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Journal of Sound and Vibration 283 (2005) 891–914
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Inverse problems for damped vibrating systems

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Received 13 November 2003; received in revised form 6 May 2004; accepted 18 May 2004
Available online 8 October 2004

Abstract

Linear damped vibrating systems are defined by three real definite matrices, $M>0$, $D\geqslant 0$, and $K>0$; the mass, damping, and stiffness matrices, respectively. It is assumed that all eigenvalues of the system are simple and nonreal so that the eigenvectors (columns of a matrix $X_c \in \mathbb{C}^{n\times n}$) are also complex. It is shown that, when properly defined, the eigenvectors have a special structure consistent with $X_c = X_R(I - \mathrm{i}\Theta)$ where $X_R, \Theta \in \mathbb{R}^{n\times n}$, X_R is nonsingular and Θ is orthogonal. By taking advantage of this structure solutions of the inverse problem are obtained: i.e., given complete information on the eigenvalues and eigenvectors, it is shown how M, D, and K can be found. Three points of view are developed and compared (namely, using spectral theory, structure preserving similarities, and factorisation theory).

1. Introduction

In this paper a "vibrating system" is understood to be the classical model of a linear, viscously damped elastic system with finitely many degrees of freedom. The unrestrained system has equations of motion

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = 0, \tag{1}$$

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doi:10.1016/j.jsv.2004.05.003

where M, D, and K are $n \times n$ *real symmetric* matrices. The mass matrix, M, and stiffness matrix, K, are *positive definite* (written $M > 0,\ K > 0$), and the damping matrix D may be *positive definite* or *positive semi-definite* (written $D \geqslant 0$).

It is well-known that all solutions of this differential equation can be obtained via the *algebraic* equation

$$(\lambda^2 M + \lambda D + K)x = 0. \tag{2}$$

Complex numbers λ and nonzero vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. It will be convenient to define the "λ-matrix", or "matrix polynomial",

$$L(\lambda) = \lambda^2 M + \lambda D + K. \tag{3}$$

The detailed structure of the eigenvalues and eigenvectors can be quite complicated and a careful treatment under minimal assumptions can be found in Chapters 10 and 13 of [1]. The "forward" problem is, of course, to find the eigenvalues and eigenvectors when the coefficient matrices are given. Our main interest in this paper is the corresponding inverse problem: Given complete information [1] about eigenvalues and eigenvectors, re-construct the coefficient matrices. This also requires analysis of the conditions which are *necessarily* satisfied by eigenvalues and eigenvectors of problems of this type, for these conditions must, of course, be satisfied by admissible data sets. Thus, ideas are examined which may admit the design of a system having prescribed (complex) natural frequencies and modes of vibration. There is quite a long literature on this topic, some of which will be introduced subsequently.

It is our objective to provide a synthesis of existing results, with some extensions, in an accessible analysis which, while making some simplifying assumptions, still provides some useful new insights on the general inverse problem. However, complete solution of the inverse problem in a computationally convenient way remains unsolved. For the reader's convenience, efforts are made to make the exposition as self-contained as reasonably possible. For clarity, some algorithms are formulated, but these are not written as high-performance software. They are summaries of computational steps which, with problems of modest size, are easily formulated and executed in "matlab" code. Problems of numerical conditioning are not considered.

It is easy to see that the $2n$ eigenvalues of Eq. (2) are either real numbers or, if not, are in complex conjugate pairs. Furthermore, all eigenvalues lie in the closed left-half of the complex plane (the system is "stable" in an appropriate sense). For the purpose of our discussion *it is assumed throughout that all eigenvalues are simple and nonreal*, and this hypothesis will not be repeated in theorem statements. This has the advantage of simplifying the algebraic theory very significantly, while retaining sufficient generality for many physical problems. Furthermore, it helps in revealing some algebraic structure which has not been closely examined to date. However a corresponding approach for general damped second-order systems (which may have real and/or defective eigenvalues and hence do not satisfy the above assumption) is the object of an ongoing research. First results will be published in the near future.

[1] It should be recognised that dominant eigenvalues and modes could be assigned as indicated by experiment, and subdominant data can be assigned in a physically plausible but otherwise arbitrary fashion.

Our hypotheses imply that the eigenvalues of $L(\lambda)$ have the form

$$\lambda_j = \mu_j \pm \mathrm{i}\omega_j \quad \text{for } j = 1, 2, \ldots, n, \tag{4}$$

where $\omega_j > 0$ for all j. It will be useful to introduce diagonal $n \times n$ matrices

$$U = \mathrm{diag}[\mu_1 \ \mu_2 \cdots \ \mu_n] \leqslant 0, \quad W = \mathrm{diag}[\omega_1 \ \omega_2 \ \cdots \ \omega_n] > 0,$$

$$\Lambda = \mathrm{diag}[\lambda_1 \ \lambda_2 \ \cdots \ \lambda_n] = U + \mathrm{i}W, \quad \Omega^2 = \Lambda^*\Lambda = U^2 + W^2 > 0.$$

(Here, and subsequently, A^* denotes the transposed complex conjugate of matrix A.) Notice that, because zero eigenvalues are not admitted, the stiffness matrix K is necessarily nonsingular.

It is well-understood that the quadratic eigenvalue problem of Eq. (3) can be studied via the *linearised* $2n \times 2n$ eigenvalue problem $\lambda A - B$ where

$$A := \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \quad B := \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}. \tag{5}$$

Thus, if σ denotes the set of all eigenvalues (the spectrum), then

$$\sigma(\lambda^2 M + \lambda D + K) = \Lambda \cup \bar{\Lambda} = \sigma\left(\lambda \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\right). \tag{6}$$

Also, $(\lambda^2 M + \lambda D + K)x = 0$ is equivalent to

$$\left(\lambda \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\right) \begin{bmatrix} x \\ \lambda x \end{bmatrix} = 0.$$

Another simplifying assumption that can frequently be made without losing the essence of the matter, is to suppose that the problem of Eq. (2) is *monic*, i.e. that $M = I_n$. For example, if this is not the case, let $M^{1/2}$ be the positive definite square-root of M and modify the problem by writing $y = M^{1/2}x$ and observe that Eq. (2) reduces to the monic problem

$$(\lambda^2 I_n + \lambda\widehat{D} + \widehat{K})y = 0, \tag{7}$$

where $\widehat{D} = M^{-1/2}DM^{-1/2}$ and $\widehat{K} = M^{-1/2}KM^{-1/2}$. This procedure has the advantage of preserving symmetries in the coefficients. Simply premultiplying Eq. (2) by M^{-1} also produces a monic problem, of course, but the symmetry is obscured. Recently, there has been an interesting contribution to this problem area in which only *partial* spectral structure is prescribed, and attention is focussed on monic systems (see Ref. [2]).

In the paper [3] related problems are considered (using techniques closely related to those developed below) in which given real coefficients M, D, K are modified by feedback in such a way that "eigenstructures" are assigned. However, symmetry is not imposed on the coefficients, which is a main concern of the present contribution. A similar feedback strategy is employed in Ref. [4], but the symmetry of an initially real symmetric system is lost in the process.

The inverse problem for *undamped* systems (i.e. when $D = 0$ in the above discussion) is relatively simple. For the forward problem, it is a classical result that the two quadratic forms defined by $M > 0$ and $K > 0$ can be simultaneously diagonalised by a real congruence. That is, there is a real nonsingular Q such that

$$Q^{\mathrm{T}}MQ = I_n \quad \text{and} \quad Q^{\mathrm{T}}KQ = W^2,$$

for a diagonal $W > 0$; the diagonal matrix of natural frequencies. In contrast to more general symmetric matrix pairs (as in Eq. (5), for example) the reduction to real diagonal forms by a simultaneous congruence transformation is always possible in the undamped case. (The reduction of some real pairs, including Eq. (5), to *complex* diagonal form is discussed in the Appendix to this paper.)

Our inverse problem for undamped systems is resolved as follows: Given any diagonal $W > 0$ and any real nonsingular Q, the system whose eigenvalues are given by W and whose eigenvectors are the columns of Q is determined by taking

$$M = Q^{-\mathrm{T}}Q^{-1} \quad \text{and} \quad K = Q^{-\mathrm{T}}W^2Q^{-1}.$$

From an algebraic point of view, the difficulties with the *damped* systems can be traced back to the fact that, in general, *three* quadratic forms cannot be simultaneously reduced to diagonal form by congruence.

2. The modal approach: Jordan pairs and triples

Associated with each eigenvalue λ_j is an eigenvector (or mode) $x_j \in \mathbb{C}^n$ which, a priori, is not defined to within a scalar multiplier. There is a natural association between the diagonal matrix Λ of n eigenvalues and an $n \times n$ matrix X_c with corresponding eigenvectors as its columns. Then it is easy to see that the n columns of the matrix

$$MX_c\Lambda^2 + DX_c\Lambda + KX_c = 0$$

summarise n separate eigenvalue–eigenvector relations of type (2).

