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Stabilization and destabilization of a circulatory system by small velocity-dependent forces

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Abstract

A linear autonomous mechanical system under non-conservative positional forces is considered. The influence of small forces proportional to generalized velocities on the stability of the system is studied. Necessary and sufficient conditions are obtained for the matrix of dissipative and gyroscopic forces to make the system asymptotically stable. A system with two degrees of freedom is studied in detail. Explicit formulae describing the structure of the stabilizing matrix and the stabilization domain in the space of the matrix elements are found and plotted. As a mechanical example, a problem of stability of the Ziegler–Herrmann–Jong pendulum is analyzed.

1. Introduction

Consider a linear mechanical system with non-conservative positional forces proportional to the vector of generalized coordinates and small forces proportional to the vector of generalized velocities

$$\mathbf{M}\ddot{\mathbf{q}} + \varepsilon\mathbf{D}\dot{\mathbf{q}} + \mathbf{A}\mathbf{q} = 0, \tag{1}$$

where **M**, **D** and **A** are constant real square matrices of order m, corresponding to inertial, dissipative plus gyroscopic, and non-conservative positional forces, respectively, $\varepsilon \geqslant 0$ is a small

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parameter, $\mathbf{q}$ is a vector of generalized coordinates, and dot indicates differentiation with respect to time t. The matrix $\mathbf{M}$ is assumed to be non-singular.

Separating time with $\mathbf{q} = \mathbf{u}e^{\lambda t}$ we get the eigenvalue problem

$$(\mathbf{M}\lambda^2 + \varepsilon\mathbf{D}\lambda + \mathbf{A})\mathbf{u} = 0. \tag{2}$$

The eigenvalues $\lambda_1, \ldots, \lambda_{2m}$ are solutions of the characteristic equation

$$\det(\mathbf{M}\lambda^2 + \varepsilon\mathbf{D}\lambda + \mathbf{A}) = 0. \tag{3}$$

Consider now system (1) in the absence of velocity-dependent forces ($\varepsilon = 0$). Such a system is called *circulatory*. In this case it follows from Eq. (3) that if λ is an eigenvalue, then $-\lambda, \bar{\lambda}, -\bar{\lambda}$ are eigenvalues too. Therefore, a circulatory system is marginally stable if and only if all the eigenvalues $\pm i\omega_j, \omega_j \geqslant 0$ are purely imaginary and semi-simple. The semi-simple eigenvalue means that the number r of linearly independent eigenvectors corresponding to that eigenvalue is equal to its algebraic multiplicity k. If $r<k$, then *secular* terms proportional to $t^\alpha e^{\omega_j t}$, $\alpha \leqslant k-1$ (instability) appear in the general solution of Eq. (1). Thus, existence of a pair of algebraically double eigenvalues $\pm i\omega_0, \omega_0 > 0$ with only one eigenvector, other eigenvalues being purely imaginary and simple, corresponds to the boundary between stability and flutter instability (i.e., oscillations with growing amplitude).

Perturbation of the circulatory system by small velocity-dependent forces $\varepsilon\mathbf{D}$ destroys the symmetry of eigenvalues. The eigenvalues can move to the right or left half of the complex plane. As the result, the non-conservative system can become unstable or asymptotically stable depending on the behavior of perturbed eigenvalues. It is important and practical to know what kind of matrices of velocity-dependent forces $\mathbf{D}$ stabilize or destabilize the unperturbed circulatory system.

The dependence of stability of a linear autonomous mechanical system on the structure of forces acting on the system is a classical subject going back to the works by Thomson and Tait [1]. However, the general interest to the problem of influence of small velocity-dependent forces on the stability of a linear non-conservative system arose in the early 1950s due to the famous work by Ziegler [2].

The destabilizing effect of viscous damping in the specific linear systems with two degrees of freedom subjected to non-conservative forces was first recognized by Ziegler [2,3] and Bolotin [4]. This effect was further explored by Herrmann and Jong [5], Nemat-Nasser and Herrmann [6], Bolotin and Zhinzher [7], Kounadis [8], Beletsky [9], Bolotin, Grishko and Panov [10], Gallina and Trevisani [11] and others. However, this effect was not placed into the framework of theorems of sufficient generality. Nor was it shown whether a more general system with many degrees of freedom can also exhibit such behavior. The destabilization paradox has attracted much attention in the world literature; see review papers by Herrmann [12], Seyranian [13], Bloch et al. [14], and a recent work by Kirillov [15].

However, already in 1960s Bolotin [4] and Herrmann and Jong [5] found that for some specific problems there exist damping configurations for which the destabilization paradox does not take place. Done [16] tried to find the stabilizing matrices in a general formulation, however, he did not succeed in getting results in an explicit form even for 2×2 matrices. Walker [17] formulated a stabilization problem: assuming that the unperturbed circulatory system is stable how to get the asymptotic stability due to small velocity-dependent forces. He found a class of "non-destabilizing" *mTimesm* matrices $\varepsilon\mathbf{D}$ by the Lyapunov direct method. It should be noted that his theorems provided some sufficient but not necessary conditions for the matrix $\mathbf{D}$ to be "non-destabilizing". Another attempt to find stabilizing configurations of the matrix $\mathbf{D}$ was done in a cluster of works by Banichuk, Bratus and Myshkis published in the early 1990s; see for example Refs. [18,19]. They succeeded in getting one of the necessary conditions for the matrix $\mathbf{D}$ of velocity-dependent forces but failed in constructing stabilizing matrices even for 2×2 case. The next clarifying step was taken by O'Reilly et al. [20,21] who obtained in explicit form the domain of stabilization for general systems with two degrees of freedom assuming that the unperturbed system has only distinct purely imaginary eigenvalues. In important papers by Seyranian and Pedersen [22], and Seyranian [23], two-dimensional stabilization domains for the classical examples by Bolotin [4] and Herrmann-Jong [5], assuming that the unperturbed system is on the boundary between flutter and stability, were found.

The purpose of the present paper is to find the necessary and sufficient conditions that the matrix $\mathbf{D}$ of velocity-dependent forces must satisfy in order to stabilize the unperturbed circulatory system with arbitrary degrees of freedom for sufficiently small ε. The paper is organized in the following way:

In Section 2, we derive main formulae for perturbations of eigenvalues of a circulatory system due to small velocity-dependent forces. Both general and degenerate cases are considered.

Based on these results, Section 3 treats stabilization conditions for the matrix $\mathbf{D}$ of velocity-dependent forces of a system with arbitrary degrees of freedom. It is assumed either that the circulatory system is stable or it is taken at the boundary between stability and flutter domains. Two theorems for the necessary and sufficient stabilization conditions are formulated and proved. These conditions imply linear and quadratic constraints on the elements of the stabilizing matrix $\mathbf{D}$. To compute the coefficients of the linear and quadratic forms, one only needs to know the spectrum of the circulatory system with the corresponding right and left eigenvectors and the so-called associated vectors.

Section 4 is devoted to synthesis of stabilizing matrices using Walker's results [17]. We formulate and prove Theorem 3 giving a class of matrices $\mathbf{D}$ making a circulatory system asymptotically stable.

Section 5 treats systems with two degrees of freedom. Here we find stabilizing matrices in explicit form with the inequalities implied on the elements of the matrix $\mathbf{D}$. Both cases of non-symmetric and symmetric matrices are considered.

Finally, in Section 6, we discuss the Ziegler–Herrmann–Jong pendulum loaded by tangential follower force. The form of the stabilizing matrix $\mathbf{D}$ is found and the inequalities on its elements are derived.

