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Stability and bifurcation analysis in Van der Pol's oscillator with delayed feedback ☆

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Abstract

The classical Van der Pol equation with delayed feedback and a modified equation where a delayed term provides the damping are considered. Linear stability of the equations is investigated by analyzing the associated characteristic equations. It is found that there exist the stability switches when delay varies, and the Hopf bifurcation occurs when the delay passes through a sequence of critical values. The bifurcation diagram is drawn in (ε, k)-plane, and the stability and direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem.

1. Introduction

Van der Pol's equation provides an example of an oscillator with nonlinear damping, energy being dissipated at large amplitudes and generated at low amplitudes. Such systems typically possess limit cycles, sustained oscillations around a state at which energy generation and dissipation balance, and they arise in many physical problems, see Refs. [1,2]. It is well known that the limit cycle oscillations with strong stability property are important in applications, hence, being able to modify their behavior through feedback is a question of interest. On the other hand, most practical implementations of feedback have inherent delays, the presence of which leads to

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an infinite-dimension system, thus complicating the analysis, see Refs. [3–6]. Recently, Atay [3] employing the averaging method introduced by Hale [7] has studied the behavior of the limit cycle to the Van der Pol's equation with nonlinear damping

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = f(t), \quad x \in R, \ \varepsilon > 0, \tag{1}$$

where the forcing f is a delayed feedback of the position x. Precisely, Atay [3] investigated the equations

$$\ddot{x}(t) + \varepsilon(x(t)^2 - 1)\dot{x}(t) + x(t) = \varepsilon k x(t - \tau) \tag{2}$$

and

$$\ddot{x}(t) + \varepsilon(x(t)^2 - 1)x(t - \tau) + x(t) = 0. \tag{3}$$

Clearly, Eq. (2) is the classical Van der Pol's equation with delayed feedback of position x, and Eq. (3) is a modification of Van der Pol's equation

$$\ddot{x}(t) + \varepsilon(x(t)^2 - 1)\dot{x}(t) + x(t) = 0. \tag{4}$$

In fact, it is supposed that the delayed position $x(t - \tau)$ incorporates some characteristics of the derivative $\dot{x}(t)$ in Eq. (4), and hence Eq. (3) follows.

Using the averaging methods Atay [3] shows that Eqs. (2) and (3) have stable and unstable periodic solutions when $\varepsilon \ll 1$ and other conditions are satisfied. An interesting question is how the delay affects the dynamics to Eqs. (2) and (3). Particularly, when $\varepsilon > 0$ and $\tau = 0$, the characteristic equations associated with the linearization of Eqs. (2) and (3) around $x = 0$ have a root with positive real part and a pair of purely imaginary roots, respectively. How is the stability to be changed as the delay varies, and how are periodic solutions to arise? The aim of the present paper is to answer these questions partially from bifurcation. It is found that there are stability switches when the delay varies, and the system undergoes a Hopf bifurcation at the origin when τ passes through a sequence of critical values. Furthermore, using the normal form and center manifold theory, the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation are determined. It is interesting that there are stability switches for Eqs. (2) and (3) even though the zero solution of Eq. (3) without delay is unstable and the characteristic equation of Eq. (2) without delay has a pair of imaginary roots. With regards to the stability switches we refer the reader to Cooke and Grossman [8].

Recently, there has been similar works in this research topic. For example, for a Van der Pol–Duffing oscillator and a co-dimension two-bifurcation system which possesses one zero eigenvalue and a pair of purely imaginary eigenvalues that are excited parametrically by a real noise with small intensity, which assume to be the first component of an output of a linear filter system and conforms to the detailed balance condition [9], Liu and Liew [10,11] have obtained the asymptotical expansions of the top Lyapunov exponents for the relevant systems. Furthermore, Liu and Liew [12] extended the research work in Ref. [11], where they have investigated the almost-sure stability condition for a co-dimension two-bifurcation system on a three-dimensional center manifold, which was parametrically excited by a real noise. The results they obtained are very interesting.

We would like to mention that, in the recent paper, the properties of Hopf bifurcation such as the direction of bifurcation and stability of bifurcating periodic solutions from the origin are determined precisely. Meanwhile, the bifurcation diagram for Van der Pol's oscillator with delayed feedback is drawn in the parameters, (ε, k)-plane. On the other hand, $\varepsilon > 0$ (resp. $0 < \varepsilon < 1$) with other conditions can ensure the existence of periodic solutions for Eq. (2) (resp. Eq. (3)) other than $0 < \varepsilon \ll 1$ like Atay [3].

The remainder of the present paper is as follows: in Section 2, we investigate the stability of the zero solution and the occurrence of Hopf bifurcation. In Section 3, direction and stability of the Hopf bifurcation are determined. Some numerical simulations to support the analysis results are given in Section 4.

2. Stability and Hopf bifurcation

For convenience, we write Eqs. (2) and (3) as the following forms, respectively:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) + \varepsilon k x(t-\tau) - \varepsilon(x(t)^2 - 1)y(t) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) - \varepsilon(x(t)^2 - 1)x(t-\tau). \end{aligned} \tag{6}$$

Firstly we consider Eq. (5). The linearization of Eq. (5) around the origin $(0, 0)$ is given by

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) + \varepsilon k x(t-\tau) + \varepsilon y(t). \end{aligned} \tag{7}$$

Its characteristic equation is

$$\lambda^2 - \varepsilon\lambda - \varepsilon k e^{-\lambda\tau} + 1 = 0. \tag{8}$$

Note that when $\tau = 0$ Eq. (8) becomes

$$\lambda^2 - \varepsilon\lambda - \varepsilon k + 1 = 0,$$

and its roots are

$$\lambda_{1,2} = \tfrac{1}{2}[\varepsilon \pm \sqrt{\varepsilon^2 - 4(1 - \varepsilon k)}].$$