In full generality, the question of linear independence of eigenvectors is subtle (see Ref. [1]). However, with our hypotheses on the distribution of eigenvalues, it can be shown that the eigenvectors associated with the n eigenvalues in the upper half of the complex plane are necessarily linearly independent. So if the matrix X_c is associated with these eigenvalues it has linearly independent columns and is therefore nonsingular.

It follows from Eq. (2) that if λ_j, x_j form an eigenvalue–eigenvector pair, then so do their complex conjugates $\overline{\lambda_j}$, $\overline{x_j}$. Thus, the $n \times 2n$ matrix

$$X = [X_c\ \overline{X_c}] \tag{8}$$

contains complete information on the (un-normalised) eigenvectors, and the $2n \times 2n$ matrix

$$J = \begin{bmatrix} \Lambda & 0 \\ 0 & \overline{\Lambda} \end{bmatrix} \tag{9}$$

contains complete information on corresponding eigenvalues. Such a pair (X, J) is known as a *Jordan pair* for the system provided that the matrix

$$Q := \begin{bmatrix} X \\ XJ \end{bmatrix} = \begin{bmatrix} X_c & \overline{X_c} \\ X_c\Lambda & \overline{X_c\Lambda} \end{bmatrix} \tag{10}$$

is nonsingular. Thus, if full sets of eigenvalues and eigenvectors are to be prescribed they must respect these structures and ensure that Q is nonsingular.

Now some basic ideas and (special cases of) results from the theory of matrix polynomials, $L(\lambda)$, are introduced. (See Refs. [1] and [5, Chapter 14]; also Ref. [6] for an early account.) First a $2n \times n$ matrix Y is defined by

$$Y = Q^{-1} \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}. \tag{11}$$

It will be shown below that the rows of Y can be interpreted as left eigenvectors of $L(\lambda)$, and the triple (X, J, Y) is known as a *Jordan triple* for $L(\lambda)$. Our subsequent theory is developed from a fundamental realisation of $(L(\lambda)^{-1})$, which is presented without proof. It has the important property that it generalises readily to a general eigenvalue distribution and to matrix polynomials of higher degree. Notice that the only essential hypotheses are that M is nonsingular and (for the first statement) (X, J) form a Jordan pair.

Theorem 1. *If* (X, J, Y) *is a Jordan triple for* $L(\lambda)$, *then*

$$L(\lambda)^{-1} = X(\lambda I_{2n} - J)^{-1} Y. \tag{12}$$

Conversely, if Eq. (12) *holds for a diagonal matrix* J, *then* (X, J, Y) *is a Jordan triple for* $L(\lambda)$.

Observe that, given a Jordan triple (X, J, Y) for $L(\lambda)$, it also holds that

$$(L^{\mathrm{T}}(\lambda))^{-1} = Y^{\mathrm{T}}(\lambda I_{2n} - J)^{-1} X^{\mathrm{T}}.$$

Thus, using the converse statement of the theorem, $(Y^{\mathrm{T}}, J, X^{\mathrm{T}})$ is a Jordan triple for $L^{\mathrm{T}}(\lambda) = M^{\mathrm{T}}\lambda^2 + D^{\mathrm{T}}\lambda + K^{\mathrm{T}}$, and the columns of Y^{T} (rows of Y) do, indeed, determine *left* eigenvectors of $L(\lambda)$.

Multiplying Eq. (11) on the left by Q immediately gives $XY = 0$ (an orthogonality property) and $XJY = M^{-1}$, giving M in terms of the spectral data. The next result extends this statement to give all three coefficients, M, D, K in terms of the Jordan triple. A proof is given which may contain some novel features.

Theorem 2. *If* (X, J, Y) *is a Jordan triple for* $L(\lambda)$, *then*[2]

$$XY = 0, \quad XJY = M^{-1}, \quad XJ^2Y = -M^{-1}DM^{-1}$$

and also $XJ^{-1}Y = -K^{-1}$.

Proof. Using linearisation (5), it is not difficult to verify that

$$(\lambda A - B)^{-1} = (\lambda I_{2n} - A^{-1}B)^{-1}A^{-1} = \begin{bmatrix} L(\lambda)^{-1} & \lambda L(\lambda)^{-1} \\ \lambda L(\lambda)^{-1} & \lambda^2 L(\lambda)^{-1} - M^{-1} \end{bmatrix}. \tag{13}$$

Now integrate around a smooth contour Γ containing all eigenvalues of $L(\lambda)$. Using the theory of residues and Theorem 1, it follows that

$$A^{-1} = \begin{bmatrix} XY & XJY \\ XJY & XJ^2Y \end{bmatrix}. \tag{14}$$

[2] Formulae of this kind may have first appeared in Ref. [7].

But it is easily seen that

$$A^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}DM^{-1} \end{bmatrix},$$

and it only remains to compare matrix entries to obtain the first three results. The last statement follows immediately on putting $\lambda = 0$ in Theorem 1. $\square$

Observe that a Jordan pair (X, J) cannot be expected to define the triple (M, D, K) uniquely. Indeed, $L(\lambda) = M\lambda^2 + D\lambda + K$ is consistent with (X, J) if and only if $AL(\lambda)$ is consistent with (X, J) for any nonsingular $n \times n$ matrix A, i.e. the *right* eigenvectors do not change under this transformation. However, once M is specified, the Jordan *triple* can be defined, and the two remaining coefficient matrices are uniquely determined by Theorem 2.

Theorem 3. *If* (X, J) *is a Jordan pair with forms* (8) *and* (9) *then there is a corresponding system* $L(\lambda)$ *with real* (*not necessarily symmetric*) *coefficients*, M, D, K.

Proof. First assign M to be any real nonsingular matrix. Define the matrix

$$P = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \tag{15}$$

and observe that $P^2 = I_{2n}$. Then $XP = [\overline{X_c}\ X_c] = \bar{X}$ and, similarly, $PJP = \bar{J}$, $QP = \bar{Q}$. Consequently, for any integer r,

$$XJ^rQ^{-1} = (XP)(PJ^rP)(QP)^{-1} = \bar{X}\bar{J}^r\overline{Q^{-1}},$$

and the product XJ^rQ^{-1} is always real. Since M is assumed real and nonsingular it follows from the formulae of Theorem 2 that D and K are real as well. $\square$

The results of this section give a solution to the inverse problem for quadratic eigenvalue problems with *real* coefficients. The matrix X_c and diagonal matrix Λ can be assigned quite generally provided only that Q of Eq. (10) is nonsingular. In particular, if Λ is fixed, then the family of quadratic polynomials obtained as X_c takes values in the nonsingular complex matrices (for which Q is nonsingular) is real and isospectral.

Example 1. Let

$$X_c = \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1+3\mathrm{i} & 0 \\ 0 & -2+4\mathrm{i} \end{bmatrix}$$

and $M = I_n$. Then Y can be calculated from Eq. (11) and Theorem 2 gives

$$D = -MXJ^2YM = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \tag{16}$$

$$K = -(XJ^{-1}Y)^{-1} = \begin{bmatrix} 14 & 2 \\ -2 & 14 \end{bmatrix}, \tag{17}$$

and demonstrates that the resulting coefficient matrices are not necessarily symmetric.

3. The symmetry condition

In the last section the eigenvectors (making up the columns of X) are not normalised; their lengths can be chosen arbitrarily. But the lengths of the *left* eigenvectors (rows of Y) are then fixed in terms of those of the right eigenvectors by Eqs. (10) and (11). When the further condition of symmetry is imposed on M, D, K, the theory must admit the possibility that $Y = X^{\mathrm{T}}$; i.e. right and left eigenvectors are identical when suitably normalised. This is implicit in the next theorem.

Theorem 4. *Let the real, symmetric, nonsingular matrix M be given. Then a Jordan pair X, J of form* (8), (9) *determines real symmetric matrices K and D if*

$$\begin{bmatrix} X \\ XJ \end{bmatrix} X^{\mathrm{T}} = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}. \tag{18}$$

Conversely, let real symmetric matrices M, D, K be given and $\sigma(L(\lambda)) = \Lambda \cap \bar{\Lambda}$, as above. Then the eigenvectors of $L(\lambda)$ can be normalised in such a way that Eq. (18) *holds.*

Proof. Use the Jordan pair and the given M to define Y as in Eq. (11) and obtain a Jordan triple (X, J, Y). Then K and D are defined as in Theorem 2, and it follows from Theorem 3 that they are real. If Eq. (18) holds then, because Q is nonsingular, it follows from Eq. (11) that $Y = X^{\mathrm{T}}$ and, substituting in the formulae of Theorem 2, it is seen that D and K are symmetric.