2. Behavior of eigenvalues due to perturbation $\varepsilon\mathbf{D}$

Let at $\varepsilon = 0$ the spectrum of eigenvalue problem (2) contain a complex-conjugate pair of double purely imaginary eigenvalues $\pm i\omega_0$ with the Jordan chain of length 2. The left and right eigenvectors and associated vectors $\mathbf{u}_0, \mathbf{u}_1$ and $\mathbf{v}_0, \mathbf{v}_1$ corresponding to the double eigenvalue $\lambda_0 =$

$\mathrm{i}\omega_0$ satisfy the equations [24–27]

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{u}_0 = 0, \quad (\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{u}_1 = -2\mathrm{i}\omega_0\mathbf{M}\mathbf{u}_0, \tag{4}$$

$$\mathbf{v}_0^{\mathrm{T}}(\mathbf{A} - \omega_0^2\mathbf{M}) = 0, \quad \mathbf{v}_1^{\mathrm{T}}(\mathbf{A} - \omega_0^2\mathbf{M}) = -2\mathrm{i}\omega_0\mathbf{v}_0^{\mathrm{T}}\mathbf{M}. \tag{5}$$

In addition, these vectors are related by the following conditions:

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 = 0, \quad \mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 = \mathbf{v}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 \neq 0. \tag{6}$$

The vectors $\mathbf{u}_0, \mathbf{u}_1$ and $\mathbf{v}_0, \mathbf{v}_1$ are not uniquely defined. Since the matrix $\mathbf{A} - \omega_0^2\mathbf{M}$ is real the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ in Eqs. (4) and (5) can be chosen real. Then, the associated vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are purely imaginary. Other eigenvalues $\pm\mathrm{i}\omega_j$, $j = 3, \ldots, m$ are assumed to be simple and purely imaginary. The unperturbed circulatory system is therefore on the boundary between the stability and flutter [28].

It should be noted that if a circulatory system depends on parameters (for example, non-conservative load parameter), then it generally has simple eigenvalues, and only at some specific values of parameters its spectrum contains multiple eigenvalues. The case of a pair of double purely imaginary eigenvalues with the Jordan chain of length 2 is typical (generic) for one and more parameter systems [29]. That is why the boundary between flutter and stability domains is generally characterized by the pair of double purely imaginary eigenvalues with the Jordan chain of length 2, while the stability domain corresponds to the systems with all simple purely imaginary eigenvalues [28].

Consider a simple eigenvalue $\lambda_j = \mathrm{i}\omega_j$ with the right and left eigenvectors $\mathbf{u}_j, \mathbf{v}_j$. A perturbation $\varepsilon\mathbf{D}$ shifts λ_j from the imaginary axis. The perturbed eigenvalue and eigenvector are smooth functions of the parameter ε and can be represented as the Taylor series [30]

$$\lambda = \lambda_j + \varepsilon\mu_j + \cdots, \quad \mathbf{u} = \mathbf{u}_j + \varepsilon\mathbf{z}_j + \cdots. \tag{7}$$

Substitution of expansions (7) into eigenvalue problem (2) yields the equation

$$(\mathbf{A} - \omega_j^2\mathbf{M})\mathbf{z}_j = -2\lambda_j\mu_j\mathbf{M}\mathbf{u}_j - \lambda_j\mathbf{D}\mathbf{u}_j. \tag{8}$$

Multiplying Eq. (8) from the left by $\mathbf{v}_j^{\mathrm{T}}$, we obtain the coefficient μ_j

$$\mu_j = -\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{2\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j}. \tag{9}$$

Due to perturbation $\varepsilon\mathbf{D}$ the double eigenvalue with the Jordan chain of length 2 as well as its eigenvector take increments, which are represented in the form of the Newton–Puiseux series [30]

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \cdots,$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{w}_1 + \varepsilon\mathbf{w}_2 + \varepsilon^{3/2}\mathbf{w}_3 + \cdots. \tag{10}$$

Substitution of these expansions into eigenvalue problem (2) and collection of the terms with the same powers of ε gives the equations

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{w}_1 = -2\mathrm{i}\omega_0\lambda_1\mathbf{M}\mathbf{u}_0, \tag{11}$$

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{w}_2 = -2\mathrm{i}\omega_0\lambda_1\mathbf{M}\mathbf{w}_1 - 2\mathrm{i}\omega_0\lambda_2\mathbf{M}\mathbf{u}_0 - \mathrm{i}\omega_0\mathbf{D}\mathbf{u}_0 - \frac{\lambda_1^2}{2}\mathbf{M}\mathbf{u}_0. \tag{12}$$

Substituting the vector $\mathbf{w}_1 = \lambda_1\mathbf{u}_1 + \gamma\mathbf{u}_0$ given by (11) into Eq. (12) and then multiplying both parts of Eq. (12) by $\mathbf{v}_0$, we get the coefficient λ_1 in expansions (10)

$$\lambda_1^2 = \mathrm{i}d, \quad d = -\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2\mathrm{i}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1}. \tag{13}$$

Note that d is a real number since the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ are real and $\mathbf{u}_1$ is purely imaginary. Thus, if $d \neq 0$, then $\lambda_1 \neq 0$, and the double eigenvalue splits into two complex eigenvalues.

In the case when the coefficient λ_1 in expansions (10) becomes zero ($d = 0$), the splitting of the double eigenvalue λ_0 is described in the first approximation by the following expression:

$$\lambda = \mathrm{i}\omega_0 + \lambda_2\varepsilon + o(\varepsilon). \tag{14}$$

Substituting expansions (10) with $\lambda_1 = 0$ into eigenvalue problem (2) and collecting the terms with the same powers of ε, we get the equations

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{w}_1 = 0, \tag{15}$$

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{w}_2 = -2\mathrm{i}\omega_0\lambda_2\mathbf{M}\mathbf{u}_0 - \mathrm{i}\omega_0\mathbf{D}\mathbf{u}_0, \tag{16}$$

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{w}_4 = -2\mathrm{i}\omega_0(\lambda_2\mathbf{M}\mathbf{w}_2 + \lambda_3\mathbf{M}\mathbf{w}_1 + \lambda_4\mathbf{M}\mathbf{u}_0) - \mathrm{i}\omega_0\mathbf{D}\mathbf{w}_2 - \lambda_2\mathbf{D}\mathbf{u}_0 - \lambda_2^2\mathbf{M}\mathbf{u}_0. \tag{17}$$

Solving Eqs. (15) and (16) we find the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$

$$\mathbf{w}_1 = \beta\mathbf{u}_0, \quad \mathbf{w}_2 = \lambda_2\mathbf{u}_1 + \gamma\mathbf{u}_0 - \mathrm{i}\omega_0\mathbf{G}(\mathbf{D}\mathbf{u}_0), \tag{18}$$

where β and γ are arbitrary constants, and $\mathbf{G}$ is the operator inverse to $\mathbf{A} - \omega_0^2\mathbf{M}$. In particular, this operator can be represented in the form [31]

$$\mathbf{G} = (\mathbf{A} - \omega_0^2\mathbf{M} + 2\mathrm{i}\omega_0\mathbf{v}_0\mathbf{v}_1^{\mathrm{T}}\mathbf{M})^{-1}, \tag{19}$$

with $\det \mathbf{G} \neq 0$. We multiply Eq. (17) by the left eigenvector $\mathbf{v}_0$, and then substitute in the result the quantity $\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{w}_2$ obtained from multiplication of Eq. (16) by the left associated vector $\mathbf{v}_1$. After this transformation substitute the vectors (18) into Eq. (17). Finally, we arrive at the quadratic equation serving for determining the coefficient λ_2

$$\lambda_2^2 + \lambda_2\frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} - \mathrm{i}\omega_0\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = 0. \tag{20}$$

This equation, derived first in Ref. [26], describes with Eq. (14) splitting of the double eigenvalue in the degenerate case $\lambda_1 = 0$.

3. Stabilization conditions

Stability of the circulatory system perturbed by the velocity-dependent forces with the matrix $\varepsilon\mathbf{D}$ depends on whether the eigenvalues shift to the left- or to the right-hand side of the complex plane. The explicit formulae describing splitting of eigenvalues derived in the previous section allow us to find constructive conditions of stability of the perturbed non-conservative system.

