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An artificial small perturbation parameter and nonlinear plate vibrations

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Abstract

Nonlinear natural in-plane vibrations of a rectangular plate are studied using three small parameters. Firstly, the nonlinearity is assumed to be small. Then, a solution to a problem of the zeroth order (linear) is sought in the form of an asymptotic series with respect to the ratio of stiffness characteristics.

For internal resonance, vibration modes are coupled via an infinite system of nonlinear algebraic equations, and the artificial small parameter approach is proposed to solve the obtained system. Analytical formulas for the amplitude–frequency characteristics are derived and the solutions are compared with numerical results.

1. Introduction

Determination of dynamic characteristics of a raft foundation, i.e. amplitudes and frequencies of vibrations, belongs to challenging tasks in the civil engineering. When known, the dynamic characteristics enable a significant decrease in both foundation vibrations and their interaction with the other two construction members, which in turn may eventually lead to a reduced need for the working personnel. Note that computations are more complicated when they include nonlinear characteristics of a foundation. In general, analytical approaches make it possible to

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take into account only the first few vibration modes. On the other hand, owing to nonlinearities various internal resonances between modes may occur. Moreover, in certain cases neglect of higher modes may yield essential errors [1]. In order to omit these drawbacks, the present work proposes a new asymptotic approach to solving nonlinear vibration problems in continuous systems, in which all modes are subject to approximation. Free vibrations of a plate within a nonlinear and elastic external medium and in conditions of in-plane deformations are studied, to finally obtain approximate relations of amplitude–frequency characteristics.

Natural in-plane vibrations of a rectangular plate $0 \leqslant x \leqslant l_1$, $0 \leqslant y \leqslant l_2$ with clamped edges surrounded by a nonlinear elastic medium are considered. The governing equations have the following form:

$$B\frac{\partial^2 u}{\partial x^2} + B_1\frac{\partial^2 u}{\partial y^2} + B_2\frac{\partial^2 \upsilon}{\partial x \partial y} - \frac{\partial^2 u}{\partial t^2} - \beta_1 u - \varepsilon\beta_2 u^3 = 0, \tag{1}$$

$$B\frac{\partial^2 \upsilon}{\partial y^2} + B_1\frac{\partial^2 \upsilon}{\partial x^2} + B_2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 \upsilon}{\partial t^2} - \beta_1 \upsilon - \varepsilon\beta_2 \upsilon^3 = 0, \tag{2}$$

where $u(\upsilon)$ are displacements in the directions Ox (Oy), respectively; $B = E/\left(\rho\left(1 - v^2\right)\right)$, E is Young's modulus, v is Poisson's coefficient, ρ is a plate material density, $B_1 = G/\rho$, G is a shear modulus, $B_2 = ((Ev/(1 - v^2)) + G)/\rho$, and ε is a small parameter ($\varepsilon \ll 1$).

The following boundary conditions are attached:

$$u = \upsilon = 0 \quad \text{for } x = 0, l_1; \quad y = 0, l_2. \tag{3}$$

The defined situation can occur when a plate is located between elastic rough surfaces, like in a so-called sheet piling (Fig. 1) where the plate edges are clamped stiffly. Since generally an external load is distributed periodically, a periodic solution is sought.

A periodic solution searched for should satisfy the following periodical conditions:

$$u(x, y, t) = u(x, y, t + T), \quad \upsilon(x, y, t) = \upsilon(x, y, t + T). \tag{4}$$

Nonlinear terms in the in-plane deformation equations are neglected, but the nonlinear medium deformation is taken into account. The described model can be applied when the stiffness characteristics of the plate and the surrounding medium differ essentially. An influence of dissipative factors is omitted in our considerations.