To prove the converse statement, consider the real symmetric pencil $\lambda A - B$ obtained by linearisation of $L(\lambda)$ as in Eq. (5). There is a *complex* $2n \times 2n$ matrix Q such that the two complex congruencies $Q^{\mathrm{T}}AQ = I_{2n}$ and $Q^{\mathrm{T}}BQ = J$ hold. (This is not obvious. An argument leading to this conclusion can be found in Ref. [6] and another proof is given in the Appendix to this paper.)

The columns of Q are, of course, the right eigenvectors of the pencil. Because the pencil has the special form of Eqs. (5), the eigenvectors have the form $\left[\begin{smallmatrix} x_j \\ \lambda_j x_j \end{smallmatrix}\right]$, where the x_j are eigenvectors of $L(\lambda)$. So, with our hypotheses on the location of eigenvalues, it may be assumed that Q has the structure presented in Eq. (10).

It follows from $Q^{\mathrm{T}}AQ = I_{2n}$ that $QQ^{\mathrm{T}} = A^{-1}$ and hence

$$\begin{bmatrix} X \\ XJ \end{bmatrix} [X^{\mathrm{T}} \;\; JX^{\mathrm{T}}] = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}DM^{-1} \end{bmatrix}.$$

Postmultiply by $\left[\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right]$ to obtain Eq. (18). $\square$

Observe that the normalisation property of eigenvectors (required to ensure that Eq. (18) holds) is now

$$[x_j^{\mathrm{T}} \;\; \lambda_j x_j^{\mathrm{T}}] \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_k x_k \end{bmatrix} = 2\lambda_j (x_j^{\mathrm{T}} M x_k) + x_j^{\mathrm{T}} D x_k = \delta_{jk}, \tag{19}$$

the Kronecker delta, where $1 \leqslant j, k \leqslant 2n$.

Consider some consequences of Eq. (18). Substituting from Eq. (10) into Eq. (18) yields

$$\begin{aligned} X_c X_c^{\mathrm{T}} + \overline{X_c X_c^{\mathrm{T}}} &= 0, \\ X_c \Lambda X_c^{\mathrm{T}} + \overline{X_c \Lambda X_c^{\mathrm{T}}} &= M^{-1}. \end{aligned} \tag{20}$$

Write X_c in real and imaginary parts, $X_c = X_R + \mathrm{i}X_I$ (i.e. X_R and X_I are real). Now the first displayed equation above says that the real part of $X_c X_c^{\mathrm{T}}$ is zero. Thus, for the properly normalised eigenvectors, $X_R X_R^{\mathrm{T}} - X_I X_I^{\mathrm{T}} = 0$, and $X_R X_R^{\mathrm{T}} = X_I X_I^{\mathrm{T}}$. Using the polar decomposition of a matrix ([5, Section 5.7], for example), it follows from this that $X_I = -X_R\Theta$, where Θ is a real orthogonal matrix. Thus,

$$X_c = X_R(I_n - \mathrm{i}\Theta). \tag{21}$$

This property has been discussed in Refs. [8,9] (see also Ref. [10], where there is a numerical confirmation) and also in Ref. [11]. Observe that, under our standing assumption that X_c is nonsingular, so is X_R, and $-\mathrm{i}$ is not an eigenvalue of Θ. In particular, Eq. (21) implies the interesting fact that the real and imaginary parts of the eigenvector matrix X_c have the same euclidean norm.

Substituting from Eq. (21) in the second equation of Eq. (20) leads to the relation

$$U + W\Theta^{\mathrm{T}} + \Theta W - \Theta U \Theta^{\mathrm{T}} = \tfrac{1}{2}(X_R^{\mathrm{T}} M X_R)^{-1}. \tag{22}$$

Note that we may also write

$$U + W\Theta^{\mathrm{T}} + \Theta W - \Theta U \Theta^{\mathrm{T}} = [I_n \;\; \Theta]\begin{bmatrix} U & W \\ W & -U \end{bmatrix}\begin{bmatrix} I_n \\ \Theta^{\mathrm{T}} \end{bmatrix}.$$

In the converse direction, it is easily seen that if X_R, Θ satisfy Eqs. (21) and (22), then Eq. (18) holds. Thus

Theorem 5. *Given a real, symmetric, nonsingular M, a standard pair X, J, as above, satisfies Eq.* (18) (*and so determines real symmetric matrices K and D*) *if and only if Eq.* (21) *holds and Θ, X_R satisfy Eq.* (22).

It is usually the case that M is positive definite. In this case the real orthogonal matrix Θ satisfies a further condition.

Theorem 6. *If $M > 0$ in the preceding theorem, then*

$$\Theta + \Theta^{\mathrm{T}} > 0. \tag{23}$$

Proof. Since Θ^{T} is a normal matrix it has an orthonormal system of eigenvectors, $x_1, x_2, \ldots, x_n$ and associated eigenvalues $\mathrm{e}^{\mathrm{i}\phi_j}$, $j = 1, 2, \ldots, n$, with $0 \leqslant \phi_j < 2\pi$. Pre- and post-multiply Eq. (22) by x_j^*, x_j, respectively, and use the fact that $M > 0$ to obtain

$$x_j^* U x_j + (x_j^* W x_j)\mathrm{e}^{\mathrm{i}\phi_j} + \mathrm{e}^{-\mathrm{i}\phi_j}(x_j^* W x_j) - \mathrm{e}^{-\mathrm{i}\phi_j}(x_j^* U x_j)\mathrm{e}^{\mathrm{i}\phi_j} > 0,$$

which yields

$$2\cos\,\phi_j(x_j^* W x_j) > 0. \tag{24}$$

Since $W > 0$, it follows that $\cos\phi_j > 0$, for $j = 1, 2, \ldots, n$.

Now let $y \in \mathbb{C}^n, y = \sum_{j=1}^n \alpha_j x_j$. Then

$$\begin{aligned} y^*(\Theta + \Theta^{\mathrm{T}})y &= \sum_{j,k} \bar{\alpha}_j x_j^* (\Theta + \Theta^{\mathrm{T}}) x_j \alpha_k \\ &= \sum_{j,k} (\bar{\alpha}_j \mathrm{e}^{-\mathrm{i}\phi_j} x_j^* x_k \alpha_k + \bar{\alpha}_j x_j^* \mathrm{e}^{\mathrm{i}\phi_k} x_k \alpha_k) \\ &= 2 \sum_j |\alpha_j|^2 \cos\phi_j. \end{aligned}$$

Since $\cos\phi_j > 0$ for each j, it follows that $y^*(\Theta + \Theta^{\mathrm{T}})y > 0$ for all vectors $y \neq 0$, i.e. $\Theta + \Theta^{\mathrm{T}} > 0$. □

When $n = 2$ and $M > 0$ this result means that Θ belongs to the family

$$\left(\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right)_{t \in \mathbb{R}} \quad \text{and not to the family} \quad \left(\begin{bmatrix} -\cos t & \sin t \\ \sin t & \cos t \end{bmatrix} \right)_{t \in \mathbb{R}}.$$

More generally, Theorem 6 means that when $M > 0$ (and for any n) Θ can be parametrised using real skew-symmetric matrices, C, in the form

$$\Theta = (I_n - C)(I_n + C)^{-1}, \tag{25}$$

(see for example [5, p. 219]).

The next example shows that the converse of Theorem 6 does not hold.

Example 2. Let Λ and X_R be as in Example 1, then for

$$\Theta = \frac{1}{5} \begin{bmatrix} 1 & -2\sqrt{6} \\ 2\sqrt{6} & 1 \end{bmatrix} \tag{26}$$

we have $\Theta + \Theta^{\mathrm{T}} = \frac{2}{5} I_2$ which is positive definite. However, evaluating the left term of Eq. (22) reveals

$$U + W\Theta^{\mathrm{T}} + \Theta W - \Theta U \Theta^{\mathrm{T}} = \frac{2}{25} \begin{bmatrix} 27 & -6\sqrt{6} \\ -6\sqrt{6} & 8 \end{bmatrix} = \tfrac{2}{25} a a^{\mathrm{T}}, \tag{27}$$

where $a^{\mathrm{T}} := (\sqrt{27}, -\sqrt{8})$. Since the matrix on the right is singular, there is no nonsingular M satisfying Eq. (22).

An interesting special case, and a check on our analysis, is obtained by setting $\Theta = I_n$. This is, of course, a real orthogonal matrix (obtained by setting $C = 0$ in Eq. (25)), and it satisfies the necessary constraints. Eq. (21) gives $X_c = (1 - \mathrm{i})X_R$. Eq. (22) reduces to $4W = (X_R^{\mathrm{T}} X_R)^{-1}$, and is satisfied by $X_R = \frac{1}{2}\Phi W^{-1/2}$ where Φ is also an arbitrary real orthogonal matrix. Following through with this argument in the case $M = I_n$ leads to diagonable systems

$$I_n\lambda^2 + \widehat{D}\lambda + \widehat{K} = \Phi(I_n\lambda^2 - 2U\lambda + \Omega^2)\Phi^{\mathrm{T}}. \tag{28}$$

By continuity, there will be a family of orthogonal matrices in a neighbourhood of I_n, and satisfying Theorem 6, which determines a family of real symmetric pencils $L(\lambda)$ with $M > 0$.