Consider first the case when the unperturbed ($\varepsilon = 0$) circulatory system is stable and its spectrum consists of simple eigenvalues $\pm\mathrm{i}\omega_j$, $j = 1, \ldots, m$ with the left and right eigenvectors

$\mathbf{u}_j, \mathbf{v}_j$. After the introduction of small velocity-dependent forces ($\varepsilon > 0$) increments of these eigenvalues are governed by the formula $\lambda_j = \mathrm{i}\omega_j + \mu_j\varepsilon + O(\varepsilon^2)$, where the real coefficient μ_j is given by Eq. (9). Therefore, the conditions

$$\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} \geqslant 0, \quad j = 1, \ldots, m \tag{21}$$

mean that in the first approximation with respect to ε all the eigenvalues λ_j do not move to the right-hand side of the complex plane due to perturbation $\varepsilon\mathbf{D}$, $\varepsilon > 0$. Strong inequalities (21) guarantee that for a sufficiently small $\varepsilon > 0$ the perturbed eigenvalues λ_j belong to the left-hand side of the complex plane.

Thus, we have proved the following:

Theorem 1. *The necessary conditions for the small velocity-dependent forces with the matrix* $\mathbf{D}$ *to make stable circulatory system* (1) *at* $\varepsilon = 0$ *asymptotically stable are*

$$\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} \geqslant 0, \quad j = 1, \ldots, m.$$

If all the weak inequalities ($\geqslant 0$) *are replaced by the strong* ($>$) *ones, then the above conditions become sufficient for asymptotic stability.*

Example 1. To illustrate Theorem 1 we consider two-dimensional non-conservative system (1) with the matrices $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ specified by the relations

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 183 & 217 \\ 21 & 25 \end{bmatrix}. \tag{22}$$

When the velocity-dependent forces are absent ($\varepsilon = 0$), the eigenvalues of the circulatory system characterized by the matrices $\mathbf{M}$ and $\mathbf{A}$ are purely imaginary and simple (stability)

$$\lambda_1 = \mathrm{i}, \quad \lambda_2 = 3\mathrm{i}, \quad \bar{\lambda}_1 = -\mathrm{i}, \quad \bar{\lambda}_2 = -3\mathrm{i}. \tag{23}$$

The left $\mathbf{v}_1, \mathbf{v}_2$ and right $\mathbf{u}_1, \mathbf{u}_2$ eigenvectors of the eigenvalues λ_1, λ_2 can be chosen real

$$\mathbf{u}_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -9 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -13 \end{bmatrix}. \tag{24}$$

With matrices (22) and eigenvectors (24) the stability conditions of Theorem 1 take the form

$$\frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{\mathbf{v}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{7}{2} > 0, \quad \frac{\mathbf{v}_2^{\mathrm{T}}\mathbf{D}\mathbf{u}_2}{\mathbf{v}_2^{\mathrm{T}}\mathbf{M}\mathbf{u}_2} = \frac{1}{2} > 0. \tag{25}$$

Thus, according to Theorem 1 the non-conservative system with small velocity-dependent forces $\varepsilon\mathbf{D}$ is asymptotically stable. Indeed, for small ε all the roots of the characteristic equation of the system

$$2\lambda^4 + 8\lambda^3\varepsilon + 20\lambda^2 - 2\varepsilon^2\lambda^2 + 64\varepsilon\lambda + 18 = 0 \tag{26}$$

have negative real parts, which can be verified by the Routh–Hurwitz conditions. For example, if $\varepsilon = 0.1$, then the eigenvalues are

$$\lambda_1, \bar{\lambda}_1 = -0.175 \pm i0.986, \quad \lambda_2, \bar{\lambda}_2 = -0.024 \pm i2.994, \tag{27}$$

which means asymptotic stability.

Consider now the case when the unperturbed circulatory system ($\varepsilon = 0$) is on the boundary between the stability and flutter and its spectrum contains a pair of the double purely imaginary eigenvalues $\pm i\omega_0$ each with only one eigenvector, other eigenvalues being purely imaginary and simple. Such circulatory system is unstable because of the secular terms in the general solution of Eq. (1).

After the introduction of small dissipative and gyroscopic forces ($\varepsilon > 0$), the double eigenvalue $\lambda_0 = i\omega_0$ with one eigenvector splits in general into two simple eigenvalues. This splitting is governed by expansion (10) where the coefficient λ_1 is found from the quadratic equation (13). Note that the quantity d is real because the vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ are real, and the vector $\mathbf{u}_1$ is purely imaginary. Therefore, for $d \neq 0$ the double eigenvalue $\lambda_0 = i\omega_0$ splits due to perturbation $\varepsilon\mathbf{D}$, $\varepsilon > 0$ into two simple eigenvalues $\lambda = i\omega_0 \pm \sqrt{id\varepsilon} + O(\varepsilon)$, one of them belonging to the right-hand half of the complex plane (Fig. 1). This means that infinitely small velocity-dependent forces generally destabilize the circulatory ($\varepsilon = 0$) system.

As a consequence, we have that the equality $d = 0$, i.e.,

$$\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2i\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = 0 \tag{28}$$

is the necessary condition of stabilization of the system by the velocity-dependent forces $\varepsilon\mathbf{D}$. When this condition is satisfied, the splitting of the double eigenvalue is governed by the expansion $\lambda = i\omega_0 + \lambda_2\varepsilon + o(\varepsilon)$, where the coefficient λ_2 is found from the quadratic equation (20).

The coefficients of Eq. (20) are real. If the circulatory system is stabilized by small velocity-dependent forces, then it is necessary for both roots λ_2 that $\mathrm{Re}\,\lambda_2 \leqslant 0$. This condition is equivalent

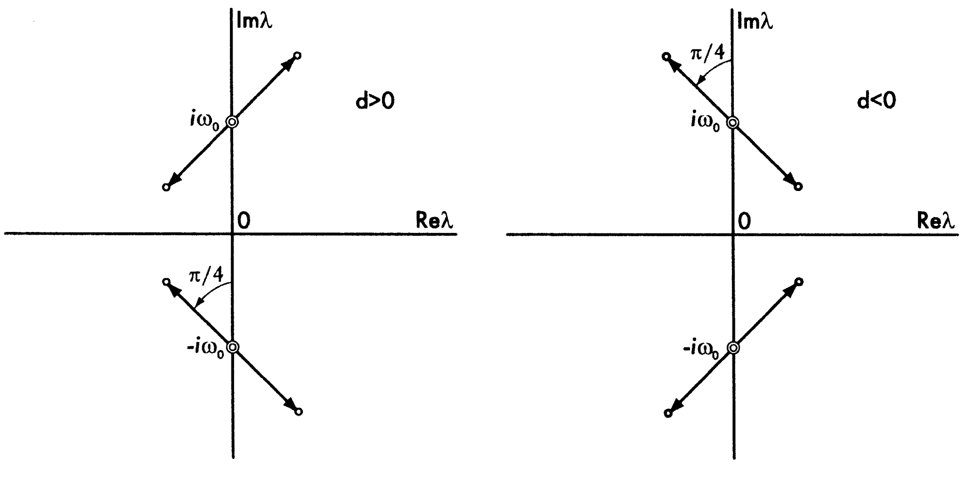

Fig. 1. Destabilization of a circulatory system by the perturbation $\varepsilon\mathbf{D}$.

to the weak Routh–Hurwitz conditions written for the polynomial (20)

$$\frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} \geqslant 0, \tag{29}$$

$$-\mathrm{i}\omega_0 \frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} \geqslant 0. \tag{30}$$

Strong inequalities (29) and (30) guarantee splitting of the double eigenvalue $\lambda_0 = \mathrm{i}\omega_0$ into two simple eigenvalues situated in the left-hand side of the complex plane for a sufficiently small $\varepsilon > 0$.

