the response and stability of system (1). Then, a uniformly approximate solution of Eq. (1) is sought in the form

$$u(t,\varepsilon) = u_0(T_0,T_1) + \varepsilon u_1(T_0,T_1) + \cdots, \tag{2}$$

where $T_0 = t, T_1 = \varepsilon t$ are fast and slow scales respectively.

By denoting $D_0 = \partial/\partial T_0, D_1 = \partial/\partial T_1$, the ordinary-time derivatives can be transformed into partial derivatives as

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots. \tag{3}$$

Substituting Eq. (2) and (3) into Eq. (1) and comparing coefficients of ε with equal powers, one obtains the following equations:

$$D_0^2 u_0 + \omega_0^2 u_0 = 0, \tag{4}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - \beta D_0 u_0 - k u_0 \cos \Omega_1 t - h u_0 \cos(\Omega_2 t + \bar{\gamma} W(t)). \tag{5}$$

The general solution of Eq. (4) can be written as

$$u_0(T_0,T_1) = A(T_1)\exp(\mathrm{i}\omega_0 T_0) + \mathrm{cc}, \tag{6}$$

where cc represents the complex conjugate of its preceding terms, and $A(T_1)$ is the slowly varying amplitude of the response. Substituting Eq. (6) into Eq. (5), one obtains

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 = & -2\mathrm{i}\omega A' \exp(\mathrm{i}\omega_0 T_0) - \mathrm{i}\omega_0 \beta A \exp(\mathrm{i}\omega_0 T_0) \\ & - \frac{k}{2} A \exp[\mathrm{i}(\Omega_1 + \omega_0)T_0] - \frac{k}{2} \bar{A} \exp[\mathrm{i}(\Omega_1 - \omega_0)T_0] \\ & - \frac{h}{2} A \exp[\mathrm{i}(\Omega_2 + \omega_0)T_0 + \gamma W(T_1)] \\ & - \frac{h}{2} \bar{A} \exp[\mathrm{i}(\Omega_2 - \omega_0)T_0 + \gamma W(T_1)] + \mathrm{cc}, \end{aligned} \tag{7}$$

where the prime stands for the derivative with respect to T_1 and the overbar stands for the complex conjugate, $\gamma = \bar{\gamma}/\sqrt{\varepsilon}$. For Wiener progress $W(t)$, $EW(t) = 0$, $EW^2(t) = t$, one has

$$\bar{\gamma} W(t) = (\bar{\gamma}/\sqrt{\varepsilon}) W(\varepsilon t) = \gamma W(T_1).$$

From the fourth and sixth terms on the right-hand side of Eq. (7), it is clear that resonance occurs when $\Omega_1 \approx 2\omega_0, \Omega_2 \approx 2\omega_0$. In what follows we shall investigate the principal resonances of system (1). To express quantitatively the nearness of these resonances, one introduces the detuning parameters σ_1 and σ_2 according to $\Omega_1 = 2\omega_0 + \varepsilon\sigma_1$, $\Omega_2 = 2\omega_0 + \varepsilon\sigma_2$. One has

$$(\Omega_1 - \omega_0)T = \omega_0 T_0 + \sigma_1 T_1, \;\; (\Omega_2 - \omega_0)T = \omega_0 T_0 + \sigma_2 T_1.$$

Using the above equation, we can transform the small-divisor terms, which arise from $\exp[\mathrm{i}(\Omega_1 - \omega_0)T]$ and $\exp[\mathrm{i}(\Omega_2 - \omega_0)T]$ in Eq. (7) into secular terms. Then, eliminating the secular terms yields

$$2\mathrm{i}\omega_0 A' + \mathrm{i}\beta\omega_0 A + \frac{k}{2}\bar{A}\exp(\mathrm{i}\sigma_1 T_1) + \frac{h}{2}\bar{A}\exp(\mathrm{i}\sigma_2 T_1 + \mathrm{i}\gamma W(T_1)) = 0. \tag{8}$$