Another interesting special case arises if all prescribed eigenvalues lie on a line parallel to the imaginary axis in the complex plane. In this case $U = -\mu I_n$ for some $\mu > 0$ and Eq. (22) reduces to a *linear* equation for Θ:

$$W\Theta^{\mathrm{T}} + \Theta W = \tfrac{1}{2}(X_R^{\mathrm{T}} M X_R)^{-1}.$$

Example 3. When $n = 2$ we may write $W = \mathrm{diag}[w_1,\ w_2] > 0$ and parametrise Θ:

$$\Theta = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

and obviously $\Theta + \Theta^{\mathrm{T}} > 0$ if and only if $\cos t > 0$, or $t \in (-\pi/2, \pi/2)$. It is straightforward to verify that $W\Theta^{\mathrm{T}} + \Theta W > 0$ for all Θ with

$$\cos t \in \left[\frac{|w_1 - w_2|}{w_1 + w_2},\ 1\right].$$

Up to now, Eqs. (20) have been interpreted as conditions to be satisfied by X_R and Θ when M is given. On the other hand, if X_R and Θ are given this can be used to define M. Thus, a method is obtained for determining real and symmetric coefficient matrices in terms of the spectral data Λ, X_R, and Θ. In summary:

Algorithm 1: Real symmetric systems

DATA: Diagonal $U \leqslant 0$, $W > 0$, nonsingular X_R and orthogonal Θ satisfying $\Theta + \Theta^{\mathrm{T}} > 0$ (all real $n \times n$ matrices).

DEFINE: $\Lambda = U + \mathrm{i}W$, $\quad X_c = X_R(I_n - \mathrm{i}\Theta)$, and then

$$J = \begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix}, \quad X = [X_c\ \overline{X_c}].$$

CONFIRM: $Q = \left[\begin{smallmatrix} X \\ XJ \end{smallmatrix}\right]$ is nonsingular. If so,

$$\text{COMPUTE:} \quad M = (XJX^{\mathrm{T}})^{-1}, \tag{29}$$

$$D = -M(XJ^2X^{\mathrm{T}})M, \tag{30}$$

$$K = -(XJ^{-1}X^{\mathrm{T}})^{-1}. \tag{31}$$

The spectrum of $L(\lambda)$ is, of course, entirely determined by Λ, and all choices of X_R and Θ (subject only to the data constraints above) lead to a real symmetric system. Thus:

Proposition 7. *The set of systems*

$$\mathscr{L}_\Lambda := \{L(\lambda) = \lambda^2 M + \lambda D + K \text{ generated as above}\} \tag{32}$$

is an isospectral set of real symmetric systems with spectrum $\Lambda \cup \bar{\Lambda}$.

Example 4. Let Λ and X_R be as in Example 1 and, in the parametrisation (25), $C = 0.75\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. This gives $\Theta = \left[\begin{smallmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{smallmatrix}\right]$ and condition (23) is satisfied. Then, using the above procedure one finds (rounded values):

$$M^{-1} = \begin{bmatrix} 5.2032 & -2.4576 \\ -2.4576 & 2.6368 \end{bmatrix},$$

$$D = \begin{bmatrix} -1.2333 & -0.3 \\ -0.3 & 4.15 \end{bmatrix},$$

$$K^{-1} = \begin{bmatrix} 0.336 & 0.192 \\ 0.192 & 0.224 \end{bmatrix}.$$

Note, that D is not positive definite. This illustrates the fact that the isospectral systems with positive definite coefficients will generally be a proper subset of $\mathscr{L}_\Lambda$—even when a spectrum in the open left half-plane is prescribed.

4. Positivity of M, D, K

In this section a remarkable fact will be revealed: briefly, that for real symmetric systems with non-real spectrum, the positive definite properties of M, D, K depend only on Θ of Eq. (21), and not on X_R.

The three coefficient matrices can be expressed in terms of a matrix function P_Θ defined as follows: For a real orthogonal matrix Θ and any complex matrix $A = A_R + \mathrm{i}A_I$ define

$$\begin{aligned} P(\Theta, A) &= (\text{Real part of})(I_n - \mathrm{i}\Theta)A(I_n - \mathrm{i}\Theta^{\mathrm{T}}) \\ &= A_R + A_I\Theta^{\mathrm{T}} + \Theta A_I - \Theta A_R \Theta^{\mathrm{T}}, \end{aligned}$$

and observe that, if A is diagonal, then $P(\Theta, A)$ is real and symmetric. In fact, in this investigation the A-domain of this function will be confined to the powers of the diagonal matrix Λ and, because this matrix is fixed in the discussion of isospectral families the abbreviation $P_r(\Theta) = P(\Theta, \Lambda^r)$ will be used. Note in particular that, since $\Lambda^0 = I_n$,

$$P_0(\Theta) = I_n - I_n = 0. \tag{33}$$

In general, the functions $P_1(\Theta), P_2(\Theta), P_{-1}(\Theta)$ determine M, D, K. Inserting $X_C = X_R(I_n - \mathrm{i}\Theta)$ into Eqs. (29)–(31) yields

$$M^{-1} = 2X_R P_1(\Theta) X_R^{\mathrm{T}}, \tag{34}$$

$$D = -\tfrac{1}{2} X_R^{-\mathrm{T}} P_1(\Theta)^{-1} P_2(\Theta) P_1(\Theta)^{-1} X_R^{-1}, \tag{35}$$

$$K^{-1} = -2X_R P_{-1}(\Theta) X_R^{\mathrm{T}}. \tag{36}$$

Now X_R in the expressions on the right of the above three equations determines a congruence transformation. Thus, by Sylvester's law of inertia, the *inertias* of M, D, K are determined by those of $P_1(\Theta)$, $-P_2(\Theta)$, and $-P_{-1}(\Theta)$, respectively. Note, that the *inertias* of M, D, K are independent of X_R. (The *inertia* of a matrix A is the triple of nonnegative integers (π, ν, δ), giving the number of eigenvalues of A in the open right half of the complex plane, the open left half-plane, and on the imaginary axis, respectively.)

Proposition 8. *When Theorem 5 holds, the inertias of* M, D, K *are equal to those of* $P_1(\Theta), -P_2(\Theta)$, *and* $-P_{-1}(\Theta)$, *respectively, and are independent of* X_R.

Example 5. Let Λ and X_c be as in Example 1, i.e. $X_R = I_2$ and $\Theta = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Then from Eq. (34) one finds

$$M^{-1} = 2(U + W\Theta^{\mathrm{T}} + \Theta W - \Theta U \Theta^{\mathrm{T}}) = \begin{bmatrix} 2 & 14 \\ 14 & -2 \end{bmatrix}. \tag{37}$$

Using this result in Eqs. (35) and (36) gives

$$D = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \qquad K^{-1} = \Theta. \tag{38}$$

The next theorem shows that the positivity property of all three matrices, M, D, K, depends primarily on $-P_2(\Theta)$.

Theorem 9. *If* $U \leqslant 0$, $-P_2(\Theta) \geqslant 0$ *and* $P_1(\Theta)$, $P_{-1}(\Theta)$ *are nonsingular, then* $M > 0$, $D \geqslant 0$, *and* $K > 0$.

Proof. Observe first of all that $D \geqslant 0$ follows immediately from Proposition 8. Suppose that $-P_2(\Theta) \geqslant 0$ and $U < 0$. Then the number of eigenvalues of $L(\lambda)$ in the open left half of the complex plane is $2n$ and it follows from Theorem 7 of [12] that $M > 0$ and $K > 0$.

Now assume that $-P_2(\Theta) \geqslant 0$ and $U \leqslant 0$, and suppose that this semidefinite matrix U is the limit of a sequence of negative definite matrices, $\{U_k\}_{k=1}^{\infty}$. With a fixed $W > 0$, it follows from Eqs. (29), (30) and (31) that M, D, K depend continuously on U and so, in the limit as $k \to \infty$, it follows that (as the limits of sequences of positive definite matrices) $M \geqslant 0$ and $K \geqslant 0$. However, the condition that $P_1(\Theta)$ and $P_{-1}(\Theta)$ are nonsingular then implies that $M > 0$ and $K > 0$. $\square$

The procedure outlined at the end of Section 3 can now be extended to admit the construction of isospectral sets of *vibrating systems* as defined in the opening paragraphs of this paper.