The conditions

$$\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} \geqslant 0, \quad j = 3, \ldots, m \tag{31}$$

mean that in the first approximation with respect to ε all the eigenvalues $\pm\mathrm{i}\omega_j$ do not shift to the right-hand side of the complex plane due to perturbation $\varepsilon\mathbf{D}$, $\varepsilon > 0$. Strong inequalities guarantee that for the sufficiently small $\varepsilon > 0$ the perturbed eigenvalues λ_j belong to the left-hand side of the complex plane.

Thus, we have proved

Theorem 2. *The necessary conditions for the small velocity-dependent forces with the matrix* $\mathbf{D}$ *to make circulatory system* (1) *at* $\varepsilon = 0$, *being on the boundary between stability and flutter, asymptotically stable are the following*

$$\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2\mathrm{i}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = 0, \quad \frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} \geqslant 0, \quad -\mathrm{i}\omega_0 \frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} \geqslant 0.$$

$$\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} \geqslant 0, \quad j = 3, \ldots, m.$$

If all the weak inequalities ($\geqslant 0$) *are replaced by the strong* ($>$) *ones, then the above conditions become sufficient for asymptotic stability.*

Conditions given by Theorems 1 and 2 are constructive, necessary conditions of stabilization of a circulatory system by small velocity-dependent forces. Correspondingly, the sufficient conditions of stabilization of system (1) follow from Eqs. (21) and (28)–(31) after the change of weak inequalities by the strong ones. These conditions imply restrictions on the elements of the matrix $\mathbf{D}$. Conditions (21), (28), (29), (31) are linear and condition (30) is quadratic with respect to the elements of the matrix $\mathbf{D}$. To compute the coefficients of the linear and quadratic forms, one needs to know the spectrum of the circulatory system with the corresponding left and right eigenvectors and associated vectors. One can see that the m^2 elements of the matrix $\mathbf{D}$ must satisfy either m inequalities (21) or one equality (28) and m inequalities (29)–(31).

Note that necessary conditions similar to (28) and (31) were obtained in Refs. [18,19], but essential inequalities (29) and (30) were missing in those papers.

Example 2. Two-dimensional circulatory system with the matrices **M** and **A** specified by the relations

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{11}\begin{bmatrix} 168 & 104 \\ 50 & 54 \end{bmatrix} \tag{32}$$

has a pair of double purely imaginary eigenvalues (stability-flutter boundary)

$$\lambda_0 = 2\mathrm{i}, \quad \bar{\lambda}_0 = -2\mathrm{i} \tag{33}$$

with the left and right Jordan chains of length 2 consisting of eigen- and associated vectors $\mathbf{u}_0, \mathbf{v}_0, \mathbf{u}_1, \mathbf{v}_1$

$$\mathbf{u}_0 = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \quad \mathbf{u}_1 = -\mathrm{i}\begin{bmatrix} 44/3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad \mathbf{v}_1 = \mathrm{i}\begin{bmatrix} 0 \\ 22 \end{bmatrix}. \tag{34}$$

Let us perturb the circulatory system by the velocity-dependent forces $\varepsilon\mathbf{D}$ with the matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{35}$$

It is easy to see that in this case the first stability condition of Theorem 2 is not satisfied

$$\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2\mathrm{i}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{19}{88} \neq 0. \tag{36}$$

Direct calculation of the roots of the characteristic equation (3) with the matrices given by Eqs. (32) and (35)

$$2\lambda^4 + 8\lambda^3\varepsilon + 16\lambda^2 - 2\varepsilon^2\lambda^2 + \frac{314}{11}\varepsilon\lambda + 32 = 0 \tag{37}$$

for $\varepsilon = 0.1$ yields

$$\lambda_1, \bar{\lambda}_1 = -0.237 \pm \mathrm{i}2.068, \quad \lambda_2, \bar{\lambda}_2 = 0.037 \pm \mathrm{i}1.921, \tag{38}$$

which means instability. It is easy to verify that the Routh–Hurwitz conditions for polynomial (37) are not fulfilled for small ε. However, small velocity-dependent forces with the matrix

$$\mathbf{D} = \frac{1}{3}\begin{bmatrix} 15 & -47 \\ 9 & 3 \end{bmatrix} \tag{39}$$

stabilize the circulatory system, because according to Theorem 2

$$\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2\mathrm{i}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = 0, \quad \frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{31}{6} > 0, \quad -\mathrm{i}\omega_0\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{13}{2} > 0. \tag{40}$$

Indeed, for small ε all the roots of the characteristic equation of the system

$$2\lambda^4 + \frac{62}{3}\lambda^3\varepsilon + 16\lambda^2 + 52\varepsilon^2\lambda^2 + \frac{248}{3}\varepsilon\lambda + 32 = 0 \tag{41}$$

have negative real parts according to the Routh–Hurwitz conditions. For example, if $\varepsilon = 0.1$, then the eigenvalues are

$$\lambda_1, \bar{\lambda}_1 = -0.300 \pm \mathrm{i}1.977, \quad \lambda_2, \bar{\lambda}_2 = -0.217 \pm \mathrm{i}1.988, \tag{42}$$

which means asymptotic stability.

4. Synthesis of the stabilizing matrix D

The necessary and sufficient conditions of stabilization of circulatory system (1) by the velocity-dependent forces given by Theorems 1 and 2 are constructive and can be used to check the stability of the system. However, in practice one often needs to synthesize the stabilizing matrix $\mathbf{D}$ explicitly by means of the coefficients of the matrices $\mathbf{M}$ and $\mathbf{A}$ of the unperturbed circulatory system.

It seems that Bolotin [4] was the first who in the early 1960s noticed that the matrix $\mathbf{D}$ proportional to the matrix $\mathbf{M}$ stabilizes circulatory system (1) for $m = 2$. In 1969 this result was extended to systems with $m > 2$ degrees of freedom by Bolotin and Zhinzher [7], being then proved independently by Done [16] in 1973. In the same year Walker [17] found that the matrix $\mathbf{D} = c_0\mathbf{M}$ belongs to the class of matrices

$$\mathbf{D} = \sum_{p=-\infty}^{\infty} c_p \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p, \quad \det \mathbf{M} \neq 0, \quad \det \mathbf{A} \neq 0, \quad c_p \geqslant 0, \tag{43}$$

which stabilize *initially stable* circulatory system (1). Note that Walker's considerations were based on the direct Lyapunov method, and he did not investigate the initially unstable circulatory systems situated on the boundary between stability and flutter.

In this section, we find how wide is the class of Walker's matrices (43) and show that they satisfy the sufficient conditions of stabilization given by Theorems 1, 2. First of all we need the following:

Lemma 1. *Let* $\lambda_0 = i\omega_0$ ($\omega_0 \neq 0$) *be a double eigenvalue with the left and right eigenvectors* $\mathbf{u}_0, \mathbf{v}_0$ *and associated vectors* $\mathbf{u}_1, \mathbf{v}_1$ *satisfying Eqs.* (4) *and* (5). *If* $\det \mathbf{M} \neq 0$, *then for arbitrary integer* $p \geqslant 0$ *the following relations take place*:

$$\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_0 = \omega_0^{2p} \mathbf{M}\mathbf{u}_0, \tag{44}$$

$$\mathbf{v}_0^{\mathrm{T}} \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_0 = 0, \tag{45}$$

$$\mathbf{v}_1^{\mathrm{T}} \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_0 = \omega_0^{2p} \mathbf{v}_1^{\mathrm{T}} \mathbf{M}\mathbf{u}_0 = \omega_0^{2p} \mathbf{v}_0^{\mathrm{T}} \mathbf{M}\mathbf{u}_1 = \mathbf{v}_0^{\mathrm{T}} \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_1 \neq 0. \tag{46}$$

Proof. For $p = 0$ the lemma is obviously true.