2. Asymptotic procedure

In order to solve boundary value problem (1)–(4), an asymptotic approach is applied. Out of three small parameters used in the study, the first one applied is parameter ε assumed to be equal to 0 in the zeroth-order approximation. In the linear case, eigenfrequencies corresponding to fundamental modes, which are to be found, are realized as half-waves in both x and y directions. Let us transform the time via the formula

$$\tau = \omega t. \tag{5}$$

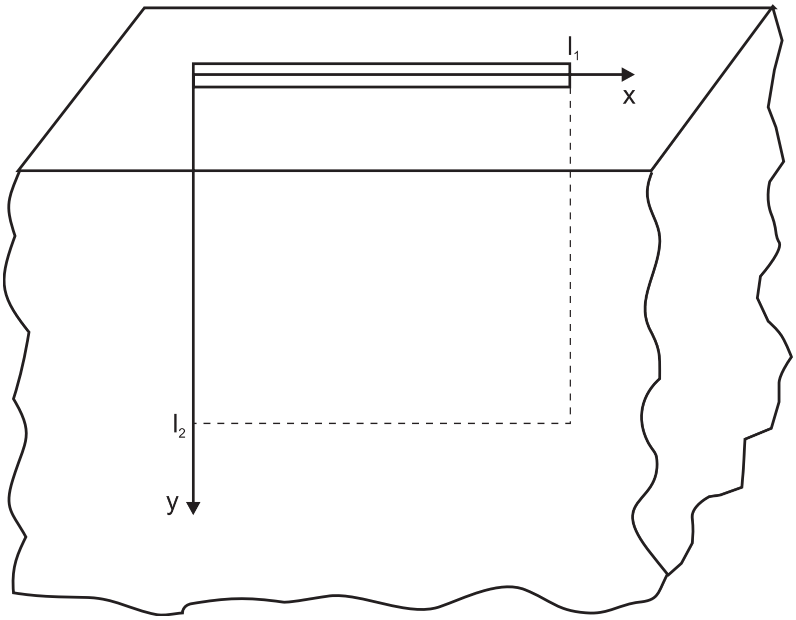

Fig. 1. Governing model: sheet piling.

The solution is sought in the form of the asymptotic series:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \tag{6}$$

$$\upsilon = \upsilon_0 + \varepsilon \upsilon_1 + \varepsilon^2 \upsilon_2 + \dots, \tag{7}$$

$$\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots . \tag{8}$$

Substituting Eqs. (5)–(8) into Eqs. (1), (2) and conditions (3)–(5), and then equating the terms with the same ε powers, the following recurrent system of equations is obtained:

$$B\frac{\partial^2 u_0}{\partial x^2} + B_1\frac{\partial^2 u_0}{\partial y^2} + B_2\frac{\partial^2 \upsilon_0}{\partial x \partial y} - \omega_0^2\frac{\partial^2 u_0}{\partial \tau^2} - \beta_1 u_0 = 0, \tag{9}$$

$$B\frac{\partial^2 \upsilon_0}{\partial y^2} + B_1\frac{\partial^2 \upsilon_0}{\partial x^2} + B_2\frac{\partial^2 u_0}{\partial x \partial y} - \omega_0^2\frac{\partial^2 \upsilon_0}{\partial \tau^2} - \beta_1 \upsilon_0 = 0, \tag{10}$$

$$B\frac{\partial^2 u_1}{\partial x^2} + B_1\frac{\partial^2 u_1}{\partial y^2} + B_2\frac{\partial^2 \upsilon_1}{\partial x \partial y} - \omega_0^2\frac{\partial^2 u_1}{\partial \tau^2} - \beta_1 u_1 = 2\omega_0\omega_1\frac{\partial^2 u_0}{\partial \tau^2} + \beta_2 u_0^3, \tag{11}$$

$$B\frac{\partial^2 \upsilon_1}{\partial y^2} + B_1\frac{\partial^2 \upsilon_1}{\partial x^2} + B_2\frac{\partial^2 u_1}{\partial x \partial y} - \omega_2^2\frac{\partial^2 \upsilon_1}{\partial \tau^2} - \beta_1 \upsilon_1 = 2\omega_0\omega_1\frac{\partial^2 \upsilon_0}{\partial \tau^2} + \beta_2 \upsilon_0^3. \tag{12}$$

Boundary conditions (3) and periodicity conditions (4) and (5) take the form

$$u_i|_{x=0,l_1} = u_i|_{y=0,l_2} = 0, \quad \upsilon_i|_{y=0,l_1} = \upsilon_i|_{x=0,l_2} = 0, \tag{13}$$

$$u_i(x, y, \tau) = u_i(x, y, \tau + 2\pi), \tag{14}$$

$$\upsilon_i(x, y, \tau) = \upsilon_i(x, y, \tau + 2\pi), \quad i = 0, 1, 2 \ldots . \tag{15}$$