Proposition 10. *Let* $\mathbf{\Theta}$ *be a family of real* $n \times n$ *orthogonal matrices satisfying the conditions*: $-P_2(\Theta) \geqslant 0$ *and* $P_1(\Theta)$, $P_{-1}(\Theta)$ *nonsingular, then*

$$\mathscr{L}_{\Lambda}^{+} := \{\mathscr{L}_{\Lambda} \colon \Theta \in \mathbf{\Theta}\}$$

is an isospectral set of vibrating systems.

From the point of view of computation, it is clear that, if Θ is first determined to satisfy the condition $-P_2(\Theta) \geqslant 0$ then, generically, the nonsingularity conditions will also be satisfied.

Algorithm 2: Vibrating systems. This is the same as Algorithm 1 except that, in addition, Θ must satisfy $-P_2(\Theta)\geqslant 0$, i.e.

$$\begin{aligned} &-\{(U^2-W^2)+2(UW)\Theta^{\mathrm{T}}+\Theta 2(UW)-\Theta(U^2-W^2)\Theta^{\mathrm{T}}\} \\ &=[I_n\ \ \Theta]\begin{bmatrix} -U^2+W^2 & -2UW \\ -2UW & U^2-W^2 \end{bmatrix}\begin{bmatrix} I_n \\ \Theta^{\mathrm{T}} \end{bmatrix}\geqslant 0. \end{aligned} \tag{39}$$

In the following example, the condition $-P_2(\Theta)\geqslant 0$ is quite sensitive to Θ. This limits the technique to the generation of systems which are "close" to diagonable systems (i.e. when $\Theta = I$). In other words, close to systems with "proportional damping".

Example 6. (1) This example is based on a four-degree-of-freedom model of a mass–spring chain system originating in Ref. [13]. The undamped system is monic, i.e. $M = I_4$, and the stiffness matrix is

$$K=\begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}.$$

The natural frequencies determine the matrix

$$W=\mathrm{diag}[0.5354\ \ 1.6031\ \ 2.4495\ \ 2.8536].$$

The strategy is to assign *this* matrix to the damped vibrating systems to be generated. Two different matrices of damping factors, U, will be assigned in cases (a) and (b).

The eigenvectors of the undamped system could be assigned to the matrix X_R defining the real parts of the eigenvectors to be generated. However, more structure is revealed if we assign $X_R = I$. Obviously, using other matrices X_R simply applies a simultaneous congruence to M, D, K. In fact, it is apparent from Eqs. (34)–(36) that, to within simultaneous congruence, we may as well assign

$$M=P_1(\Theta)^{-1},\qquad D=-P_1(\Theta)^{-1}P_2(\Theta)P_1(\Theta)^{-1},\qquad K=-P_{-1}(\Theta)^{-1}. \tag{40}$$

Orthogonal matrices Θ are generated using Eq. (25) with the (arbitrary) choice of skew-symmetric matrix,

$$C=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Thus, we assign $\Theta=(I-\alpha C)(I+\alpha C)^{-1}$ for a range of values of α increasing from zero (when $\Theta = I$).

(a) Put all the eigenvalues on the vertical line $\mu=-0.1$ parallel to the imaginary axis. i.e. $U=-(0.1)I_4$. It is found that vibrating systems are generated for all $\alpha\in[0, 0.06]$, but at $\alpha = 0.08$, the positive definite property of D is lost. The eigenvalues of the (full) matrices M, D, K of (40) at $\alpha = 0.06,\ 0.08$ are indicated in Table 1.

Table 1
Eigenvalues of M, D, K as functions of α for the case $U = \mu I_4$

α	M	D	K
0.06	0.1764 0.2069 0.3163 0.9475	0.0056 0.0340 0.0599 0.2299	0.2783 0.8213 1.2464 1.4399
0.08	0.1773 0.2091 0.3198 0.9583	−0.0172 0.0316 0.0622 0.2563	0.2793 0.8344 1.2619 1.4489

Table 2
Eigenvalues of M, D, K as functions of α for the case $U = \mu W$

α	M	D	K
0.06	0.1764 0.2069 0.3163 0.9475	0.0151 0.0923 0.1103 0.1878	0.2717 0.8263 1.2567 1.4525
0.08	0.1773 0.2091 0.3198 0.9585	−0.0141 0.0901 0.1145 0.2194	0.2727 0.8394 1.2724 1.4616

(b) For $\mu = -0.1$ assume $U = \mu W$, which means that all of the assigned eigenvalues λ_k in the second quadrant lie on the same radial line (with slope $\tan^{-1}(\mu^{-1})$). The results are similar and shown in Table 2.

The experiments indicate that, in each case, positivity of M,D, and K is maintained for $\alpha \in [0, \alpha_0)$ for some α_0 in (0.06, 0.08).

(2) In this example, the dependence on parameter α seems to be more robust. The mass and stiffness matrices, M and K are as in Example 6(1), and a damping matrix is arbitrarily assigned:

$$D = \mathrm{diag}[1.0 \;\; 0.5 \;\; 1.0 \;\; 0.5].$$

The idea now is to compute the corresponding matrices X_R and Θ_0. Then generate a family of orthogonal matrices $\Theta(\alpha)$ about Θ_0 (keeping X_R fixed) and check the corresponding matrix coefficients for positive definiteness.

It is found that

$$\Theta_0 = \begin{bmatrix} 0.9961 & -0.0421 & 0.0392 & 0.0663 \\ 0.0314 & 0.9652 & 0.2593 & -0.0123 \\ -0.0501 & -0.2573 & 0.9649 & 0.0187 \\ -0.0649 & 0.0195 & -0.0175 & 0.9975 \end{bmatrix}.$$

The corresponding skew-symmetric matrix $C_0 = (I + \Theta_0)^{-1}(I - \Theta_0)$ is

$$C_0 = \begin{bmatrix} 0.0 & 0.0188 & -0.0227 & -0.0329 \\ -0.0188 & 0.0 & -0.1315 & 0.0080 \\ 0.0227 & 0.1315 & 0.0 & -0.0093 \\ 0.0329 & -0.0080 & 0.0093 & 0.0 \end{bmatrix}.$$

Now a class of symmetric systems is generated by defining $\Theta = (I - \alpha C_0)(I + \alpha C_0)^{-1}$ for $\alpha \in [-3, 4]$. Observe that the unperturbed system then corresponds to $\alpha = 1$, and $\alpha = 0$ yields the more familiar case $\Theta = I$.

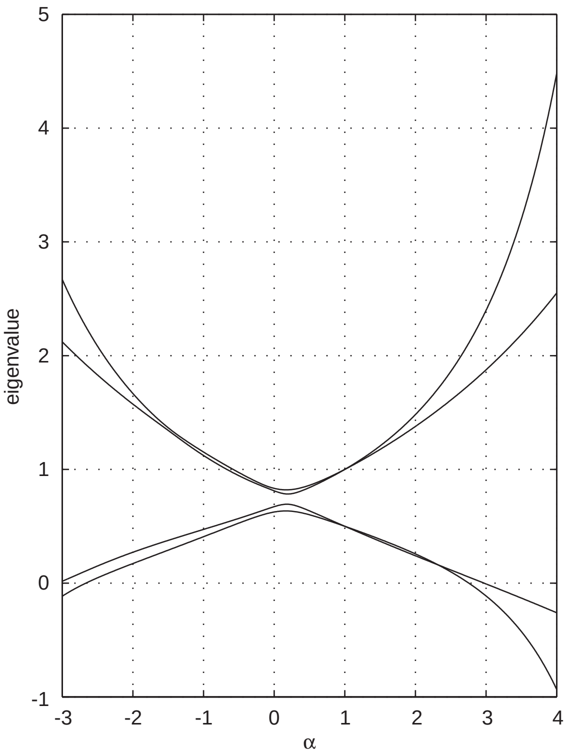

Fig. 1. Eigenvalues of D as functions of α.

It is found that vibrating systems are generated for all $\alpha \in [-2.666,\ 2.756]$, for example. The eigenvalues of D as functions of α are shown in Fig. 1. The figure shows clearly the double eigenvalues, 1 and $\frac{1}{2}$ assigned to the unperturbed problem (at $\alpha = 1$).

5. Structure preserving similarity

In this section our inverse problem is examined from a different point of view. Once again, the assigned spectrum is as before and the argument progresses from the generation of real systems, to real symmetric and vibrating systems.