Clearly, there is at least one root λ satisfying $\mathrm{Re}\,\lambda > 0$ when $\varepsilon > 0$. For convenience, we make the following assumptions:

$$(\mathrm{H}_1) \quad k^2 + \left(\frac{\varepsilon}{2}\right)^2 > 1,$$

$$(\mathrm{H}_2) \quad |k| < \frac{1}{\varepsilon}.$$

Lemma 2.1. *For Eq.* (8), *we have*

(i) *if* (H_1) *is not satisfied and* $k^2 + (\varepsilon/2)^2 \neq 1$, *or* ($H_1$), ($H_2$) *and* $\varepsilon > \sqrt{2}$ *are satisfied, then Eq.* (8) *has at least one root with positive real part for all* $\tau > 0$;
(ii) *if* (H_2) *is not satisfied and* $|k| \neq 1/\varepsilon$, *then there exists* $\tau_0^+ < \tau_1^+ < \cdots < \tau_j^+ < \cdots$, *such that Eq.* (8) *has a pair of purely imaginary roots* $\pm i\omega_+$ *when* $\tau = \tau_j^+$;
(iii) *if* (H_1), (H_2) *and* $0 < \varepsilon < \sqrt{2}$ *are satisfied, then there exist* $\tau_0^+ < \tau_1^+ < \cdots < \tau_j^+ < \cdots$ *and* $\tau_0^- < \tau_1^- < \cdots < \tau_j^- < \cdots$ *such that Eq.* (8) *has a pair of imaginary roots* $\pm i\omega_\pm$ *when* $\tau = \tau_j^\pm$,*respectively*, *where*

$$\tau_j^+ = \begin{cases} \dfrac{1}{\omega_+}\left(2\pi - \arcsin\dfrac{\omega_+}{|k|} + 2j\pi\right), & k \leqslant -1, \\ \dfrac{1}{\omega_+}\left(\pi + \arcsin\dfrac{\omega_+}{|k|} + 2j\pi\right), & -1 \leqslant k < 0, \\ \dfrac{1}{\omega_+}\left(\arcsin\dfrac{\omega_+}{k} + 2j\pi\right), & 0 < k \leqslant 1, \\ \dfrac{1}{\omega_+}\left(\pi - \arcsin\dfrac{\omega_+}{k} + 2j\pi\right), & k \geqslant 1, \; j = 0, 1, \ldots, \end{cases} \tag{9}$$

$$\tau_j^- = \begin{cases} \dfrac{1}{\omega_-}\left(\pi + \arcsin\dfrac{\omega_-}{|k|} + 2j\pi\right), & k < 0, \\ \dfrac{1}{\omega_-}\left(\arcsin\dfrac{\omega_-}{k} + 2j\pi\right), & k > 0, \; j = 0, 1, \ldots \end{cases} \tag{10}$$

and

$$\omega_\pm = \frac{1}{\sqrt{2}}\left[(2 - \varepsilon^2) \pm \varepsilon\sqrt{4(k^2 - 1) + \varepsilon^2}\right]^{1/2}. \tag{11}$$

Let

$$\lambda = \alpha(\tau) + i\omega(\tau)$$

be the root of Eq. (8) satisfying $\alpha(\tau_j^\pm) = 0, \omega(\tau_j^\pm) = \omega_\pm$, respectively. Substituting $\lambda(\tau)$ into Eq. (8) and taking the derivative with respect to τ, one can obtain the following conclusions easily.

Lemma 2.2. $\alpha'(\tau_j^+) > 0$, *and* $\alpha'(\tau_j^-) < 0$.

Claim 1. *Suppose* (H_1), (H_2), $0 < \varepsilon < \sqrt{2}$ *and* $k > 0$ *are satisfied. Then* $\tau_0^+ > \tau_0^-$, *and there exists an integer* $m \geqslant 0$ *such that*

$$\tau_0^- < \tau_0^+ < \tau_1^- < \cdots < \tau_m^- < \tau_m^+ < \tau_{m+1}^+ < \tau_{m+1}^-.$$

Proof. Conditions (H_1), (H_2) and $0<\varepsilon<\sqrt{2}$ imply that $\tau_j^{\pm}$ are defined well. It is sufficient to verify that $\tau_0^{+}>\tau_0^{-}$. From

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+1)(2n)!!} x^{2n+1}, \quad x \in [-1,1]$$

and Eqs. (9) and (10) we have

$$\tau_0^{+} = \frac{1}{k} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+1)(2n)!!} \cdot \frac{\omega_+^{2n}}{k^{2n+1}}, \quad 0<k<1,$$

$$\begin{aligned}\tau_0^{+} &= \frac{1}{\omega_+}\left(\pi - \arcsin\frac{\omega_+}{k}\right) > \frac{1}{\omega_+}\left(\arcsin\frac{\omega_+}{k}\right)\\ &= \frac{1}{k} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+1)(2n)!!} \cdot \frac{\omega_+^{2n}}{k^{2n+1}}, \quad k>1\end{aligned}$$

and

$$\tau_0^{-} = \frac{1}{k} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+1)(2n)!!} \cdot \frac{\omega_-^{2n}}{k^{2n+1}}.$$

This and Eq. (11) imply that $\tau_0^{+}>\tau_0^{-}$.

From $\tau_{j+1}^{+}-\tau_j^{+}=\frac{2\pi}{\omega_+}$, $\tau_{j+1}^{-}-\tau_j^{-}=\frac{2\pi}{\omega_-}$ and $\omega_+>\omega_-$ we have

$$\tau_{j+1}^{+}-\tau_j^{+}<\tau_{j+1}^{-}-\tau_j^{-}.$$

Hence the conclusion follows. □

Lemma 2.3. (i) *If* (H_2) *is not satisfied, then Eq.* (8) *has at least one root with positive real part for all* $\tau>0$.

(ii) *If* $0<\varepsilon<\sqrt{2}, k>0$, *and* ($H_1$) *and* ($H_2$) *are satisfied, then there exists an integer* $m\geqslant 0$ *such that Eq.* (8) *has a pair of roots with positive real parts when* $\tau\in(\tau_{j-1}^{+},\tau_j^{-})$ *for* $j=0,1,\ldots,m$ *with* $\tau_{-1}^{+}=0$, *and all roots of Eq.* (8) *have negative real parts when* $\tau\in(\tau_j^{-},\tau_j^{+})$ *for* $j=0,1,\ldots,m$, *and Eq.* (8) *has at least a pair of roots with positive real parts when* $\tau>\tau_m^{+}$.