Expressing A in the polar form $A(T_1) = a(T_1)\exp[i\varphi(T_1)]$, substituting this equation into Eq. (8) and separating the real and imaginary parts of Eq. (8), one obtains

$$\begin{aligned} a' &= -\frac{\beta}{2}a - \frac{k}{4\omega_0}a\sin\eta_1 - \frac{h}{4\omega_0}a\sin\eta_2, \\ a\eta_1' &= \sigma_1 a - \frac{k}{2\omega_0}a\cos\eta_1 - \frac{h}{2\omega_0}a\cos\eta_2, \\ a\eta_2' &= \sigma_2 a - \frac{k}{2\omega_0}a\cos\eta_1 - \frac{h}{2\omega_0}a\cos\eta_2 - \gamma a W'(T_1), \end{aligned} \tag{9}$$

where

$$\eta_1 = \sigma_1 T_1 - 2\varphi, \;\; \eta_2 = \sigma_2 T_1 - 2\varphi + \gamma W(T_1)$$

3. Largest Lyapunov exponent

It is obvious that Eq. (9) have a solution $a = 0$, which corresponds to the trivial steady-state response. Now we discuss its stability. Let $v = \ln a$; Eq. (9) can be written as:

$$\begin{aligned} dv &= \left(-\frac{\beta}{2} - \frac{k}{4\omega_0}\sin\eta_1 - \frac{h}{4\omega_0}\sin\eta_2\right) dT_1, \\ d\eta_1 &= \left(\sigma_1 - \frac{k}{2\omega_0}\cos\eta_1 - \frac{h}{2\omega_0}\cos\eta_2\right) dT_1, \\ d\eta_2 &= \left(\sigma_2 - \frac{k}{2\omega_0}\cos\eta_1 - \frac{h}{2\omega_0}\cos\eta_2\right) dT_1 - \gamma\, dW. \end{aligned} \tag{10}$$

It is clear that the stochastic processes $(\eta_1(T_1), \eta_2(T_1))$ generated on $[0, 2\pi] \times [0, 2\pi]$ by Eq. (10) are Markov, and since the diffusion process is non-singular, they are ergodic on the $[0, 2\pi] \times [0, 2\pi]$. The invariant measure (steady-state probability density function) $p(\eta_1, \eta_2)$ of the processes $(\eta_1(T_1), \eta_2(T_1))$ is governed by the following *FPK* equation:

$$\frac{\partial^2 p}{\partial \eta_2^2} - \frac{\partial}{\partial \eta_1}\left[(\bar{\sigma}_1 - \bar{k}\cos\eta_1 - \bar{h}\cos\eta_2)p\right] - \frac{\partial}{\partial \eta_2}\left[(\bar{\sigma}_2 - \bar{k}\cos\eta_1 - \bar{h}\cos\eta_2)p\right] = 0, \tag{11}$$

where

$$\bar{\sigma}_1 = \frac{2\sigma_1}{\gamma^2}, \quad \bar{\sigma}_2 = \frac{2\sigma_2}{\gamma^2}, \quad \bar{h} = \frac{h}{\omega_0\gamma^2}.$$

The unique solution satisfying both the periodicity condition

$$p(\eta_1, \eta_2) = p(\eta_1 + 2\pi, \eta_2) = p(\eta_1, \eta_2 + 2\pi)$$

and normality condition $\int_0^{2\pi}\int_0^{2\pi} p(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 = 1$, respectively.

Eq. (11) generally can be solved only numerically. The method of path integration is one such numerical procedure and it is appropriate for the present purpose. Early application of the path integration to solving *FPK* equation was made by Wehner and Wolfer [15], and recent improvements of the technique can be found in Ref. [16]. According to Oseledec's multiplicative ergodic theorem [1], the exponential growth rate (i.e., the Lyapunov exponent) of the corresponding solution $a(T_1; a_0, \eta_0)$ of Eq. (9) for any initial values (a_0, η_0) is given by

$$\lambda(a_0, \eta_0) = \lim_{T_1 \to \infty} \frac{1}{T_1} \ln\left|a\left(T_1; a_0, \eta_0\right)\right|, \quad \text{w.p.1}$$