Definition. Given $\Lambda = U + iW$ with $U \leqslant 0$ and $W > 0$ as above, there is an associated *canonical vibrating system*:

$$L_0(\lambda) := \lambda^2 I_n - 2\lambda U + \Omega^2. \tag{41}$$

A particular linearisation of $L_0(\lambda)$ is

$$\lambda I_{2n} - C_0,$$

where C_0 is the associated *companion matrix* defined by

$$C_0 := \begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix}.$$

In the literature on vibrating systems there are various notions of "structure preserving" and are sometimes defined in terms of either pairs of matrices satisfying a *simultaneous congruence* with the linearising pair of (5) or, (more generally) satisfying a *strict equivalence* relation with that pair (see for example Refs. [14,15]). Here, a narrower definition is made in terms of similarity of the companion matrix of a system to that of the canonical system. It is convenient to make the definition in two parts:

Definition. (a) A matrix $V \in \mathbb{R}^{2n\times 2n}$ is said to define a *weakly structure preserving similarity* if the matrix

$$C := VC_0V^{-1} = V\begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix}V^{-1} \tag{42}$$

is a block companion matrix, i.e. C can be partitioned into $n \times n$ blocks:

$$C = \begin{bmatrix} 0 & I_n \\ C_{21} & C_{22} \end{bmatrix}.$$

We refer to such a V as a *weakly structure preserving matrix.*

(b) A weakly structure preserving matrix V defines a *structure preserving similarity* if there are real $n \times n$ matrices $M>0$, $D \geqslant 0$, $K>0$ such that $C_{21} = -M^{-1}K$, $C_{22} = -M^{-1}D$. We refer to such a V as a *structure preserving matrix*.

It is clear that all matrices C (and hence the underlying quadratic systems) determined by weakly structure preserving similarities are isospectral with spectrum $\Lambda \cup \bar{\Lambda}$.

A simple lemma can be useful:

Lemma 11. *A nonsingular* $V \in \mathbb{R}^{2n\times 2n}$ (*with* $n \times n$ *partitions* V_{ij}) *is a weakly structure preserving matrix if and only if*

$$V_{21} = -V_{12}\Omega^2 \quad \text{and} \quad V_{22} = V_{11} + 2V_{12}U. \tag{43}$$

Proof. With V nonsingular equation (42) is equivalent to $CV = V\left[\begin{smallmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{smallmatrix}\right]$. Comparing blocks it is found that $C_{11} = 0$ and $C_{12} = I_n$ if and only if Eq. (43) holds. $\square$

Example 7. (a) Note that a simple class of weakly structure preserving matrices consists of matrices

$$V = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

where A is nonsingular. These transformations generate diagonable systems which are similar to the canonical system and are not generally useful.

(b) Another simple class of structure preserving matrices is generated by nonsingular matrices V which commute with C_0. This class is also of no immediate interest because $VC_0V^{-1} \equiv C_0$ leaves C_0 invariant.

Our next objective is to express a weakly structure preserving matrix, V, in terms of the spectral data, Λ and X_c employed earlier in this paper. Observe first that the *complex* matrix

$$Z := \begin{bmatrix} \bar{\Lambda} & -I_n \\ \Lambda & -I_n \end{bmatrix}, \tag{44}$$

reduces the canonical companion matrix to diagonal form, thus

$$\begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix} = Z^{-1} \begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix} Z.$$

Note that the columns of Z^{-1} are eigenvectors of C_0. If V is a weakly structure preserving matrix then, using the defining equation (42),

$$C = VZ^{-1} \begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix} (VZ^{-1})^{-1}. \tag{45}$$

Thus, the columns of VZ^{-1} are eigenvectors of C and, with our hypotheses on the spectrum, we may write

$$VZ^{-1} = \begin{bmatrix} X_c & \bar{X}_c \\ X_c\Lambda & \overline{X_c\Lambda} \end{bmatrix}, \tag{46}$$

whence,

$$V = \begin{bmatrix} X_c & \bar{X}_c \\ X_c\Lambda & \overline{X_c\Lambda} \end{bmatrix} Z = \begin{bmatrix} X_c\overline{\Lambda} + \bar{X}_c\Lambda & -(X_c + \bar{X}_c) \\ (X_c + \bar{X}_c)\Omega^2 & -(X_c\Lambda + \overline{X_c\Lambda}) \end{bmatrix}, \tag{47}$$

and is clearly a real matrix (as the definition requires). Thus, V is expressed in terms of the eigenvalues and eigenvectors making up Λ and X_c. (It is easily verified that this expression for V is consistent with Lemma 11.)

Our construction ensures that C is real and (from Eqs. (45) and (46)) the relations

$$X_c\Lambda^2 - C_{22}X_c\Lambda - C_{21}X_c = 0 \quad \text{and} \quad \overline{X_c\Lambda^2} - C_{22}\overline{X_c\Lambda} - C_{21}\bar{X}_c = 0$$

hold. Observe also that V nonsingular ensures that Q of Eq. (10) is nonsingular and so Λ and X_c determine a Jordan pair. The following statements summarises the position and provides an alternative to the use of Theorems 2 and 3.

First let $\mathscr{X}$ denote the set of $X_c \in \mathbb{C}^{n\times n}$ for which V is nonsingular.

Theorem 12. *Assume that* $U \leqslant 0$, $W > 0$ *and* $\Lambda = U + \mathrm{i}W$ *are fixed* (*as above*). *Then the set*

$$\{L_{X_c}(\lambda) = \lambda^2 I_n - \lambda C_{22} - C_{21}: X_c \in \mathscr{X}\}$$

generated by Eqs. (42) *and* (47) *consists of real and isospectral systems with spectrum* $\Lambda \cup \bar{\Lambda}$ *and eigenvector matrix* $X = [X_c\ \bar{X}_c]$.

The next objectives are, of course, to determine the structure preserving matrices V which generate *symmetric* systems and, especially, those which generate vibrating systems in the sense of our formal definition. Naturally, we turn to Eqs. (21) and see that, by assigning $X_c = X_R(I_n - \mathrm{i}\Theta)$, as before, a necessary condition for symmetry is satisfied and that the coefficients

M, D, K are determined in terms of Λ, X_R, and Θ by Eqs. (34)–(36). Furthermore, the orthogonality properties of the eigenvectors are ensured. In this case, Eq. (47) takes the simple form

$$V = V(X_R, \Theta) = 2\begin{bmatrix} X_R & 0 \\ 0 & X_R \end{bmatrix}\begin{bmatrix} U - \Theta W & -I_n \\ \Omega^2 & -(U + \Theta W) \end{bmatrix}. \tag{48}$$

As before, positivity of M is equivalent to that of $P_1(\Theta)$, i.e. in order to generate a vibrating system, Θ must be chosen so that $P_1(\Theta) > 0$. Indeed, if Θ is chosen so that the hypotheses of Theorem 8 are satisfied, then there is a vibrating system $L(\lambda) = \lambda^2 M + \lambda D + K$ determined by this data and, as the right eigenvectors are fixed by $X_c = X_R(I_n - \mathrm{i}\Theta)$, the matrices C_{21}, C_{22} must have the form $-M^{-1}K$ and $-M^{-1}D$, respectively, and Eqs. (34)–(36) hold. Thus, V defines a structure preserving similarity.

Define a subset $\mathscr{X}^+$ of $\mathscr{X}$ to be the set of $X_c = X_R(I_n - \mathrm{i}\Theta)$ for which $X_R \in \mathbb{R}^{n\times n}$ is nonsingular, and $\Theta \in \mathbf{\Theta}$ as defined in Proposition 10. Then we have:

Theorem 13. *If $X_c \in \mathscr{X}^+$ then V of Eq.* (48) *defines a structure preserving similarity $C := VC_0V^{-1}$, with*

$$C_{21} = X_R P_1(\Theta)(P_{-1}(\Theta))^{-1} X_R^{-1} \quad \textit{and} \quad C_{22} = X_R P_2(\Theta)(P_1(\Theta))^{-1} X_R^{-1}.$$

Another formal definition will assist in linking the results of this and the preceding sections. Let

$$\mathscr{V}_\Lambda^+ = \{V(X_R, \Theta) \text{ generated by (48)}: X_c \in \mathscr{X}^+\}.$$

Each V in $\mathscr{V}_\Lambda^+$ defines a structure preserving similarity and hence generates a vibrating system, say $\phi(V) := L(\lambda) = \lambda^2 M + \lambda D + K$. Then, recalling Proposition 9,

$$\mathscr{L}_\Lambda^+ = \{\phi(V): V \in \mathscr{V}_\Lambda^+\}.$$

Note that, because similarity transformations are invertible, any pair of systems $L_1(\lambda)$ and $L_2(\lambda)$ in $\mathscr{L}_\Lambda^+$ have the property that their companion matrices C_1 and C_2 are similar.

In the spirit of Algorithms 1 and 2, an "algorithm" for generating real symmetric systems could be formulated using Eq. (42) together with Eq. (48). However, it is less direct than the preceding algorithms and so is not formulated explicitly.