Note that Eq. (8) with $\tau=0$ has at least one root with positive real part, and hence the conclusion of (i) follows from (ii) in Lemmas 2.1 and 2.2. The conclusion of (ii) follows from (iii) in Lemmas 2.1, 2.2 and Claim 1 and Rouché's Theorem [14, Theorem 9.17.4].

From Lemmas 2.1–2.3 and the Hopf bifurcation theorem for functional differential equations [5, Chapter 11, Theorem 1.1], we have the following results on stability and bifurcation to system (5).

Theorem 2.4. *For system* (5),

(i) *if* (H_1) *is not satisfied and* $k^2+(\varepsilon/2)^2\neq 1$, *or* ($H_1$),($H_2$) *and* $\varepsilon>\sqrt{2}$ *are satisfied, then the zero solution is unstable for all* $\tau\geqslant 0$;

(ii) *if* (H_2) *is not satisfied and* $|k|\neq 1/\varepsilon$, *then the zero solution is unstable for all* $\tau\geqslant 0$, *and system* (5) *undergoes a Hopf bifurcation at the origin* (0,0) *when* $\tau=\tau_j^{+}$, $j=0,1,\ldots$;

(iii) *if* (H_1), (H_2) *and* $0<\varepsilon<\sqrt{2}$ *are satisfied, then system* (5) *undergoes a Hopf bifurcation at the origin* (0,0) *when* $\tau=\tau_j^{\pm}$, $j=0,1,\ldots$. *Particularly, when* $k>0$, *there exists an integer* $m\geqslant 0$ *such that the zero solution is unstable when* $\tau\in(\tau_{j-1}^{+},\tau_j^{-})$ *for* $j=0,1,\ldots,m$ *and* $\tau>\tau_m^{+}$, *and asymptotically stable when* $\tau\in(\tau_j^{-},\tau_j^{+})$ *for* $j=0,1,\ldots,m$.

Remark. From the conclusions of Theorem 2.4, we can draw the bifurcation diagram in the parameter plane as Fig. 1. Under hypotheses (H_1), (H_2) and $0<\varepsilon<\sqrt{2}$, we have proved that $\tau_0^-<\tau_0^+$ when $k>0$. But in the case $k<0$, either $\tau_0^-<\tau_0^+$ or $\tau_0^-<\tau_0^+$ may occur. The numerical simulations in Fig. 2 show this.

Next, we consider system (6). Its linearization around the origin is given by

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=-x(t)+\varepsilon x(t-\tau).\end{aligned}\tag{12}$$

The characteristic equation associated with Eq. (12) is

$$\lambda^2-\varepsilon e^{-\lambda\tau}+1=0.\tag{13}$$

One can easily get the following conclusion.

Lemma 2.5. *Suppose that* $0<\varepsilon<1$ *is satisfied. Then there exist*

$$\tau_0^+<\tau_1^+<\cdots<\tau_j^+<\cdots$$

and

$$\tau_0^-<\tau_1^-<\cdots<\tau_j^-<\cdots$$

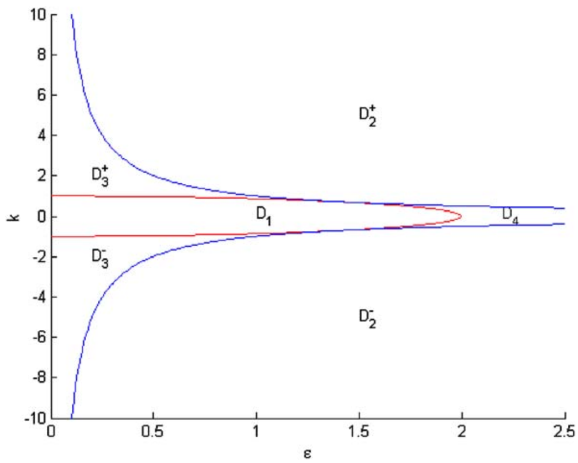

Fig. 1. The bifurcation diagram for Eq. (5). The curves $k^2+(\varepsilon/2)^2=1$ and $k=\pm 1/\varepsilon$ divide the right half (ε,k)-plane into six regions. The zero solution is unstable for all $\tau\geqslant 0$ when $(\varepsilon,k)\in D_1\cup D_4$. When $(\varepsilon,k)\in D_2^+\cup D_2^-$, the zero solution is unstable for all $\tau\geqslant 0$, and system (5) undergoes a Hopf bifurcation when $\tau=\tau_j^+$, $j=0,1,\ldots$; $D_3^+\cup D_3^-$ is a conditional stability region. Particularly, when $(\varepsilon,k)\in D_3^+$, there exists an integer $m\geqslant 0$ such that the zero solution is asymptotically stable when $\tau\in\bigcup_{j=0}^{m}(\tau_j^-,\tau_j^+)$ and unstable when $\tau\in(\bigcup_{j=0}^{m}(\tau_j^+,\tau_j^-))\cup(\tau_m^+,\infty)$.