Consider the case $p > 0$. Let us multiply the first of Eqs. (4) from the left by $\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^{p-1}\mathbf{M}^{-1}$. This yields the sequence of equalities, which prove Eq. (44)

$$\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_0 = \cdots = \omega_0^{2k} \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^{p-k} \mathbf{u}_0 = \cdots = \omega_0^{2p} \mathbf{M}\mathbf{u}_0, \tag{47}$$

where $1 \leqslant k \leqslant p$. Multiplying Eq. (44) by $\mathbf{v}_0^{\mathrm{T}}$ from the left and taking into account orthogonality condition (6) we prove Eq. (45). Multiplication of both sides of Eq. (44) by $\mathbf{v}_1^{\mathrm{T}}$ from the left yields

$$\mathbf{v}_1^{\mathrm{T}} \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p \mathbf{u}_0 = \omega_0^{2p} \mathbf{v}_1^{\mathrm{T}} \mathbf{M}\mathbf{u}_0. \tag{48}$$

To prove the remaining half of Eq. (46) we multiply the second of Eqs. (4) by $\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^{p-1}\mathbf{M}^{-1}$ and obtain

$$(\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p - \omega_0^2 \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^{p-1})\mathbf{u}_1 = -2i\omega_0 \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^{p-1}\mathbf{u}_0. \tag{49}$$

Taking into account orthogonality condition (6) we get from Eq. (49) for $p = 1$

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})\mathbf{u}_1 = \omega_0^2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1. \tag{50}$$

Suppose that for any k from 2 to $p-1$ we have

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^k\mathbf{u}_1 = \omega_0^{2k}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1. \tag{51}$$

Then, multiplying both sides of Eq. (49) by $\mathbf{v}_0^{\mathrm{T}}$ from the left and taking into account Eq. (45) and hypothesis (51) we establish the equality

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_1 = \omega_0^{2p}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1, \tag{52}$$

which completes the proof.

Now we formulate the statement which gives the stabilizing matrix $\mathbf{D}$ of arbitrary order m in an explicit form.

Theorem 3. *Velocity-dependent forces with the matrix*

$$\mathbf{D} = \sum_{p=0}^{m-1} c_p\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p, \quad \det\mathbf{M}\neq 0, \quad c_p \geqslant 0,$$

where p is an integer number, make circulatory system (1) *at* $\varepsilon = 0$ *asymptotically stable if the unperturbed circulatory system is stable or situated on the flutter boundary.*

Proof. First of all we note that summation from $-\infty$ to ∞ in expression (43) for Walker's matrix is superfluous because for any $p \geqslant m$ and $p < 0$ (if $\det\mathbf{A}\neq 0$) the matrix $(\mathbf{M}^{-1}\mathbf{A})^p$ is a linear combination of the matrices $(\mathbf{M}^{-1}\mathbf{A})^q$, where $0 \leqslant q \leqslant m-1$. This is an obvious consequence of the classical Cayley–Hamilton theorem [24], which states that any matrix satisfies its characteristic equation. Thus, in formula (43) we should replace the summation limits by 0 and $m-1$.

To check the sufficient conditions provided by Theorems 1 and 2 of the present paper, it is enough to consider the matrix $\mathbf{D} = \mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p$ since the coefficients $c_p \geqslant 0$. Then, from Lemma 1 and the second of Eqs. (4) we have

$$\mathbf{v}_1^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_0 = \mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_1 = \omega_0^{2p}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1, \tag{53}$$

$$\mathbf{G}(\mathbf{D}\mathbf{u}_0) = \mathbf{G}(\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_0) = \omega_0^{2p}\mathbf{G}(\mathbf{M}\mathbf{u}_0) = \omega_0^{2p}\left(\gamma\mathbf{u}_0 - \frac{\mathbf{u}_1}{2\mathrm{i}\omega_0}\right), \tag{54}$$

where γ is the arbitrary constant and the matrix $\mathbf{G}$ is given by Eq. (19). Substituting Eqs. (53) and (54) into conditions (28)–(30) we get for the double eigenvalue

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 = \mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_0 = \omega_0^{2p}\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 = 0, \tag{55}$$

$$\frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_1}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = (\omega_0^p)^2 > 0, \tag{56}$$

$$-\mathrm{i}\omega_0\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = -\mathrm{i}\omega_0\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\omega_0^{2p}\mathbf{G}(\mathbf{M}\mathbf{u}_0)}{2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1} = \frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_1}{4\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1}\,\omega_0^{2p} = \left(\frac{\omega_0^{2p}}{2}\right)^2 > 0 \tag{57}$$

and for the simple eigenvalues we obtain

$$\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} = \omega_j^{2p}\frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} = (\omega_j^p)^2 > 0. \tag{58}$$

Therefore, the sufficient conditions given by Theorems 1 and 2 are satisfied. This completes the proof.

According to Theorem 3 Walker's matrix has no more than m free parameters, while the matrix $\mathbf{D}$ satisfying conditions of Theorems 1 and 2 has either m^2 or m^2-1 free parameters. This means that the class of Walker's matrices is much narrower than that given by Theorems 1 and 2 of the present paper. Thus, Theorem 3 simplifies Walker's Theorem 1 from [17] making the summation in Eq. (43) finite and extends it to the initially unstable systems located on the flutter boundary.

Example 3. Finally, we return to the circulatory systems of Examples 1 and 2 and synthesize the stabilizing matrices $\mathbf{D}$ according to Theorem 3. The first circulatory system characterized by the matrices

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 183 & 217 \\ 21 & 25 \end{bmatrix} \tag{59}$$

has two pairs of simple eigenvalues $\pm \mathrm{i}, \pm 3\mathrm{i}$. It turns out that the velocity-dependent forces with the matrix $\varepsilon\mathbf{D}$,

$$\mathbf{D} = \mathbf{M} + \mathbf{A} = \begin{bmatrix} 186 & 218 \\ 22 & 26 \end{bmatrix} \tag{60}$$

make the circulatory system asymptotically stable, because all the roots of the characteristic equation

$$2\lambda^4 + 24\lambda^3\varepsilon + 20\lambda^2 + 40\varepsilon^2\lambda^2 + 56\varepsilon\lambda + 18 = 0 \tag{61}$$

have for small ε negative real parts. For example, if $\varepsilon = 0.1$, then

$$\lambda_1, \bar{\lambda}_1 = -0.100 \pm \mathrm{i}0.995, \quad \lambda_2, \bar{\lambda}_2 = -0.500 \pm \mathrm{i}2.958. \tag{62}$$

The circulatory system specified by the matrices

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{11}\begin{bmatrix} 168 & 104 \\ 50 & 54 \end{bmatrix} \tag{63}$$

has a pair of the double eigenvalues $\pm 2\mathrm{i}$ with the Jordan chain of length 2. This system is stabilized by the small velocity-dependent forces with the matrix

$$\mathbf{D} = \mathbf{M} + \mathbf{A} = \frac{1}{11}\begin{bmatrix} 201 & 115 \\ 61 & 65 \end{bmatrix}. \tag{64}$$

Indeed, all the roots of the characteristic equation for small ε

$$2\lambda^4 + 20\lambda^3\varepsilon + 16\lambda^2 + 50\varepsilon^2\lambda^2 + 80\varepsilon\lambda + 32 = 0 \tag{65}$$

have negative real parts according to the Routh–Hurwitz conditions. For $\varepsilon = 0.1$ the eigenvalues of the perturbed system are

$$\lambda_1, \bar{\lambda}_1 = -0.250 \pm \mathrm{i}1.984, \quad \lambda_2, \bar{\lambda}_2 = -0.250 \pm \mathrm{i}1.984, \tag{66}$$

which means asymptotic stability.

5. Stabilization of a system with two degrees of freedom

In this section we will show that for systems with two degrees of freedom it is possible to find the structure of stabilizing matrices $\mathbf{D}$ in an explicit form, and therefore obtain full description of the set of stabilizing matrices.