6. Factorisation and the eigenmatrix

Under the hypotheses made on the eigenvalues of a vibrating system $L(\lambda)$ it is not difficult to see that $L(\lambda)$ is positive definite whenever $\lambda \in \mathbb{R}$. This, in turn, is sufficient to show that, in the monic case, there is a factorisation of $L(\lambda)$ of the form

$$L(\lambda) = (\lambda I_n - Z^*)(\lambda I_n - Z) = \lambda^2 I_n - \lambda(Z^* + Z) + Z^*Z, \tag{49}$$

where Z has all its eigenvalues in the upper half of the complex plane (see for e.g. Ref. [1, Chapter 13]). Since eigenvalues and eigenvectors of Z are clearly eigenvalues and eigenvectors of $L(\lambda)$ (Z is sometimes known as an *eigenmatrix*), Eq. (49) can be used to prescribe a set of n eigenvalues and corresponding right eigenvectors. However, observe that Z cannot be a *real* matrix, or our hypotheses on $\sigma(L)$ will not be satisfied. In this section, the use of this factored form as an approach to the inverse eigenvalue problem is explored. However, there may be a disadvantage in the fact that the systems discussed are monic (see the Introduction).

The factored form implies that $\widehat{D} = -(Z + Z^*)$ and $\widehat{K} = Z^*Z$ are automatically hermitian, and the condition $\widehat{K} \geqslant 0$ is automatically fulfilled. (However, there are no zero eigenvalues and so, in fact, $\widehat{K}$ is positive definite.) This approach is appealing, however, although the matrices $\widehat{D} = -(Z + Z^*)$ and $\widehat{K} = Z^*Z$ are (complex) hermitian, in general *they are not real and symmetric*. This approach has been followed up at some length in Ref. [16], and Theorem 9 from that paper is re-examined here. The theorem is specialised in one direction (the nature of the spectrum) and extended in another.

Theorem 14. *A pair of matrices X, J as defined in Eqs.* (8) *and* (9) *determine a monic matrix polynomial $L(\lambda)$ with $\widehat{K}$ and $\widehat{D}$ real and symmetric and $\widehat{K} > 0$ if and only if there are real symmetric matrices S and T, with T positive definite, such that*

$$X_c X_c^{\mathrm{T}} = -\tfrac{1}{2}\mathrm{i}T, \tag{50}$$

$$X_c \Lambda X_c^{\mathrm{T}} = \tfrac{1}{2}(I_n + \mathrm{i}S). \tag{51}$$

Furthermore, $\widehat{D} \geqslant 0$ only if $S \geqslant 0$, and $\widehat{D} > 0$ if and only if $S > 0$.

Proof. If K and D are real and symmetric then X_c can be defined so that Eqs. (20) hold. The first of these implies that $X_c X_c^{\mathrm{T}}$ is skew-hermitian. Obviously, this matrix is also symmetric. It follows easily from these observations that $X_c X_c^{\mathrm{T}} = -\frac{1}{2}\,\mathrm{i}T$ for some real symmetric T.

Using the abbreviation $Y = X_c \Lambda X_c^{\mathrm{T}}$, the second equations of Eq. (20) reads $Y + Y^* = I_n$ and has the general solution $Y = \frac{1}{2} I_n + E$, where E is an arbitrary skew-hermitian matrix. But Y is also symmetric and it follows that $E = \frac{1}{2}\,\mathrm{i}S$ for some real symmetric S, and Eq. (51) is obtained.

To see that $T > 0$ recall that X_c is nonsingular and write the second equation of Eq. (20) in the form

$$(X_c \Lambda X_c^{-1})(X_c X_c^{\mathrm{T}}) + (\bar{X}_c X_c^*)(X_c^{-*} \bar{\Lambda} X^*) = I_n.$$

For brevity, put $A = X_c \Lambda X_c^{-1}$ and use Eq. (50) to obtain $A(-\frac{1}{2}\,\mathrm{i}T) + (\frac{1}{2}\,\mathrm{i}T)A^* = I_n$, or

$$(\mathrm{i}A)T + T(\mathrm{i}A)^* = -2I_n.$$

But $\mathrm{i}\Lambda_c$ (and hence $\mathrm{i}A$) has all its eigenvalues in the open left half of the complex plane. It follows from the Lyapunov theorem (see for example [5, Section 13.1]) that T is positive definite.

Conversely, let Eqs. (50) and (51) hold with S and T real symmetric. Then it is easily verified that Eqs. (20) hold. It is necessary to show that the pair $X,\ J$ is a Jordan pair. But

$$\begin{bmatrix} X \\ XJ \end{bmatrix} \begin{bmatrix} X_c^{\mathrm{T}} & \Lambda X_c^{\mathrm{T}} \\ X_c^* & \Lambda^* X_c^* \end{bmatrix} = \begin{bmatrix} X_c & \overline{X_c} \\ X_c \Lambda & \overline{X_c \Lambda} \end{bmatrix} \begin{bmatrix} X_c^{\mathrm{T}} & \Lambda X_c^{\mathrm{T}} \\ X_c^* & \Lambda^* X_c^* \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_n & \otimes \end{bmatrix},$$

where $\otimes$ denotes a matrix of no immediate concern. Since the matrix on the right is obviously nonsingular, so is the left-most matrix and $X,\ J$ do, indeed, form a Jordan pair.

Note that, when Eqs. (50) and (51) hold and T is nonsingular, $X_c^{\mathrm{T}} = -\frac{\mathrm{i}}{2} X_c^{-1} T$ can be substituted in the second equation to obtain

$$Z = X_c \Lambda X_c^{-1} = (-S + \mathrm{i} I_n) T^{-1}. \tag{52}$$

With this Z, Eq. (49) holds, and it is easily verified that, indeed,

$$\widehat{D} = -(Z + Z^*) = S T^{-1} + T^{-1} S \tag{53}$$

and

$$\widehat{K} = Z^* Z = T^{-1}(S^2 + I_n) T^{-1} \tag{54}$$

are real symmetric matrices. Certainly $\widehat{K} \geqslant 0$ and, because each factor on the right is nonsingular, $\widehat{K} > 0$.

Now suppose that $\widehat{D} \geqslant 0$ and let μ, x be an eigenvalue/eigenvector pair of S, i.e. $Sx = \mu x$, $x \neq 0$. Write

$$2T^{-1}S = (T^{-1}S + ST^{-1}) + (T^{-1}S - ST^{-1}).$$

Since $T^{-1}S - ST^{-1}$ is skew-symmetric, $x^{\mathrm{T}}(T^{-1}S - ST^{-1})x = 0$ and, from Eq. (53),

$$x^{\mathrm{T}} \widehat{D} x = x^{\mathrm{T}}(T^{-1}S + ST^{-1})x = 2x^{\mathrm{T}} T^{-1} S x = 2\mu(x^{\mathrm{T}} T^{-1} x).$$

Since $x^{\mathrm{T}} T^{-1} x > 0$ and $\widehat{D} \geqslant 0$ it follows that $\mu \geqslant 0$. Thus, all eigenvalues of S are nonnegative and $S \geqslant 0$, as required.

Examples are easily constructed to show that $S \geqslant 0$ does not necessarily imply that $\widehat{D} \geqslant 0$. However, our argument does show that $\widehat{D} > 0$ if $S > 0$. Conversely, it follows from Theorem 3 of Section 13.1 of [16], for example, that $S > 0$ implies $\widehat{D} > 0$. $\square$

From the point of view of parametrisation of the monic problem, observe that, when $\Lambda = U + \mathrm{i}W$ is prescribed, the free parameters of S and T match exactly those of $\widehat{D}$ and $\widehat{K}$, to be determined. Also, the important case in which $\widehat{D} > 0$ can be ensured by choosing $S > 0$. The eigenvalues and eigenvectors of the monic system will be those of $Z = (-S + \mathrm{i} I_n) T^{-1}$, together with their complex conjugates. Precisely how to assign S and T to produce desired eigenvalues and eigenvectors is, however, an open question.

In the special case of diagonable systems of form (28) (with $\Theta = I_n$ in Section 3), it is found that

$$S = \Phi(W^{-1}U)\Phi^{\mathrm{T}}, \qquad T = \Phi W^{-1} \Phi^{\mathrm{T}} > 0.$$

An alternate characterisation of Z in Eq. (52) is

Proposition 15. *A matrix $Z \in \mathbb{C}^{n \times n}$ has form* (52) *if and only if it has a polar decomposition $Z = YH$ where* (*in real and imaginary parts*) *$Y = Y_R + \mathrm{i} Y_I$ is unitary with Y_I nonsingular and $H \in \mathbb{R}^{n \times n}$ with $H > 0$.*

The proof is straightforward and is omitted. This can be seen as providing a "normalised" version of Eq. (52). The fact that H is real makes this form rather special. Also, because Z is not generally normal (otherwise eigenvectors would be orthogonal), the analogous properties do not hold for the alternate polar form, $Z = \widehat{H}\widehat{Y}$. This result suggests that, by prescribing Z in this way we get more direct information about the *singular values* of Z (the square-roots of eigenvalues of H) than we do about the eigenvalues of Z.