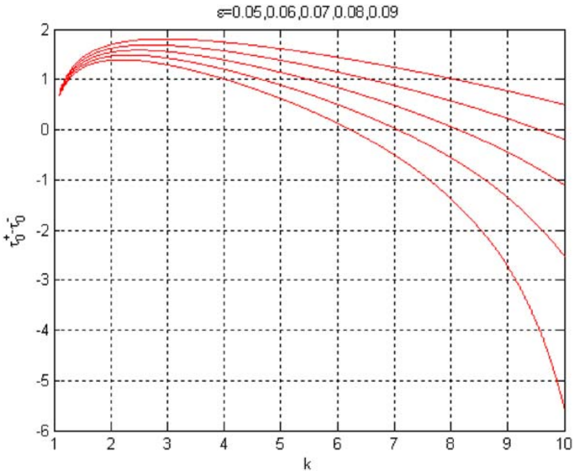

Fig. 2. These $(|k|, \tau_0^+ - \tau_0^-)$-curves are for $k<0$, and $\varepsilon = 0.05, 0.06, 0.07, 0.08$, and 0.09, respectively, which show that for $(\varepsilon, k) \in D_3^-$, $\tau_0^- < \tau_0^+$ or $\tau_0^- > \tau_0^+$ may occur.

such that Eq. (13) *has a pair of imaginary roots* $\pm i\omega_\pm$ *when* $\tau = \tau_j^\pm$, $j = 0, 1, \ldots$, *respectively*, *where*

$$\omega_\pm = \sqrt{1 \pm \varepsilon}, \tag{14}$$

$$\tau_j^+ = \frac{(2j+1)\pi}{\sqrt{1+\varepsilon}}, \quad j = 0, 1, 2, \ldots \tag{15}$$

and

$$\tau_j^- = \frac{2j\pi}{\sqrt{1-\varepsilon}}, \quad j = 0, 1, 2, \ldots \ . \tag{16}$$

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$$

be the root of Eq. (13), satisfying $\alpha(\tau_j^\pm) = 0$ and $\omega(\tau_j^\pm) = \omega_\pm$. Similar to Lemma 2.2, we have following:

Lemma 2.6. *If* $0<\varepsilon<1$ *is satisfied*, *then*

$$\alpha'(\tau_j^+)>0 \quad \textit{and} \quad \alpha'(\tau_j^-)<0.$$

For convenience, we denote

$$\{\tau_n\}_{n=0}^\infty = \{\tau_j^+\} \cup \{\tau_j^-\} \tag{17}$$

satisfying

$$\tau_{2j+1} = \tau_j^+, \quad \tau_{2j} = \tau_j^-.$$

Claim 2. *For the sequence* $\{\tau_n\}_{n=0}^{\infty}$, *we have*

$$0=\tau_0<\tau_1<\tau_2<\cdots<\tau_{2m}<\tau_{2m+1}<\tau_{2m+3}<\tau_{2m+2},$$

where

$$m=\left[\frac{\sqrt{1-\varepsilon}}{2(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon})}\right].$$

Proof. Clearly, $\tau_{2j-1}<\tau_{2j+1}, \tau_{2j}<\tau_{2j+2}$ and $\tau_{2j-1}<\tau_{2j}$ for $j=1,2,\ldots$. For $j\leqslant m$, we have

$$\begin{aligned}\frac{\tau_{2j+1}}{\tau_{2j}}&=\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\left(1+\frac{1}{2j}\right)\geqslant\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\left(1+\frac{1}{2m}\right)\\&>\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\left[1+\frac{2(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon})}{2\sqrt{1-\varepsilon}}\right]=1,\end{aligned}$$

which implies that $\tau_{2j}<\tau_{2j+1}$ for $j\leqslant m$. Hence

$$\tau_0<\tau_1<\tau_2<\cdots<\tau_{2m}<\tau_{2m+1}.$$

Meanwhile, from $m+1>\sqrt{1-\varepsilon}/(2(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon}))$ it follows that

$$\frac{\tau_{2m+3}}{\tau_{2m+2}}=\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\left(1+\frac{1}{2(m+1)}\right)<\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\left(1+\frac{\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon}}\right)=1,$$

which means that $\tau_{2m+3}<\tau_{2m+2}$. Thus $\tau_{2m+1}<\tau_{2m+3}<\tau_{2m+2}$, and this completes the proof. □

Noting the roots of Eq. (13) with $\tau=0$ are

$$\lambda=\pm\mathrm{i}\sqrt{1-\varepsilon}$$

and applying Lemmas 2.5, 2.6 and Claim 2, we get the following conclusion.

Lemma 2.7. *Suppose that* $0<\varepsilon<1$ *is satisfied. Then there exists a sequence* $\{\tau_j\}_{j=0}^{\infty}$ *which is defined by Eq.* (17) *such that all the roots of Eq.* (13) *have negative real parts when* $\tau\in\bigcup_{j=0}^{m}(\tau_{2j},\tau_{2j+1})$, *and Eq.* (13) *has at least a pair of roots with positive real parts when* $\tau\in(\bigcup_{j=1}^{m}(\tau_{2j-1},\tau_{2j}))\cup(\tau_{2m+1},\infty)$, *where*

$$m=\left[\frac{\sqrt{1-\varepsilon}}{2(\sqrt{1+\varepsilon}-\sqrt{1-\varepsilon})}\right].$$

Theorem 2.8. *Suppose that* $0<\varepsilon<1$ *is satisfied. Then there exists a sequence* $\{\tau_j\}_{j\geqslant 0}$ *such that the zero solution of system* (6) *is asymptotically stable when* $\tau\in\bigcup_{j=0}^{m}(\tau_{2j},\tau_{2j+1})$, *and unstable when* $\tau\in(\bigcup_{j=1}^{m}(\tau_{2j-1},\tau_{2j}))\cup(\tau_{2m+1},\infty)$, *as well as system* (6), *undergoes a Hopf bifurcation at the origin when* $\tau=\tau_j$, $j=0,1,\ldots$, *where* $\{\tau_j\}_{j\geqslant 0}$ *and m is defined by Eq.* (17) *and Lemma* 2.7, *respectively.*

It is easy to obtain the conclusions of Theorem 2.8 from Lemmas 2.6, 2.7 and the Hopf bifurcation theorem for functional differential equation [5, Chapter 11, Theorem 1.1].

3. Direction and stability of Hopf bifurcation

In Section 2 we have obtained some conditions which guarantee that Van der Pol's equation with delayed feedback and its modification undergo Hopf bifurcation at some critical values of τ. In this section we shall study the direction, stability, and the period of the bifurcating periodic solutions. The method we use is based on the normal form method and the center manifold theory introduced by Hassard et al. [13].