3. Zeroth order approximation

To find an approximation of the zeroth order with respect to small parameter ε, the boundary value problem (9), (10), (13)–(15) should be solved. It is worth noting that for this problem variables cannot be separated. The Galerkin approach may be used, but, on the other hand, the problem, which is governed by PDEs of the fourth order with respect to special variables, would become essentially simpler if it could be described by PDEs of the second order instead. In what follows, the second asymptotic procedure is applied in order to reduce the input PDEs of the fourth order to two PDEs of the second order. The parameter $\delta = B/B_2 < 1$ serves as a small parameter.

In the aircraft and rocket designing [2,3], as well as in the theory of composites [4–7], the following intuitively clear simplification of the plane problem of elasticity is widely used: if a load acts in the direction of Ox (Oy), then one may take $\upsilon = 0\,(u = 0)$. The asymptotic character of the described simplifications is well clarified in Refs. [8–11]; Refs. [10–13] describe construction of singular asymptotics, and a rigorous mathematical treatment is addressed in Refs. [14,15]. Unfortunately, nowhere does this engineering approach refer to the questions related to the solution accuracy improvement, formulation of boundary conditions, etc.

Investigations that have been carried out show that as far as practical purposes are concerned the introduced simplifications exhibit a sufficient accuracy. As an example, consider a static model of concentrated force acting on an elastic half-plane. In this case both exact and approximate solutions have a similar structure. Namely, in order to obtain the approximate solution, intensity coefficient 2 in the exact solution should be substituted by 1.69. Since in practice all coefficients are generally burdened with their estimation errors, the introduced approximation seems to be an appropriate one.

Our problem will be dealt with using the approach described in Refs. [11,16,17]. First, the following transformation of the coordinates is introduced:

$$x = \sqrt{B}x_1, \quad y = \sqrt{B_1}y_1, \tag{16}$$

$$x = \sqrt{B_1}x_2, \quad y = \sqrt{B}y_2. \tag{17}$$

On substituting Eq. (16) into Eqs. (9), (13), and Eq. (17) into Eqs. (10), (13), boundary value problem (9), (10), (13)–(15) takes the form

$$\frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial y_1^2} + \delta \frac{\partial^2 \upsilon_0}{\partial x_1 \partial y_1} - \omega_0^2 \frac{\partial^2 u_0}{\partial \tau^2} - \beta_1 u_0 = 0, \tag{18}$$

$$\frac{\partial^2 \upsilon_0}{\partial y_2^2}+\frac{\partial^2 \upsilon_0}{\partial x_2^2}+\delta\frac{\partial^2 u_0}{\partial x_2 \partial y_2}-\omega_0^2\frac{\partial^2 \upsilon_0}{\partial \tau^2}-\beta_1\upsilon_0=0, \tag{19}$$

$$\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial y_1^2}+\delta\frac{\partial^2 \upsilon_1}{\partial x_1 \partial y_1}-\omega_0^2\frac{\partial^2 u_1}{\partial \tau^2}-\beta_1 u_1=2\omega_0\omega_1\frac{\partial^2 u_0}{\partial \tau^2}+\beta_2 u_0^3, \tag{20}$$

$$\frac{\partial^2 \upsilon_1}{\partial y_2^2}+\frac{\partial^2 \upsilon_1}{\partial x_2^2}+\delta\frac{\partial^2 u_1}{\partial x_2 \partial y_2}-\omega_0^2\frac{\partial^2 \upsilon_1}{\partial \tau^2}-\beta_1\upsilon_1=2\omega_0\omega_1\frac{\partial^2 \upsilon_0}{\partial \tau^2}+\beta_2\upsilon_0^3, \tag{21}$$

$$u_i|_{x_1=0,l_1/\sqrt{B}}=u_i|_{y_1=0,l_2/\sqrt{G}}=u_i|_{x_2=0,l_1/\sqrt{B_1}}=u_i|_{y_2=0,l_2/\sqrt{B}}=0, \tag{22}$$