Algorithm 3: Monic vibrating systems

DATA: Real matrices $T>0$ and $S\geqslant 0$ chosen so that $(-S+\mathrm{i}I_n)T^{-1}$ has only nonreal eigenvalues.

$$\text{COMPUTE}: \quad \widehat{D} = ST^{-1} + T^{-1}S, \tag{55}$$

$$\widehat{K} = T^{-1}(S^2 + I_n)T^{-1}. \tag{56}$$

Eigenvalues and eigenvectors assigned to $L(\lambda)$ in this algorithm are, of course, those of $(-S+\mathrm{i}I_n)T^{-1}$. In general, one must apply a standard eigenvalue/eigenvector technique to find $\sigma((-S+\mathrm{i}I_n)T^{-1})$ (and hence Λ) and the matrix of eigenvectors, X_c. To obtain a nonmonic vibrating system, an arbitrary mass matrix $M>0$ can be assigned, and then D and K are obtained by reversing the process leading to Eq. (7).

Connections between the factorisation and the modal approaches for symmetric vibrating systems can be made by inserting $X_c = X_R(I_n - \mathrm{i}\Theta)$ into Eqs. (50) and (51) to obtain

$$T = 2X_R(\Theta + \Theta^{\mathrm{T}})X_R^{\mathrm{T}}, \tag{57}$$

$$I_n = 2X_R P_1(\Theta) X_R^{\mathrm{T}}, \tag{58}$$

$$S = 2X_R(W - \Theta W \Theta^{\mathrm{T}} - \Theta U - U\Theta^{\mathrm{T}})X_R^{\mathrm{T}}. \tag{59}$$

The first equation relates Theorem 6 to the property $T>0$ of Theorem 13 and, in the light of Eq. (34), the second equation above is consistent with the monic case considered here. However, Eq. (59) reveals that the condition $S\geqslant 0$ of Theorem 13 is equivalent to (see Section 4)

$$W - \Theta W \Theta^{\mathrm{T}} - \Theta U - U\Theta^{\mathrm{T}} = [I \;\; \Theta]\begin{bmatrix} W & -U \\ -U & -W \end{bmatrix}\begin{bmatrix} I \\ \Theta^{\mathrm{T}} \end{bmatrix} \geqslant 0,$$

in contrast to the condition $-P_2(\Theta)\geqslant 0$ of Theorem 8.

Example 8. Positive definite matrices T and S are constructed to closely reproduce the eigenvalues of Example 1. Thus, $T = \mathrm{diag}[0.207, 0.459]$ and

$$S = \begin{bmatrix} 0.350 & 0.404 \\ 0.404 & 0.602 \end{bmatrix}$$

are positive definite, and the eigenvalues of $Z = (-S+\mathrm{i}I)T^{-1}$ are $-2.0003 + 4.0087\mathrm{i}$, and $-1.0021 + 3.0009\mathrm{i}$. Then Eqs. (55) and (56) yield

$$\widehat{D} = \begin{bmatrix} 3.3816 & 2.8319 \\ 2.8319 & 2.6231 \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} 30.0057 & 4.0480 \\ 4.0480 & 7.2414 \end{bmatrix}$$

which are, indeed, positive definite.

Notice also that the data of Example 1 is consistent with the following data for the modal approach:

$$\Theta = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \quad X_R = \begin{bmatrix} -0.248 & 0.205 \\ 0.305 & 0.370 \end{bmatrix}.$$

7. Conclusions

The inverse spectral problem for vibrating systems has been discussed from three points of view: spectral theory, structure preserving similarity transformations, and factorisation properties. Throughout, it has been assumed that the spectrum of the systems investigated is made up of simple, nonreal eigenvalues. The first two points of view are the most direct and admit the construction of isospectral sets of quadratic matrix functions $L(\lambda) = \lambda^2 M + \lambda D + K$. Some light has been cast on the construction of successively more intricate systems with first, real coefficients; second, real symmetric coefficients; and third, real and positive definite (or semi-definite) coefficients.

For real symmetric systems, the eigenvectors have a special structure. The "eigenmatrix" $X_c \in \mathbb{C}^{n\times n}$ can be written in the form $X_c = X_R(I - \mathrm{i}\Theta)$ where $X_R, \Theta \in \mathbb{R}^{n\times n}$, X_R is nonsingular and Θ is orthogonal. This structure plays a vital role in these investigations (see Theorem 5). In particular, the inertias of M, D, K are shown to depend only on Θ (Proposition 8), and a technique has been developed to determine vibrating systems (when $M > 0,\ D \geqslant 0,\ K > 0$), see Theorem 8, Algorithm 2 and Example 6.

In Section 5 it has been shown how structure preserving similarities for real symmetric systems can be constructed (see Eqs. (42) and (48)).

The approach via factorisation properties (Section 6) effectively replaces the inverse quadratic eigenvalue problem for vibrating systems by the *linear* inverse spectral problem for matrices of the form $(-S + \mathrm{i}I)T^{-1}$ where $T > 0$ and $S \geqslant 0$. Further investigation of this linear problem would be useful.

For the future, it is important to obtain a deeper understanding of the role of the orthogonal matrix Θ. In particular, clarification of the solutions sets of the inhomogeneous equation (22) (when the mass matrix M is prescribed) would be useful.

Research should also be directed to resolution of this whole complex of problems when both real and nonreal eigenvalues are admitted.

Acknowledgements

Research of the first-named author was partially funded by the Natural Sciences and Engineering Research Council of Canada. Dr. Prells acknowledges the funding of the Engineering and Physical Science Research Council (EPSRC) through the two linked grants GR/M93062 and GR/M93079, entitled "The Application of Geometric Algebra to Second Order Dynamic Systems in Engineering".

Appendix A. Normalising the eigenvectors

This appendix concerns real symmetric pencils $\lambda A - B$ and their reduction to diagonal form by congruence. Application is made in the main text to pencils obtained from a "linearisation" process, but the argument here applies more generally. However, attention is confined to pencils with the distribution of eigenvalues assumed throughout this paper, namely that all eigenvalues are simple and nonreal (and are written $\lambda_j = \mu_j \pm \mathrm{i}\omega_j$, $j = 1, 2, \ldots, n$ with $\omega_j > 0$ for each j). In particular, this implies that A and B are of even size, say $2n$. It will be shown that:

Proposition 16. *Under the hypotheses on A and B above, there is a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that $X^{\mathrm{T}}AX = I_n$ and $X^{\mathrm{T}}BX$ is a diagonal matrix of eigenvalues of the pencil $\lambda A - B$.*

Proof. First, a general result concerning the simultaneous reduction of pairs of real symmetric matrices by a *real* congruence transformation is used. With our assumptions on the eigenvalue distribution, Theorem I.5.4 of [17] (especially equations (I.5.12) and (I.5.11)) (see also [18, Section 2.6]) asserts the existence of a nonsingular real symmetric X_1 such that

$$X_1^{\mathrm{T}}AX_1 = \mathrm{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \tag{A.1}$$

and

$$X_1^{\mathrm{T}}BX_1 = \mathrm{diag}\left(\begin{bmatrix} -\omega_1 & \mu_1 \\ \mu_1 & \omega_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\omega_n & \mu_n \\ \mu_n & \omega_n \end{bmatrix}\right). \tag{A.2}$$

The further reduction to diagonal (rather than tri-diagonal) form now reduces to examination of the primitive 2×2 pencil

$$\lambda A_0 - B_0 := \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -\omega & \mu \\ \mu & \omega \end{bmatrix}, \tag{A.3}$$

with $\omega > 0$. It is easily verified that the eigenvalues of this pencil are $\mu \pm \mathrm{i}\omega$. Let $v = \mathrm{e}^{-\mathrm{i}\pi/4}$ and form the complex symmetric matrix

$$X_{00} = \frac{1}{\sqrt{2}}\begin{bmatrix} v & \bar{v} \\ \bar{v} & v \end{bmatrix}.$$

It can be verified that

$$X_{00}^{\mathrm{T}}A_0X_{00} = I_2 \quad \text{and} \quad X_{00}^{\mathrm{T}}B_0X_{00} = \begin{bmatrix} \mu + \mathrm{i}\omega & 0 \\ 0 & \mu - \mathrm{i}\omega \end{bmatrix},$$

and a simultaneous complex congruence of the required form is obtained in this primitive case.

Now write X_0 as a direct sum of n primitive matrices X_{00} associated with the eigenvalues λ_j for $j = 1, 2, \ldots, n$. Thus,

$$X_0 = \begin{bmatrix} X_{00,1} & 0 & 0 & \ldots & 0 \\ 0 & X_{00,2} & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & X_{00,n} \end{bmatrix},$$

and it is easily verified that, if $X = X_1 X_0^{-1}$ then X is, of course, complex and $X^T AX = I_{2n}$ and $X^T BX = D$, a diagonal matrix of eigenvalues, as required. $\square$

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