We first re-scale the time by $t \mapsto (t/\tau)$ to normalize the delay so that system (5) and (6) can be written as the form

$$\dot{x}(t) = F(x_t, \tau).$$

In fact, system (5) and (6) become, respectively,

$$\begin{aligned} \dot{x}(t) &= \tau y(t), \\ \dot{y}(t) &= -\tau x(t) + \varepsilon k \tau x(t-1) - \varepsilon\tau(x^2(t) - 1)y(t) \end{aligned} \tag{5'}$$

and

$$\begin{aligned} \dot{x}(t) &= \tau x(t), \\ \dot{y}(t) &= -\tau x(t) - \varepsilon\tau(x^2(t) - 1)x(t-1). \end{aligned} \tag{6'}$$

3.1. *The properties of Hopf bifurcation to Eq.* (5′)

In this subsection, we consider Eq. (5′). The characteristic equation associated with the linearization of Eq. (5′) around the origin is given by

$$z^2 - \varepsilon\tau z - \varepsilon k \tau^2 \mathrm{e}^{-z} + \tau^2 = 0. \tag{18}$$

Comparing Eq. (18) with Eq. (8), one can find that $z = \tau\lambda$. So from conclusions (iii) in Lemma 2.1 and (ii) in Lemma 2.3 we have that there are $\tau_0^+ < \tau_1^+ < \cdots < \tau_j^+ < \cdots$ and $\tau_0^- < \tau_1^- < \cdots < \tau_j^- < \cdots$, and an integer $m \geqslant 0$ such that Eq. (18) has a pair of imaginary roots $\pm \mathrm{i}\tau_j^\pm\omega_\pm$ when $\tau = \tau_j^\pm$, respectively, and all the roots of Eq. (18) with $\tau = \tau_j^\pm$ for $0 \leqslant j \leqslant m$, except $\pm\mathrm{i}\tau_j^\pm\omega_\pm$, have negative real parts. Let $z(\tau)$ be the root of Eq. (18) satisfying $\operatorname{Re} z(\tau_j^\pm) = 0$ and $\operatorname{Im} z(\tau_j^\pm) = \tau_j^\pm\omega_\pm$. By Lemma 2.2, we have

$$\frac{\mathrm{d}\operatorname{Re} z(\tau_j^+)}{\mathrm{d}\tau} = \tau_j^+ \alpha'(\tau_j^+) > 0; \quad \frac{\mathrm{d}\operatorname{Re} z(\tau_j^-)}{\mathrm{d}\tau} = \tau_j^- \alpha'(\tau_j^-) < 0, \tag{19}$$

where $\alpha(\tau)$ is defined by Lemma 2.2.

Clearly, the phase space is $C = C([-1, 0], R^2)$. For convenience, let $\tau = \tau_0 + \mu$, $\mu \in R$ and τ_0 be taken in $\{\tau_j^+\} \cup \{\tau_j^-\}$. Then $\mu = 0$ is the Hopf bifurcation value for Eq. (5′). Let $\mathrm{i}\tau_0\omega_0$ be the root of Eq. (18) when $\tau = \tau_0$, where either $\omega_0 = \omega_+$ or $\omega_0 = \omega_-$. For $\varphi \in C$, let

$$L_\mu \varphi = (\tau_0 + \mu)B_1\varphi(0) + \varepsilon k(\tau_0 + \mu)B_2\varphi(-1), \tag{20}$$

where

$$B_1 = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$f(\mu, \varphi) = \begin{pmatrix} 0 \\ -\varepsilon(\tau_0 + \mu)\varphi_1^2(0)\varphi_2(0) \end{pmatrix}. \tag{21}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu \varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta, \mu)\varphi(\theta) \quad \text{for } \varphi \in C. \tag{22}$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_0 + \mu)B_1, & \theta = 0, \\ -\varepsilon k(\tau_0 + \mu)\mathsf{B}_2\delta(\theta), & \theta \in [-1, 0). \end{cases}$$

Then Eq. (22) is satisfied.

For $\varphi \in C^1([-1, 0], R^2)$, define

$$A(\mu)\varphi = \begin{cases} \mathrm{d}\varphi(\theta)/\mathrm{d}\theta, & \theta \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\eta(t, \mu)\varphi(t), & \theta = 0 \end{cases} \tag{23}$$

and

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases} \tag{24}$$

Hence, we can rewrite Eq. (5′) as the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{25}$$

where $u = (u_1, u_2)^{\mathrm{T}}$, $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0, 1], R^2)$, define

$$A^*\psi(s) = \begin{cases} -\mathrm{d}\psi(s)/\mathrm{d}s, & s \in (0, 1], \\ \int_{-1}^{0} \psi(-t)\,\mathrm{d}\eta(t, 0), & s = 0. \end{cases}$$

For $\varphi \in C[-1, 0]$ and $\psi \in C[0, 1]$, define the bilinear form

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\,\mathrm{d}\eta(\theta)\varphi(\xi)\,\mathrm{d}\xi, \tag{26}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and $A(0)$ are adjoint operators, and $\pm \mathrm{i}\tau_0\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

By direct computation, we obtain that

$$q(\theta) = \begin{pmatrix} 1 \\ \mathrm{i}\omega_0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}$$

is the eigenvector of $A(0)$ corresponding to $\mathrm{i}\tau_0\omega_0$, and

$$q^*(s) = D(-\varepsilon + \mathrm{i}\omega_0, 1)\mathrm{e}^{\mathrm{i}\tau_0\omega_0 s}$$

is the eigenvector of A^* corresponding to $-\mathrm{i}\tau_0\omega_0$, where

$$D = (-\varepsilon + \varepsilon k\tau_0 \mathrm{e}^{\mathrm{i}\tau_0\omega_0})^{-1}.$$

Moreover,

$$\langle q^*, q\rangle = 1, \quad \langle q^*, \bar{q}\rangle = 0.$$

Using the same notation as in Ref. [13], we first compute the coordinates to describe the center manifold $\mathscr{C}_0$ at $\mu = 0$. Let u_t be the solution of Eq. (5′) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t\rangle, \quad w(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold $\mathscr{C}_0$ we have

$$w(t,\theta) = w(z(t), \bar{z}(t), \theta),$$

where

$$w(z,\bar{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

z and $\bar{z}$ are local coordinates for center manifold $\mathscr{C}_0$ in the direction of q^* and $\bar{q}^*$. Note that w is real if u_t is real. We consider only real solutions.