We consider system (1) with $m = 2$. Multiplying Eq. (1) by $\mathbf{M}^{-1}$ from the left and introducing the notation

$$\tilde{\mathbf{D}} = \mathbf{M}^{-1}\mathbf{D}, \quad \tilde{\mathbf{A}} = \mathbf{M}^{-1}\mathbf{A}, \tag{67}$$

we get the equation

$$\ddot{\mathbf{q}} + \varepsilon\tilde{\mathbf{D}}\dot{\mathbf{q}} + \tilde{\mathbf{A}}\mathbf{q} = 0. \tag{68}$$

For $\varepsilon = 0$ we have a circulatory system. Consider the case when the circulatory system is situated on the boundary between the stability and flutter. Then its spectrum consists only of a pair of double purely imaginary eigenvalues $\pm \mathrm{i}\omega_0$. Since there are no simple eigenvalues, stability of the system depends on the behavior of this pair.

The necessary and sufficient condition of existence of the double eigenvalue $\lambda_0 = \mathrm{i}\omega_0$ in the spectrum of the circulatory system can be written in the form

$$4\tilde{a}_{12}\tilde{a}_{21} + (\tilde{a}_{22} - \tilde{a}_{11})^2 = 0, \tag{69}$$

which is equivalent to the following equality $\det \tilde{\mathbf{A}} = (\mathrm{tr}\, \tilde{\mathbf{A}}/2)^2$ [15].

Therefore,

$$-\lambda_0^2 = \omega_0^2 = \frac{\tilde{a}_{11} + \tilde{a}_{22}}{2} > 0, \quad \tilde{a}_{12}\tilde{a}_{21} \leqslant 0. \tag{70}$$

Taking into account conditions (69) and (70), we find from Eqs. (4) and (5) the eigenvectors and associated vectors $\mathbf{u}_0, \mathbf{v}_0, \mathbf{u}_1, \mathbf{v}_1$ of the double eigenvalue $\lambda_0 = \mathrm{i}\omega_0$:

$$\mathbf{u}_0 = \begin{bmatrix} 2\tilde{a}_{12} \\ \tilde{a}_{22} - \tilde{a}_{11} \end{bmatrix}, \quad \mathbf{v}_0 = \begin{bmatrix} 2\tilde{a}_{21} \\ \tilde{a}_{22} - \tilde{a}_{11} \end{bmatrix}, \tag{71}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -4\mathrm{i}\omega_0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -4\mathrm{i}\omega_0 \end{bmatrix}. \tag{72}$$

With the use of these vectors we find the denominator of formulae (28)–(30)

$$2\mathbf{v}_0^{\mathrm{T}}\mathbf{u}_1 = -8\mathrm{i}\omega_0(\tilde{a}_{22} - \tilde{a}_{11}), \tag{73}$$

since the mass matrix is equal to the identity matrix. Taking into account (69) and (73), we get from the necessary condition (28)

$$\frac{\mathbf{v}_0^{\mathrm{T}}\tilde{\mathbf{D}}\mathbf{u}_0}{2\mathrm{i}\mathbf{v}_0^{\mathrm{T}}\mathbf{u}_1} = \frac{(\tilde{d}_{22}-\tilde{d}_{11})(\tilde{a}_{22}-\tilde{a}_{11})+2(\tilde{d}_{12}\tilde{a}_{21}+\tilde{d}_{21}\tilde{a}_{12})}{8\omega_0} = 0. \tag{74}$$

This condition can be written in the compact form: $2\,\mathrm{tr}(\tilde{\mathbf{A}}\tilde{\mathbf{D}}) = \mathrm{tr}\,\tilde{\mathbf{A}}\,\mathrm{tr}\,\tilde{\mathbf{D}}$.

Let us find the coefficients of quadratic equation (20). For the coefficient at the linear term, we obtain

$$\mathbf{v}_1^{\mathrm{T}}\tilde{\mathbf{D}}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\tilde{\mathbf{D}}\mathbf{u}_1 = -8\mathrm{i}\omega_0(\tilde{d}_{22}(\tilde{a}_{22}-\tilde{a}_{11})+\tilde{d}_{12}\tilde{a}_{21}+\tilde{d}_{21}\tilde{a}_{12}) = -4\mathrm{i}\omega_0\,\mathrm{tr}\,\tilde{\mathbf{D}}(\tilde{a}_{22}-\tilde{a}_{11}). \tag{75}$$

To find the free term of Eq. (20) we must solve the non-homogeneous equation for the vector $\mathbf{w}$

$$(\tilde{\mathbf{A}} - \omega_0^2\mathbf{I})\mathbf{w} = \tilde{\mathbf{D}}\mathbf{u}_0, \tag{76}$$

where the eigenvector $\mathbf{u}_0$ is taken from Eq. (71). Solving Eq. (76), we get

$$\mathbf{G}(\tilde{\mathbf{D}}\mathbf{u}_0) \equiv \mathbf{w} = \begin{bmatrix} -2\tilde{d}_{12} \\ 2\tilde{d}_{11} \end{bmatrix}, \tag{77}$$

$$\mathbf{v}_0^{\mathrm{T}}\tilde{\mathbf{D}}\mathbf{G}(\tilde{\mathbf{D}}\mathbf{u}_0) = 2(\tilde{a}_{22}-\tilde{a}_{11})\det\tilde{\mathbf{D}}. \tag{78}$$

Substitution of Eqs. (73), (75), and (78) into quadratic equation (20) yields

$$\lambda_2^2 + \lambda_2\tfrac{1}{2}\mathrm{tr}\,\tilde{\mathbf{D}} + \tfrac{1}{4}\det\tilde{\mathbf{D}} = 0. \tag{79}$$

Thus, for the system with two degrees of freedom ($m = 2$) the necessary conditions (28)–(30) written in the compact form are the following:

$$2\,\mathrm{tr}(\tilde{\mathbf{A}}\tilde{\mathbf{D}}) = \mathrm{tr}\,\tilde{\mathbf{A}}\,\mathrm{tr}\,\tilde{\mathbf{D}}, \tag{80}$$

$$\mathrm{tr}\,\tilde{\mathbf{D}} \geqslant 0, \quad \det\tilde{\mathbf{D}} \geqslant 0. \tag{81}$$

Note that analogous conditions and Eq. (79) were obtained in Ref. [15] by the analysis of the characteristic polynomial (3) of system (1).

Let us find the stabilization domain in the space of the elements of the matrix $\tilde{\mathbf{D}}$, which satisfies conditions (80) and (81). Two cases naturally arise. In the first case $\tilde{a}_{12} \neq 0$ we find the term $\tilde{d}_{21}$ from equality (80) and obtain the structure of the matrix of dissipative and gyroscopic forces as

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \dfrac{(\tilde{d}_{22}-\tilde{d}_{11})(\tilde{a}_{11}-\tilde{a}_{22})-2\tilde{a}_{21}\tilde{d}_{12}}{2\tilde{a}_{12}} & \tilde{d}_{22} \end{bmatrix}. \tag{82}$$

Using condition (69) of existence of the double eigenvalue λ_0, we transform inequalities (81) for matrix (82) in the following way:

$$\tilde{d}_{11} + \tilde{d}_{22} \geqslant 0, \tag{83}$$

$$\left(\tilde{d}_{11} - \tilde{d}_{12}\frac{\tilde{a}_{11}-\tilde{a}_{22}}{2\tilde{a}_{12}}\right)\left(\tilde{d}_{22} - \tilde{d}_{12}\frac{\tilde{a}_{22}-\tilde{a}_{11}}{2\tilde{a}_{12}}\right) \geqslant 0. \tag{84}$$

These inequalities are equivalent to the following conditions:

$$\tilde{d}_{11} \geqslant \tilde{d}_{12} \frac{\tilde{a}_{11} - \tilde{a}_{22}}{2\tilde{a}_{12}}, \quad \tilde{d}_{22} \geqslant \tilde{d}_{12} \frac{\tilde{a}_{22} - \tilde{a}_{11}}{2\tilde{a}_{12}}. \tag{85}$$

Inequalities (85) define two half-spaces in the space of three parameters $\tilde{d}_{11}, \tilde{d}_{22}, \tilde{d}_{12}$. The intersection of these half-spaces yields a dihedral angle, which is the domain of stabilization of a circulatory system by small dissipative and gyroscopic forces with the matrix $\tilde{\mathbf{D}}$ from Eq. (82) (Fig. 2).