where w.p.1 means with probability one (almost sure). $\lambda(a_0, \eta_0)$ can take only the following deterministic values:$\lambda_{\min} = \lambda_2 < \lambda_1 = \lambda_{\max}$. The almost certain stability of the trivial solution (9) can be determined by the largest Lyapunov $\lambda = \lambda_{\max}$, i.e., when $\lambda < 0$ the trivial solution is almost certainly stable and when $\lambda > 0$ the trivial one is unstable, hence $\lambda = 0$ is the bifurcation point of the stability of the trivial solution. From Eq. (10), one has

$$\begin{aligned} \lambda &= \lim_{T_1 \to \infty} \frac{1}{T_1} \ln\left|\frac{a(T_1)}{a(0)}\right| = \lim_{T_1 \to \infty} \frac{1}{T_1}(v(T_1) - v(0)) \\ &= -\frac{\beta}{2} - \lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \left[\frac{k}{4\omega_0} \sin \eta_1(\tau) + \frac{h}{4\omega_0} \sin \eta_2(\tau)\right] \mathrm{d}\tau \\ &= -\frac{\beta}{2} - \frac{k}{4\omega_0} E\left[\sin \eta_1\right] - \frac{h}{4\omega_0} E\left[\sin \eta_2\right] \\ &= -\frac{\beta}{2} - \frac{k}{4\omega_0} \int_0^{2\pi} \int_0^{2\pi} p(\eta_1, \eta_2) \sin \eta_1 \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 - \frac{h}{4\omega_0} \int_0^{2\pi} \int_0^{2\pi} p(\eta_1, \eta_2) \sin \eta_2 \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2. \end{aligned} \quad (12)$$

Herein, the steady-state probability density function $p(\eta_1, \eta_2)$ can be solved numerically from Eq. (11) by the method of path integration; then, the largest Lyapunov λ can be solved numerically from Eq. (12).

4. Numerical results and conclusions

For the first representative case $\beta = 0.0$, $\omega_0 = 1.0, \sigma_1 = k = 0.0$, in which the parametric excitations are only the random ones, the variations of λ governed by Eq. (12) with σ_2 and h are shown in Fig. 1.

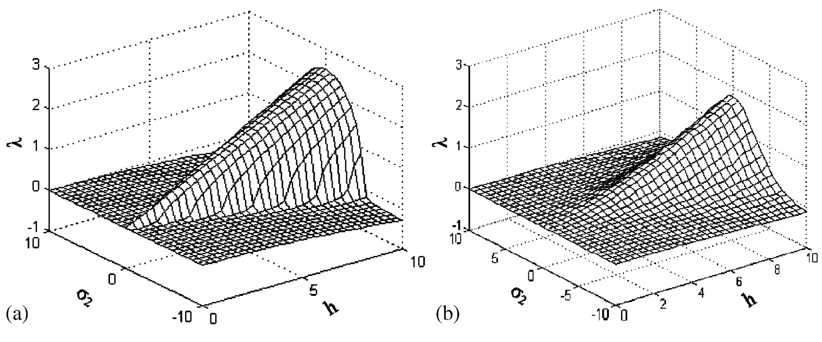

Fig. 1. Largest Lyapunov exponent of system (10): (a) $\gamma = 0.1$; (b) $\gamma = 2.0$.

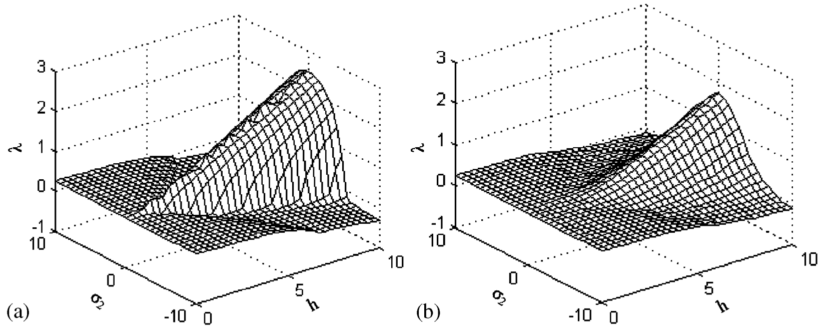

Fig. 2. Largest Lyapunov exponent of system (10): (a) $\gamma = 0.1$; (b) $\gamma = 2.0$.