For solution u_t in $\mathscr{C}_0$ of Eq. (5′), since $\mu = 0$,

$$\begin{aligned}\dot{z}(t) &= \mathrm{i}\tau_0\omega_0 z + \langle q^*(\theta),\ f(w + 2\,\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &= \mathrm{i}\tau_0\omega_0 z + \bar{q}^*(0) f(w(z,\bar{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\}) \\ &\stackrel{\text{def}}{=} \mathrm{i}\tau_0\omega_0 z + \bar{q}^*(0) f_0(z,\bar{z}).\end{aligned}$$

We rewrite this as

$$\dot{z}(t) = \mathrm{i}\tau_0\omega_0 z(t) + g(z,\bar{z}), \tag{27}$$

where

$$\begin{aligned}g(z,\bar{z}) &= \bar{q}^*(0) f(w(z,\bar{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots\ .\end{aligned} \tag{28}$$

By Eqs. (25) and (27), we have

$$\begin{aligned}\dot{w} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q}\\ &= \begin{cases} Aw - 2\,\mathrm{Re}\{\bar{q^*}(0)f_0 q(\theta)\}, & \theta \in [-1,0),\\ Aw - 2\,\mathrm{Re}\{\bar{q^*}(0)f_0 q(0)\} + f_0, & \theta = 0,\end{cases}\\ &\stackrel{\text{def}}{=} Aw + H(z,\bar{z},\theta),\end{aligned}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \,. \tag{29}$$

Expanding the above series and comparing the coefficients, we obtain

$$(A - 2\mathrm{i}\tau_0\omega_0 I)w_{20}(\theta) = -H_{20}(\theta), \quad Aw_{11}(\theta) = -H_{11}(\theta), \ldots \,. \tag{30}$$

Notice that

$$q^*(0) = D(-\varepsilon + \mathrm{i}\omega_0, 1),$$

$$x(t) = z + \bar{z} + w^{(1)}(z,\bar{z},0)$$

and

$$y(t) = \mathrm{i}\omega_0 z - \mathrm{i}\omega_0\bar{z} + w^{(2)}(z,\bar{z},0),$$

where

$$w^{(1)}(z,\bar{z},0) = w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$w^{(2)}(z,\bar{z},0) = w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

and

$$f_0 = \begin{pmatrix} 0 \\ -\varepsilon\tau_0 x^2(t)y(t)\end{pmatrix},$$

we have

$$\begin{aligned} g(z,\bar{z}) = \bar{q^*}(0)f_0 &= -\bar{D}\varepsilon\tau_0 x^2(t)y(t)\\ &= -\bar{D}\varepsilon\tau_0\left(z + \bar{z} + w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots\right)^2\\ &\quad\cdot\left(\mathrm{i}\omega_0 z - \mathrm{i}\omega_0\bar{z} + w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots\right)\\ &= -\bar{D}\varepsilon\tau_0(-\mathrm{i}\omega_0 z^2\bar{z} + 2\mathrm{i}\omega_0 z^2\bar{z} + \cdots)\\ &= -\bar{D}\varepsilon\tau_0(\mathrm{i}\omega_0 z^2\bar{z} + \cdots).\end{aligned}$$

Comparing the coefficients with Eq. (28), we have

$$g_{20} = g_{11} = g_{02} = 0,$$

$$g_{21} = -2\mathrm{i}\bar{D}\varepsilon\tau_0\omega_0.$$

Hence, from

$$C_1(0) = \frac{\mathrm{i}}{2\tau_0\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},$$

we get

$$C_1(0) = -\mathrm{i}\bar{D}\varepsilon\tau_0\omega_0.$$

Notice

$$\bar{D} = \frac{1}{-\varepsilon + \varepsilon k\tau_0 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}}$$

and

$$\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} = \frac{1 - \omega_0^2 - \mathrm{i}\varepsilon\omega_0}{\varepsilon k},$$

we have

$$\bar{D} = \frac{1}{\tau_0(1-\omega_0^2) - \varepsilon - \mathrm{i}\varepsilon\tau_0\omega_0}.$$

Hence

$$C_1(0) = \frac{1}{\triangle}[\varepsilon^2\tau_0^2\omega_0^2 - \mathrm{i}\varepsilon\tau_0\omega_0(\tau_0(1-\omega_0^2) - \varepsilon)],$$

where

$$\triangle = (\tau_0(1-\omega_0^2) - \varepsilon)^2 + \varepsilon^2\tau_0^2\omega_0^2.$$

Thus

$$\mathrm{Re}\, C_1(0) = \frac{\varepsilon^2\tau_0^2\omega_0^2}{\triangle} > 0,$$

$$\mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\alpha'(\tau_0)} = -\frac{1}{\triangle}\frac{\varepsilon^2\tau_0^2\omega_0^2}{\alpha'(\tau_0)} = \begin{cases} >0, & \tau_0 = \tau_j^-, \\ <0, & \tau_0 = \tau_j^+, \end{cases}$$

$$\beta_2 = 2\,\mathrm{Re}\{C_1(0)\} = \frac{2\varepsilon^2\tau_0^2\omega_0^2}{\triangle} > 0.$$

Theorem 3.1. *Suppose that* (H_1), (H_2) *and* $0<\varepsilon<\sqrt{2}$ *are satisfied. For* $0\leqslant j\leqslant m$, *we have*

(i) *the Hopf bifurcation at the origin when* $\tau=\tau_j^+$ *is subcritical and the bifurcating periodic solutions are unstable*;
(ii) *the Hopf bifurcation at the origin when* $\tau=\tau_j^-$ *is supercritical and the bifurcating periodic solutions are unstable.*

The conclusions follow from the general theorem in Ref. [13].