$$\upsilon_i|_{y_1=0,l_2/\sqrt{B_1}}=\upsilon_i|_{x_1=0,l_1/\sqrt{B}}=\upsilon_i|_{y_2=0,l_2/\sqrt{B}}=\upsilon_i|_{x_2=0,l_1/\sqrt{B_1}}=0, \tag{23}$$

$$u_i(x_1,y_{1,},\tau)=u_i(x_1,y_{1,},\tau+2\pi), \tag{24}$$

$$\upsilon_i(x_2,y_2,\tau)=\upsilon_i(x_2,y_2,\tau+2\pi). \tag{25}$$

Since usually $\delta<1$, δ may be treated as a small parameter. A solution to boundary value problem (18)–(25) is sought in the form of the asymptotic series:

$$u_i=u_{i,0}+\delta u_{i,1}+\delta^2 u_{i,2}+\ldots,\quad \upsilon_i=\upsilon_{i,0}+\delta\upsilon_{i,1}+\delta^2\upsilon_{i,2}+\ldots.$$

Taking into account only the approximations of the zeroth order with respect to δ, one arrives at input variables:

$$B\frac{\partial^2 u_{0,0}}{\partial x^2}+B_1\frac{\partial^2 u_{0,0}}{\partial y^2}-\omega_0^2\frac{\partial^2 u_{0,0}}{\partial \tau^2}-\beta_1 u_{0,0}=0, \tag{26}$$

$$B\frac{\partial^2 \upsilon_{0,0}}{\partial y^2}+B_1\frac{\partial^2 \upsilon_{0,0}}{\partial x^2}-\omega_0^2\frac{\partial^2 \upsilon_{0,0}}{\partial \tau^2}-\beta_1\upsilon_{0,0}=0, \tag{27}$$

$$B\frac{\partial^2 u_{1,0}}{\partial x^2}+B_1\frac{\partial^2 u_{1,0}}{\partial y^2}-\omega_0^2\frac{\partial^2 u_{1,0}}{\partial \tau^2}-\beta_1 u_{1,0}=2\omega_0\omega_1\frac{\partial^2 u_{0,0}}{\partial \tau^2}+\beta_2 u_{0,0}^3, \tag{28}$$

$$B\frac{\partial^2 \upsilon_{1,0}}{\partial y^2}+B_1\frac{\partial^2 \upsilon_{1,0}}{\partial x^2}-\omega_2^2\frac{\partial^2 \upsilon_{1,0}}{\partial \tau^2}-\beta_1\upsilon_{1,0}=2\omega_0\omega_1\frac{\partial^2 \upsilon_{0,0}}{\partial \tau^2}+\beta_2\upsilon_{0,0}^3, \tag{29}$$

$$u_{i,0}\big|_{x=0,l_1}=0,\quad u_{i,0}\big|_{y=0,l_2}=0, \tag{30}$$

$$\upsilon_{i,0}\big|_{x=0,l_1}=0,\quad \upsilon_{i,0}\big|_{x=0,l_2}=0. \tag{31}$$

Further problems for functions $u_{i,0}$ and $\upsilon_{i,0}$ will be analyzed separately.

For Eqs. (26) and (30) the solution reads

$$u_{0,0}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{1(m,n)}\sin\left(\frac{\omega_{1(m,n)}^{\text{lin}}}{\omega_{1,0}}\tau_1\right)\sin\left(\frac{\pi m\sqrt{B}}{l_1}x_1\right)\sin\left(\frac{\pi n\sqrt{B_1}}{l_2}y_1\right) \tag{32}$$

and for Eqs. (27) and (31) one has

$$v_{0,0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{2(m,n)} \sin\left(\frac{\omega_{2(m,n)}^{\text{lin}}}{\omega_{2,0}}\tau_2\right) \sin\left(\frac{\pi m\sqrt{B_1}}{l_1}x_2\right) \sin\left(\frac{\pi n\sqrt{B}}{l_2}y_2\right), \tag{33}$$

where

$$\omega_{1(m,n)}^{\text{lin}} = \sqrt{\frac{\pi^2 m^2 B}{l_1^2} + \frac{\pi^2 n^2 G}{l_2^2} + \beta_1}, \quad \omega_{2(m,n)}^{\text{lin}} = \sqrt{\frac{\pi^2 m^2 G}{l_1^2} + \frac{\pi^2 n^2 B}{l_2^2} + \beta_1}, \quad m, n = 1, 2, 3 \ldots,$$

are vibration frequencies of the linear system, $\omega_{\alpha,0} = \omega_{\alpha(1,1)}^{\text{lin}}$, $\alpha = 1, 2$.