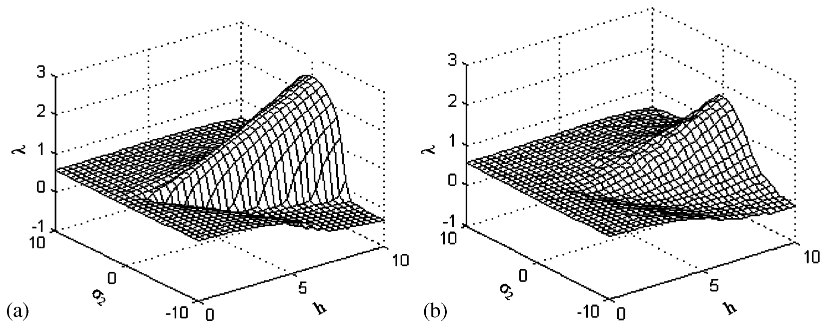

Fig. 3. Largest Lyapunov exponent of system (10): (a) $\gamma = 0.1$; (b) $\gamma = 2.0$.

Fig. 1 shows three-dimensional plots of the Lyapunov number λ over the parameter range $0 \leqslant h \leqslant 10$ and $-10 \leqslant \sigma_2 \leqslant 10$. There are obviously two different solution ranges for the Lyapunov exponents. Near the parameter resonance at excitation frequency $\Omega_2 = 2\omega_0$, the Lyapunov exponents increase, reaching their maximum values in the center of the instability region. Outside the mountain, there exists a complete plane where the Lyapunov exponents possess the constant value $\lambda = 0$. Herein, they are independent of frequency Ω_2 and amplitude h of the random parameter excitation. Obviously, the sharp separation between both parameter ranges is smoothed out for increasing frequency fluctuation.

For the second representative case $\beta = 0.0$, $\omega_0 = 1.0, \sigma_1 = 0.0, k = 1.0$, in which the parametric excitations are combined harmonic and random ones, the variations of λ governed by Eq. (12) are shown in Fig. 2. There is a mountain in Fig. 2, which is similar to Fig. 1. However, outside the mountain, there is not a complete plane in Fig. 2, which is different from Fig. 1. Near the area of a small value of h, the Lyapunov exponent is greater than zero, which means that the deterministic excitation makes the system attain almost sure instability. However, in the area of big value of $h > 7$, the Lyapunov exponents (outside the mountain) is smaller than zero, which means that the random excitation helps the system become stability. It is something interesting that the random noise can sometimes stabilize the system.

For the third representative case $\beta = 0.0$, $\omega_0 = \gamma = 1.0, \sigma_1 = 1.0, k = 3.0$, the variations of λ governed by Eq. (12) are shown in Fig. 3.

For the first time, the largest Lyapunov exponent of the system under combined harmonic and stochastic bounded excitations is calculated. Numerical calculation shows that λ is a decreasing function of $|\sigma_1|, |\sigma_2|$, and reaches its maximum value when $\sigma_1 = \sigma_2 = 0$, which means that the trivial solution will lose its stability and become unstable as the frequencies of the harmonic and random excitations are near the principal resonance frequencies $\Omega_1 = \Omega_2 = 2\omega_0$. In some parameter areas, the random noise can stabilize the system.

Acknowledgements

The work reported in this paper was supported by the National Natural Science Foundation of China through Grant No. 10332030, the Natural Science Foundation of Guangdong Province through Grant Nos. 034071 and 04011640, and the Open Fund of the State Key Laboratory of Vibration, Shock and Noise, Shanghai Jiaotong University through Grant No. VSN-2002-04.

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