3.2. The properties of Hopf bifurcation to Eq. (6′)

In this subsection, we consider Eq. (6′). The characteristic equation associated with the linearization of Eq. (6′) around the origin is

$$z^2-\varepsilon\tau^2\mathrm{e}^{-z}+\tau^2=0. \tag{18′}$$

Similar to Section 3.1 we have

$$L_\mu\varphi=(\tau_0+\mu)B_1\varphi(0)+\varepsilon(\tau_0+\mu)B_2\varphi(-1), \tag{20′}$$

where

$$B_1=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad B_2=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

and

$$f(\mu,\varphi)=\begin{bmatrix}0\\ -\varepsilon(\tau_0+\mu)\varphi_1^2(0)\varphi_1(-1)\end{bmatrix}. \tag{21′}$$

There exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $\theta\in[-1,0]$ such that

$$L_\mu\varphi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\varphi(\theta)\quad\text{for }\varphi\in C, \tag{22′}$$

where

$$\eta(\theta,\mu)=\begin{cases}(\tau_0+\mu)B_1, & \theta=0,\\ -\varepsilon(\tau_0+\mu)B_2\delta(\theta+1), & \theta\in[-1,0).\end{cases}$$

By direct computation, we obtain that

$$q(\theta)=\begin{bmatrix}1\\ \mathrm{i}\omega_0\end{bmatrix}\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}$$

is an eigenvector of $A(0)$ corresponding to $\mathrm{i}\tau_0\omega_0$, and

$$q^*(s)=D(\mathrm{i}\omega_0,1)\mathrm{e}^{\mathrm{i}\tau_0\omega_0 s}$$

is an eigenvector of A^* corresponding to $-\mathrm{i}\tau_0\omega_0$. Moreover,

$$\langle q^*,q\rangle=1\quad\text{and}\quad\langle q^*,\bar{q}\rangle=0,$$

where

$$D = \frac{1}{\varepsilon\tau_0} e^{-i\tau_0\omega_0}.$$

Notice that

$$q^*(0) = D(i\omega_0, 1),$$

$$\begin{aligned} x(t) &= zq^{(1)}(0) + \bar{z}\bar{q}^{(1)}(0) + w^{(1)}(z, \bar{z}, 0) \\ &= z + \bar{z} + w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots, \end{aligned}$$

$$\begin{aligned} x(t-1) &= zq^{(1)}(-1) + \bar{z}\bar{q}^{(1)}(-1) + w_{20}^{(1)}(-1)\frac{z^2}{2} + w_{11}^{(1)}(-1)z\bar{z} + w_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots \\ &= ze^{-i\tau_0\omega_0} + \bar{z}e^{i\tau_0\omega_0} + \cdots, \end{aligned}$$

$$f_0 = \begin{bmatrix} 0 \\ -\varepsilon\tau_0 x^2(t)x(t-1) \end{bmatrix},$$

we have

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^* f_0 \\ &= -\bar{D}\varepsilon\tau_0 x^2(t)x(t-1) \\ &= -\bar{D}\varepsilon\tau_0(e^{i\tau_0\omega_0} z^2\bar{z} + 2e^{-i\tau_0\omega_0} z^2\bar{z} + \cdots) \end{aligned}$$

Comparing the coefficients, we have

$$g_{20} = g_{11} = g_{02} = 0,$$

$$g_{21} = -2\bar{D}\varepsilon\tau_0(e^{i\tau_0\omega_0} + 2e^{-i\tau_0\omega_0}).$$

Since $\pm i\tau_0\omega_0$ is a pair of imaginary roots of Eq. (18′) with $\tau = \tau_0$,

$$e^{\pm i\tau_0\omega_0} = \frac{1}{\varepsilon}(1 - \omega_0^2).$$

We have

$$g_{21} = -\frac{2}{\varepsilon^2}[(1 - \omega_0^2)^2 + 2\varepsilon^2].$$

Clearly,

$$C_1(0) = \frac{i}{2\tau_0\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} = \frac{g_{21}}{2},$$

and hence

$$\operatorname{Re} C_1(0) = -\frac{1}{\varepsilon^2}[(1 - \omega_0^2)^2 + 2\varepsilon^2] < 0.$$

Thus

$$\beta_2 = 2\,\mathrm{Re}\,C_1(0) < 0$$

and

$$\mu_2 = -\frac{\mathrm{Re}\,C_1(0)}{\alpha'(\tau_0)} = \begin{cases} > 0, & \tau_0 = \tau_j^+, \\ < 0, & \tau_0 = \tau_j^-. \end{cases}$$

Theorem 3.2. *Suppose that* $0 < \varepsilon < 1$ *is satisfied. Then, for* $0 \leqslant j \leqslant m$, *the Hopf bifurcation at the origin when* $\tau = \tau_j^+(\tau_j^-)$ *is supercritical* (*subcritical*), *and the bifurcating periodic solutions are orbitally asymptotically stable.*

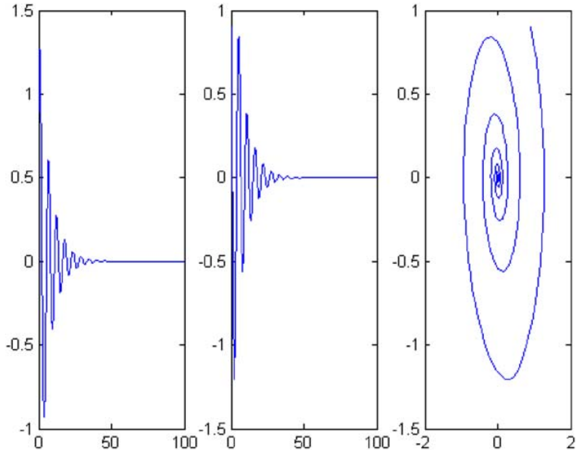

Fig. 3. For Eq. (31), when $\tau = 2$ the equilibrium $(0, 0)$ is asymptotically stable.