4. Artificial small-parameter method

The next approximation with respect to ε is found by solving boundary value problem (28) and (29). As usual, in order to avoid secular terms, in the right-hand sides of Eqs. (28) and (29) the coefficients standing by the terms

$$\sin\left(\frac{\omega_{1(m,n)}^{\text{lin}}}{\omega_{1,0}}\tau_1\right) \sin\left(\frac{\pi m\sqrt{B}}{l_1}x_1\right) \sin\left(\frac{\pi n\sqrt{B_1}}{l_2}y_1\right)$$

and

$$\sin\left(\frac{\omega_{2(m,n)}^{\text{lin}}}{\omega_{2,0}}\tau_2\right) \sin\left(\frac{\pi m\sqrt{B_1}}{l_1}x_2\right) \sin\left(\frac{\pi n\sqrt{B}}{l_2}y_2\right), \quad m, n = 1, 2, 3 \ldots,$$

should be equal to zero. These conditions yield two infinite systems of nonlinear algebraic equations of the form

$$\frac{2A_{\alpha(m,n)}\omega_{\alpha,1}}{\beta_2\omega_{\alpha,0}}\left(\omega_{\alpha(m,n)}^{\text{lin}}\right)^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=1}^{\infty} C_{\alpha(m,n)}^{(ijklps)} A_{\alpha(i,j)}A_{\alpha(k,l)}A_{\alpha(p,s)}, \tag{34}$$

where coefficients $C_{\alpha(m,n)}^{(ijklps)}$ are found by substituting expressions (33) and (34) into the right-hand sides of Eqs. (28) and (29), and performing the corresponding simplifications. To this aim, 'Mathematica' package is applied.

Furthermore, a solution to the analyzed problem is sought in the form

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{1(m,n)} \sin\left(\Omega_{1(m,n)}t\right) \sin\left(\frac{\pi m}{l_1}x\right) \sin\left(\frac{\pi n}{l_2}y\right) + O(\varepsilon) + O(\delta), \tag{35}$$

$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{2(m,n)} \sin\left(\Omega_{2(m,n)}t\right) \sin\left(\frac{\pi m}{l_1}x\right) \sin\left(\frac{\pi n}{l_2}y\right) + O(\varepsilon) + O(\delta), \tag{36}$$

where $\Omega_{\alpha(m,n)} = (\omega_{\alpha(m,n)}^{\text{lin}}/\omega_{\alpha(1,1)}^{\text{lin}})\omega_{\alpha}$.

The first series term with respect to ε represents an influence of nonlinearity. The solution of Eq. (34) yields quantities $\omega_{\alpha,1}$ and eigenfrequencies $\Omega_{\alpha(m,n)}$. Although the system can be solved using reduction, an increase in the number of equations causes computational difficulties, and in addition, higher modes are not taken into account. To avoid these problems, the method of artificial small parameter is applied [18–20]. Let us discuss this third asymptotic procedure.

Birkhoff [21] has wondered "How well is Nature simulated by the varied asymptotics models that imaginative scientists have invented?" Any new asymptotic model is linked with a certain small parameter, which sometimes is far from being trivial. For instance, a parameter that has found a wide application in the quantum mechanics is $1/N$, where N stands for the number of spatial dimensions [22]. Note that not always does a real (physically realized) small parameter assure a proper solution to a problem. This is because for small parameter values, the analytical dependence of a solution that is looked for on the given parameter possesses singularities. In this situation it is an artificial parameter that is recommended. There exist a large number of ways in which an artificial small parameter can be introduced (see Refs. [18–20,22–29]). For example, ε may be assumed to stand by the terms to be neglected (assuming ε to be small), and consequently ε can be taken to be equal to 1.