When $\tilde{a}_{21} \neq 0$ the necessary stability conditions are

$$\tilde{d}_{11} \geqslant \tilde{d}_{21} \frac{\tilde{a}_{11} - \tilde{a}_{22}}{2\tilde{a}_{21}}, \quad \tilde{d}_{22} \geqslant \tilde{d}_{21} \frac{\tilde{a}_{22} - \tilde{a}_{11}}{2\tilde{a}_{21}}, \tag{86}$$

which correspond to the dihedral angle in the space of three parameters $\tilde{d}_{11}, \tilde{d}_{22}$, and $\tilde{d}_{21}$. In this case the structure of the matrix $\tilde{\mathbf{D}}$ is defined by the expression

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{d}_{11} & \dfrac{(\tilde{d}_{22} - \tilde{d}_{11})(\tilde{a}_{11} - \tilde{a}_{22}) - 2\tilde{a}_{12}\tilde{d}_{21}}{2\tilde{a}_{21}} \\ \tilde{d}_{21} & \tilde{d}_{22} \end{bmatrix}. \tag{87}$$

If $\tilde{a}_{12} \neq 0$ and $\tilde{a}_{21} \neq 0$, then conditions (85) and (86), corresponding to matrices (82) and (87), are equivalent.

Consider separately the case when $\tilde{a}_{11} = \tilde{a}_{22}$. Then, $\tilde{a}_{12} = 0$ or $\tilde{a}_{21} = 0$. Note that these two equalities cannot be satisfied simultaneously because this means that the double eigenvalue λ_0 would have two linearly independent eigenvectors in contradiction with the initial assumption. From Eq. (80) we find $\tilde{d}_{12} = 0$ or $\tilde{d}_{21} = 0$. In accordance with the strong conditions (85) and (86)

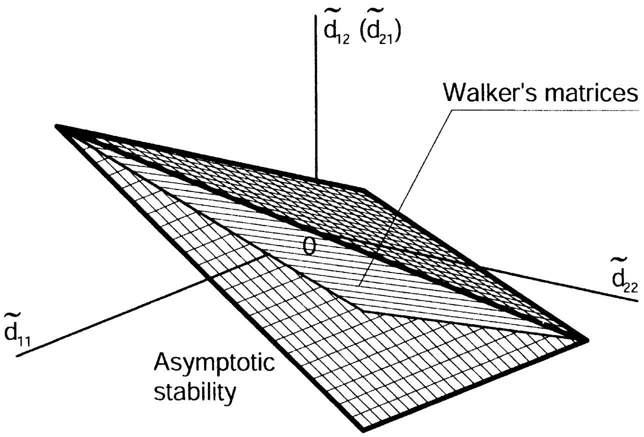

Fig. 2. Asymptotic stability domain (its boundary is double-hatched) for the non-symmetric matrices $\tilde{\mathbf{D}}$ given by Eqs. (57) and (62) for $(\tilde{a}_{11} - \tilde{a}_{22})/2\tilde{a}_{12} > 0$ (or $(\tilde{a}_{11} - \tilde{a}_{22})/2\tilde{a}_{21} > 0$).

the domain of stabilization in the space of three parameters $\tilde{d}_{11}, \tilde{d}_{22}$ and $\tilde{d}_{12}$ (or $\tilde{d}_{21}$) is the dihedral angle $\tilde{d}_{11} > 0$, $\tilde{d}_{22} > 0$.

Let us find now the structure of the *symmetric* matrices $\tilde{\mathbf{D}}$ stabilizing a circulatory system. Then, gyroscopic forces are absent and inequalities (81) mean the non-negativeness of the matrix $\tilde{\mathbf{D}}$. Note that the strong inequalities (81) mean that the dissipation is full. Isolating the coefficient $\tilde{d}_{12} = \tilde{d}_{21}$ in Eq. (80), we get

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{d}_{11} & \dfrac{(\tilde{a}_{22} - \tilde{a}_{11})(\tilde{d}_{11} - \tilde{d}_{22})}{2(\tilde{a}_{12} + \tilde{a}_{21})} \\ \dfrac{(\tilde{a}_{22} - \tilde{a}_{11})(\tilde{d}_{11} - \tilde{d}_{22})}{2(\tilde{a}_{12} + \tilde{a}_{21})} & \tilde{d}_{22} \end{bmatrix}. \tag{88}$$

Note that $\tilde{a}_{12} + \tilde{a}_{21} \neq 0$. Otherwise, the double eigenvalue λ_0 would have two linearly independent eigenvectors in contradiction with the initial assumption. Calculating the determinant and the trace of matrix (88) and assuming their non-negativeness we obtain the necessary conditions of stabilization in the plane of the two parameters $\tilde{d}_{11}, \tilde{d}_{22}$

$$\tilde{d}_{11}, \tilde{d}_{22} \geqslant 0, \quad \frac{\sqrt{x} - 1}{\sqrt{x} + 1}\tilde{d}_{22} \leqslant \tilde{d}_{11} \leqslant \tilde{d}_{22}\frac{\sqrt{x} + 1}{\sqrt{x} - 1}, \quad x = 1 + \left(\frac{\tilde{a}_{22} - \tilde{a}_{11}}{\tilde{a}_{12} + \tilde{a}_{21}}\right)^2. \tag{89}$$

Note that conditions (88) and (89) were first derived in Ref. [15] by the direct analysis of the characteristic equation (3) of system (1). Strong inequalities (89) describe the domain of stabilization of system (1) by small velocity-dependent forces. Therefore, the domain of stabilization of the circulatory system by the symmetric matrices $\varepsilon\tilde{\mathbf{D}}$ with the structure given by Eqs. (88) and (89) is an angle in the plane of the parameters $\tilde{d}_{11}$ and $\tilde{d}_{22}$ (Fig. 3). One can see from formula (89) that in general this angle is less than $\pi/2$, being right only for $\tilde{a}_{11} = \tilde{a}_{22}$.

Now we show how the set of stabilizing matrices $\tilde{\mathbf{D}}$ defined by Theorem 3 is located in the domain of stabilization given by Eqs. (82) and (85) or Eqs. (86) and (87). According to Theorem 3

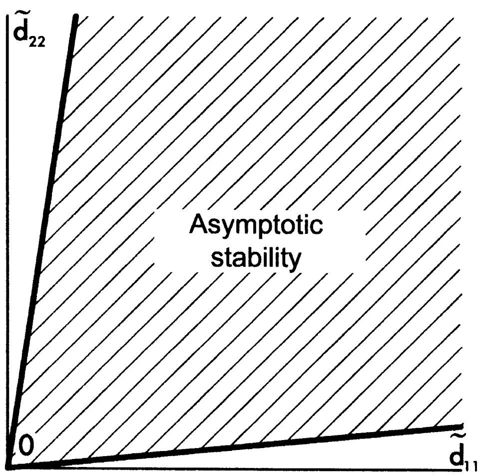

Fig. 3. Asymptotic stability domain (hatched) for the symmetric matrix $\tilde{\mathbf{D}}$ given by (63).

for $m = 2$ the stabilizing matrix $\tilde{\mathbf{D}}$ has the form

$$\tilde{\mathbf{D}} = c_0\mathbf{I} + c_1\tilde{\mathbf{A}} = \begin{bmatrix} c_0 + c_1\tilde{a}_{11} & c_1\tilde{a}_{12} \\ c_1\tilde{a}_{21} & c_0 + c_1\tilde{a}_{22} \end{bmatrix}. \tag{90}$$

Denoting $\tilde{d}_{11} = c_0 + c_1\tilde{a}_{11}$, $\tilde{d}_{22} = c_0 + c_1\tilde{a}_{22}$ and expressing the coefficient c_1 we obtain the following matrix of velocity-dependent forces