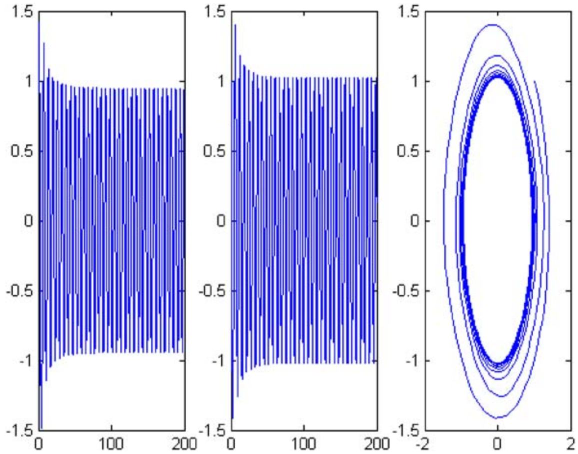

Fig. 4. For Eq. (31), when $\tau_1 < \tau = 3 < \tau_2$ and is sufficiently near τ_1 the bifurcating periodic solution from $(0, 0)$ occurs and is asymptotically stable.

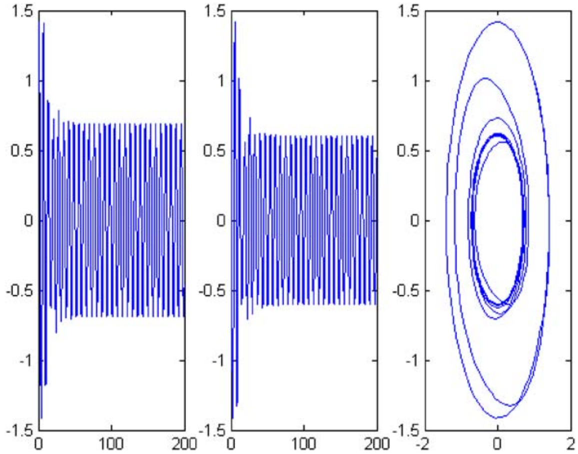

Fig. 5. For Eq. (31), when $\tau_1 < \tau = 7 < \tau_2$ and is sufficiently near τ_2 the bifurcating periodic solution from $(0, 0)$ occurs and is asymptotically stable.

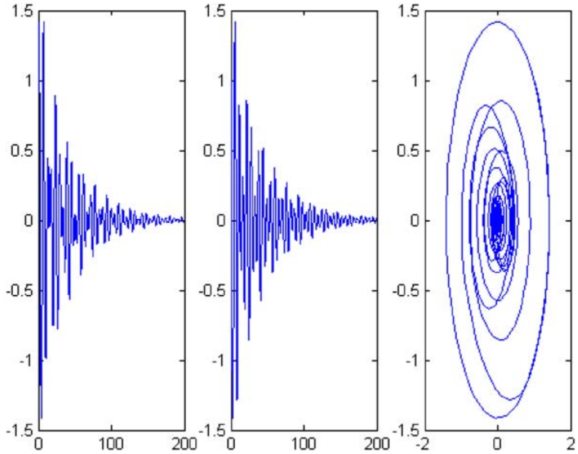

Fig. 6. For Eq. (31), when $\tau_2 < \tau = 8 < \tau_3$ equilibrium $(0, 0)$ is asymptotically stable.

4. Numerical examples

In this section, we shall carry out numerical simulation on system (6) at special values of ε and τ. We consider the following system:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= -x(t) - 0.5(x^2(t) - 1)x(t - \tau). \end{aligned} \tag{31}$$

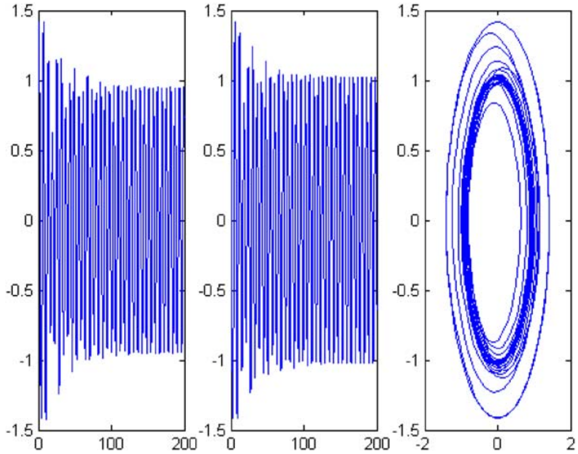

Fig. 7. For Eq. (31), when $\tau_3 < \tau = 9$ and is sufficiently near τ_3 the bifurcating periodic solution from $(0, 0)$ occurs and is asymptotically stable.

By Theorem 2.8 and $\varepsilon = 0.3$, we know that $m = [\sqrt{1-\varepsilon}/(2\sqrt{1+\varepsilon} - \sqrt{1-\varepsilon})] = 1$ and $2m + 1 = 3$. Hence, we obtain $\tau_0 = 0$, $\tau_1 \doteq 2.75396$, $\tau_2 \doteq 7.50604$, $\tau_3 \doteq 8.26189$, $\tau_5 \doteq 13.76981$, $\tau_4 \doteq 15.0127, \ldots$. Thus, the equilibrium $(0, 0)$ is asymptotically stable when $\tau \in (\tau_0, \tau_1) \cup (\tau_2, \tau_3)$, and unstable when $\tau \in (\tau_1, \tau_2) \cup (\tau_3, \infty)$. This shows that the equilibrium $(0, 0)$ switches 2 times from stability to instability and is unstable for all $\tau > \tau_3$. By the results in Section 3, it follows that $\mathrm{Re}\, C_1(0) < 0$ for any τ_j, and when $\tau = \tau_{2k}$, $\mu_2 < 0$, and $\mu_2 > 0$ for $\tau = \tau_{2k+1}$. Therefore, from Theorems 2.8 and 3.2, we conclude that the Hopf bifurcation of system (31) occurs in $\tau > \tau_1$, $\tau > \tau_3$, $\tau < \tau_2$and the bifurcating periodic solutions are orbitally asymptotically stable. These numerical simulations mentioned above are shown in Figs. 3–7.

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