In the case under consideration, in the right-hand side of each (m, n)th equation of system (34) an artificial small parameter μ is introduced, which stands by each term $A_{\alpha(I,j)}A_{\alpha(k,l)}A_{\alpha(p,s)}$ for which the following inequality chain holds: $(i>m)\cup(k>m)\cup(p>m)\cup(j>n)\cup(l>n)\cup(s>n)$. Therefore, for $\mu=0$, system (34) has a triangular structure and is reduced to a recurrent set of equations, whereas for $\mu=1$, it is transformed to the input form.

In the next step, a solution is sought in the form

$$\omega_{\alpha,1}=\omega_{\alpha,1}^{(0)}+\mu\omega_{\alpha,1}^{(1)}+\mu^2\omega_{\alpha,1}^{(2)}+\dots, \tag{37}$$

$$A_{\alpha(m,n)}=A_{\alpha(m,n)}^{(0)}+\mu A_{\alpha(m,n)}^{(1)}+\mu^2 A_{\alpha(m,n)}^{(2)}+\dots,\quad m,n=1,2,3\dots,(m,n)\neq(1,1) \tag{38}$$

and then $\mu=1$ is taken. The proposed approach makes it possible to consider an arbitrary number of equations. In our further computations, only the first two terms in series (37) and (38) are taken into account.

Note that the accuracy of the proposed approach can be significantly increased with the application of the Padé approximations [30].

5. Analysis of a general case $\beta_1\in(0;\infty)$

In the considered problem, β_1 plays the role of a bifurcation parameter. In the general case, when $\beta_1\neq0$, $\beta_1\propto O(1)$ and $\beta_2\propto O(1)$ systems (34) have the following solutions:

$$A_{\alpha(i,j)}=0,\quad i,j=1,2,3\dots,\quad (i,j)\neq(m,n),$$

$$\omega_{\alpha,1}=\frac{27}{128}\frac{A_{\alpha(m,n)}^2\beta_2\omega_{\alpha,0}}{\left(\omega_{\alpha(m,n)}^{\text{lin}}\right)^2},\quad m,n=1,2,3\dots.$$

Observe that in the zeroth order approximation with respect to ε there occurs only one mode in displacements u, and similarly one mode in displacements υ. The corresponding frequencies read

$$\Omega_{\alpha(m,n)} = \omega^{\text{lin}}_{\alpha(m,n)} + 0.2109375 \frac{A^2_{\alpha(m,n)}\beta_2}{\omega^{\text{lin}}_{\alpha(m,n)}}\varepsilon + O(\delta) + O(\varepsilon\delta) + O(\varepsilon^2), \quad \alpha = 1, 2.$$

Internal resonance occurs for $\beta_1 = 0$. In the case under discussion the internal resonance appears between two vibration modes. Solving systems (34) with the help of the artificial parameter method, one arrives at

$$A_{\alpha(m,n)} = 0, \quad m, n = 1, 2, 3\ldots, \ (m,n) \neq (1,1), \quad (m,n) \neq (2i-1,\ 2i-1), \quad i = 1, 2, 3\ldots, \tag{39}$$

$$A_{\alpha(3,3)} = -4.5662 \times 10^{-3} A_{\alpha(1,1)}, \quad A_{\alpha(5,5)} = 2.1139 \times 10^{-5} A_{\alpha(1,1)}, \ldots,$$

$$\omega_{\alpha,1} = 0.211048 \frac{A^2_{\alpha(1,1)}\beta_2}{\omega_{\alpha,0}}.$$

In the zeroth order approximation with respect to ε all odd 'diagonal' modes (1,1), (3,3), (5,5) are realized simultaneously. Furthermore, if originally the vibrations are self-excited having one of the higher-order modes, then also low-order modes are excited up to the fundamental one (1,1). In other words, a high-frequency excitation with small amplitude of the plate can cause large vibration amplitude.

The eigenfrequencies are estimated via the formula

$$\Omega_{\alpha(m,n)} = m\omega_{\alpha,0}\left(1 + 0.211048 \frac{A^2_{\alpha(1,1)}\beta_2}{\omega^2_{\alpha,0}}\varepsilon\right) + O(\delta) + O(\varepsilon\delta) + O(\varepsilon^2), \tag{40}$$

where $m = n = 1, 3, 5\ldots$, $\alpha = 1, 2$.