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{d}_{11} & \dfrac{\tilde{a}_{12}(\tilde{d}_{22} - \tilde{d}_{11})}{\tilde{a}_{22} - \tilde{a}_{11}} \\ \dfrac{\tilde{a}_{21}(\tilde{d}_{22} - \tilde{d}_{11})}{\tilde{a}_{22} - \tilde{a}_{11}} & \tilde{d}_{22} \end{bmatrix}. \tag{91}$$

Since the coefficients c_0 and c_1 are non-negative, $\operatorname{tr}\tilde{\mathbf{D}} > 0$. Besides, with Eq. (69) we get from Eq. (91)

$$\det\tilde{\mathbf{D}} = \left(\frac{\tilde{d}_{11} + \tilde{d}_{22}}{2}\right)^2 > 0. \tag{92}$$

Consider the case when $\tilde{a}_{12} \neq 0$. Then matrix (91) follows from Eq. (82) if we denote there

$$\tilde{d}_{12} = \frac{\tilde{a}_{12}}{\tilde{a}_{22} - \tilde{a}_{11}}(\tilde{d}_{22} - \tilde{d}_{11}). \tag{93}$$

This means that Walker's matrices form a two-dimensional-subset of the three-dimensional-stability domain given by inequalities (85). Eq. (93) defines a plane in the space of parameters $\tilde{d}_{11}, \tilde{d}_{22}$, and $\tilde{d}_{12}$, coming through the edge of the dihedral angle given by Eqs. (85) as shown in Fig. 2. The part of plane (93) satisfying the condition $\tilde{d}_{11} + \tilde{d}_{22} > 0$ is a set of the matrices $\tilde{\mathbf{D}}$ given by Theorem 3 stabilizing the circulatory system with two degrees of freedom. This set is shown in Fig. 2 as a hatched region. Note that the case $\tilde{a}_{21} \neq 0$ can be considered analogously.

We conclude that Walker's matrices given by Eq. (43) for the systems with two degrees of freedom are equivalent just to $\tilde{\mathbf{D}} = c_0\mathbf{I} + c_1\tilde{\mathbf{A}}$. Therefore, in this case Walker's matrices do not "enlarge considerably" the class of stabilizing damping configurations as it was announced in Ref. [17]. It turns out that Walker's matrices constitute a 2d-subset belonging to the 3d-set of stabilizing matrices given by Eqs. (85) and (86).

Finally, we return to the general problem (1) with two degrees of freedom with an arbitrary non-singular matrix $\mathbf{M}$. The structure of stabilizing matrix $\mathbf{D}$ in terms of the matrices $\mathbf{M}$ and $\mathbf{A}$ is given by the following equalities:

$$\mathbf{D} = \mathbf{M}\tilde{\mathbf{D}}(\tilde{\mathbf{A}}), \quad \tilde{\mathbf{A}} = \mathbf{M}^{-1}\mathbf{A}. \tag{94}$$

Thus, $\mathbf{D} = \mathbf{M}\tilde{\mathbf{D}}(\mathbf{M}^{-1}\mathbf{A})$ is the stabilizing matrix for system (1) expressed in terms of the matrices $\mathbf{M}, \mathbf{A}$, and $\tilde{\mathbf{D}}$ given by Eqs. (82)–(87).

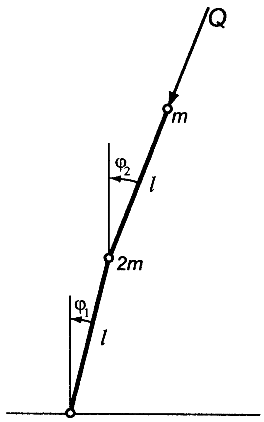

Fig. 4. Ziegler–Herrmann–Jong pendulum.

6. Mechanical example: Ziegler–Herrmann–Jong pendulum

Consider a double pendulum [2,5] composed of two rigid weightless bars of equal length l, which carry concentrated masses $m_1 = 2m$ and $m_2 = m$. The generalized coordinates φ_1 and φ_2 are assumed to be small. A follower load Q is applied at the free end as shown in Fig. 4. The visco-elastic hinges are characterized by the same stiffness c but different damping coefficients εb_1 and εb_2. Introducing the dimensionless quantities

$$q = \frac{Ql}{c}, \quad k_1 = \frac{b_1}{\sqrt{cml^2}}, \quad k_2 = \frac{b_2}{\sqrt{cml^2}}, \quad \tau = t\sqrt{\frac{c}{ml^2}}, \tag{95}$$

where τ is time, we write the equations of small vibrations of the pendulum in the form

$$\frac{d^2\mathbf{y}}{d\tau^2} + \varepsilon\mathbf{D}\frac{d\mathbf{y}}{d\tau} + \mathbf{A}\mathbf{y} = 0, \quad \mathbf{y} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \tag{96}$$

with the matrices

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} k_1 + 2k_2 & -2k_2 \\ -k_1 - 4k_2 & 4k_2 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{2}\begin{bmatrix} 3-q & q-2 \\ q-5 & 4-q \end{bmatrix}. \tag{97}$$

It is known that in the absence of viscous damping ($\varepsilon = 0$) the equilibrium of the pendulum is stable for $q < q_0 = 7/2 - \sqrt{2}$, [5]. The critical load q_0 corresponds to the boundary between stability and flutter of the circulatory system. At this point the spectrum of the system contains a pair of the double, purely imaginary eigenvalues $\pm i\omega_0, \omega_0 = 2^{-1/4}$ with only one eigenvector.

Let us find with the use of stabilization conditions (80) and (81) the values of damping parameters k_1 and k_2 making the perturbed system asymptotically stable. Calculating the invariants of the matrices $\mathbf{A}$ and $\mathbf{D}$ for $q = q_0$

$$\operatorname{tr}\mathbf{A} = \sqrt{2}, \quad \operatorname{tr}\mathbf{D} = \frac{1}{2}k_1 + 3k_2, \quad \det\mathbf{D} = \frac{1}{2}k_1k_2, \tag{98}$$

$$\mathrm{tr}(\mathbf{AD}) = \left(-\frac{1}{2}+\frac{\sqrt{2}}{2}\right)k_1 + \left(-\frac{1}{2}+3\sqrt{2}\right)k_2 \tag{99}$$

and substituting them into Eqs. (80) and (81), we obtain the necessary stabilization conditions as

$$k_1 = (5\sqrt{2}+4)k_2, \quad k_2 \geqslant 0. \tag{100}$$

Therefore, if the damping coefficients at the hinges satisfy the strong conditions (100), then the pendulum is asymptotically stable. This result coincides with that found in Ref. [5].

Let us find now the general structure of the matrix **D** stabilizing circulatory system (1) without assumption that the matrix **D** has form (97). Substituting the coefficients of the matrix **A**, evaluated at the critical point $q = q_0$ into formulae (86) and (87), we get the stabilizing matrix in the form

$$\mathbf{D} = \begin{bmatrix} d_{11} & (17-12\sqrt{2})d_{21} + (3-2\sqrt{2})(d_{22}-d_{11}) \\ d_{21} & d_{22} \end{bmatrix}, \tag{101}$$

with the coefficients satisfying the following inequalities:

$$-d_{22} \leqslant d_{21}(3-2\sqrt{2}) \leqslant d_{11}. \tag{102}$$

It is easy to verify that if the matrix **D** has form (97), then conditions (101) and (102) are equivalent to conditions (100).

7. Conclusion

Stabilization and destabilization phenomena of circulatory systems due to small velocity-dependent forces have been attracting substantial interest from researchers for half a century since the work by Ziegler [2]. In the present paper three theorems on the necessary and sufficient conditions for the matrices of velocity-dependent forces to stabilize an unperturbed circulatory system are established. These results are of general nature and have an applicable form allowing one to find elements of the stabilizing matrices.

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