Fig. 2 shows dependences of eigenfrequencies $\Omega_{\alpha(1,1)}$ of fundamental modes on amplitudes $A_{\alpha(1,1)}$ for different values of parameter ε. During computations the following relations hold:

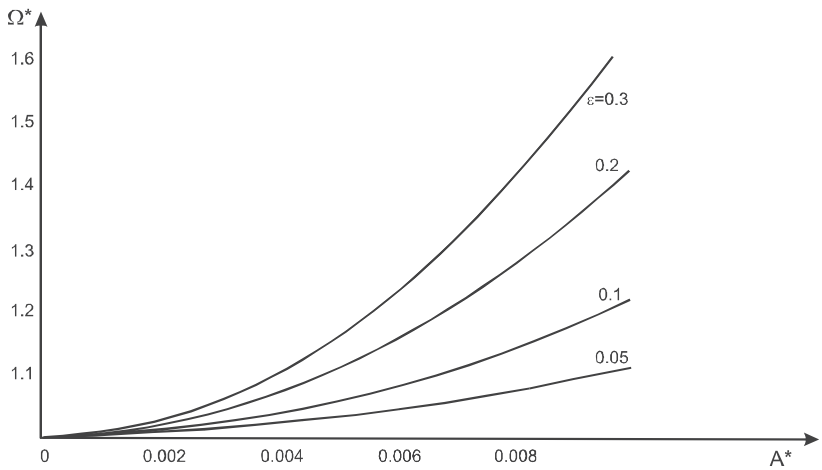

Fig. 2. Amplitude frequency dependences for some values of parameters ($\Omega^* = \Omega_{\alpha(1,1)}/\omega_{\alpha,0}$; $A^* A_{\alpha(1,1)} l_1^{-1} l_2^{-1} \left(l_1^{-2} + l_2^{-2}\right)^{1/2}$).

Table 1
Comparison of numerical and asymptotic results

Asymptotical solutions (32)	Numerical results
$\omega_{\alpha,1} = 0.211048 A^2_{\alpha(1,1)} \beta_2/\omega_{\alpha,0}$	$\omega_{\alpha,1} = 0.211044 A^2_{\alpha(1,1)} \beta_2/\omega_{\alpha,0}$
$A_{\alpha(m,n)} = 0$, $m,n = 1,2,3,\ldots$, $(m,n) \neq (1,1)$, $(m,n) \neq (2i-1, 2i-1)$, $i = 1,2,3,\ldots$	$A_{\alpha(m,n)} = 0$, $m,n = 1,2,3,\ldots$, $(m,n) \neq (1,1)$, $(m,n) \neq (2i-1, 2i-1)$, $i = 1,2,3,\ldots$
$A_{\alpha(3,3)} = -4.5662 \times 10^{-3} A_{\alpha(1,1)}$	$A_{\alpha(3,3)} = -4.5639 \times 10^{-3} A_{\alpha(1,1)}$
$A_{\alpha(5,5)} = 2.1139 \times 10^{-5} A_{\alpha(1,1)}$	$A_{\alpha(5,5)} = 2.1117 \times 10^{-5} A_{\alpha(1,1)}$

$Q/\left(Bl_2^2 + B_1 l_1^2\right) = 10^5$ for the equation concerning u and $Q/\left(B_1 l_2^2 + Bl_1^2\right) = 10^5$ for the equation concerning v, where $Q = \beta_2 l_1^4 l_2^4 / \left(\pi^2 \left(l_1^2 + l_2^2\right)\right)$.

Observe that the applied method of artificial small parameter guarantees good accuracy of the obtained results. In Table 1, relations (40) are compared with those obtained via numerical solution of nonlinear systems (34). Again the computations are carried out using the 'Mathematica' package with only the first few equations including the mode (5,5) taken into account.

6. Concluding remarks

Natural nonlinear in-plane plate vibrations are analyzed. The Relations between amplitudes of vibration modes are obtained and asymptotic formulas for eigenfrequencies are displayed. Among others, it is shown that when parameter β_1 is equal to zero, all odd diagonal modes in the zeroth order approximation are excited. Moreover, even a weak excitation of high frequency may yield low-frequency vibrations with large amplitude.

The proposed approach can be used to solve approximately many other problems of nonlinear vibrations in various systems with distributed parameters.

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