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An efficient technique for multi-frequency acoustic analysis by boundary element method

Sheng Li*

Department of Naval Architecture, Dalian University of Technology, Dalian, Liaoning Province 116024, People's Republic of China

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Abstract

A new technique for multi-frequency acoustic analysis by boundary element method is proposed. This new technique is based on the elimination of the frequency-dependent character of the boundary element coefficient matrices. By making use of the algebraic polynomials and taking out the frequency term from integrands, numerical integration involved in setting up the coefficient matrices is only confined to a frequency-independent part and the boundary element global coefficient matrices obtained by this technique are frequency independent. The final global coefficient matrices at any frequency can be simply formed by a summation of the frequency-independent global matrices and the need to compute the numerical integration at each frequency is eliminated. The technique therefore seems to be especially interesting when the bulk of the computational effort is spent on the integration phase of the solution process for a multi-frequency analysis. The test case of sound radiation from a pulsating sphere is given to demonstrate the technique. The numerical results show that this new technique can significantly save the CPU time for a multi-frequency problem.

1. Introduction

The boundary element method (BEM) has long been a very powerful numerical technique for acoustic analysis. One potential shortcoming of the BEM is the frequency-dependent character of

*Tel.: +86-411-84707319 ext. 301; fax: +86-411-84707337.
E-mail address: shengli@dlut.edu.cn (S. Li).

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the boundary element coefficient matrices. Consequently, for each different frequency, all the components in the coefficient matrices have to be recalculated. The recalculating procedure will be very time consuming if solutions for a large number of individual frequencies are required for analysis because heavy numerical integration is involved in setting up the matrices. As a result, The BEM is somewhat less attractive when used for a multi-frequency analysis.

To overcome this drawback, techniques have been developed for the solution of multi-frequency problems, such as the frequency interpolation technique [1,2], the Green's function interpolation procedure [3], the frequency interpolated transfer function [4], the matrix interpolation and solution iteration process [5] and the frequency response function approximation [6].

In this paper, a new technique for the multi-frequency analysis by the BEM is presented. The technique is based upon the elimination of the frequency-dependent character of the BEM matrix coefficients. The boundary element global matrices obtained by this technique are frequency independent. The final global coefficient matrices at any frequency can be simply formed by a summation of the frequency-independent global matrices. Therefore, heavy numerical integration is not involved. The computational effort spent on the integration time is saved.

2. Conventional boundary element formulation

For acoustic radiation problems in free space, the BEM formulation is based on the Helmholtz integral equation

$$C(P)p(P) = \int_S \left(\frac{\partial G(Q,P)}{\partial n} p(Q) - G(Q,P) \frac{\partial p(Q)}{\partial n} \right) \mathrm{d}S(Q), \tag{1}$$

where S denotes the surface of the structure, the function

$$G(Q,P) = \mathrm{e}^{-\mathrm{i}kR}/4\pi R \tag{2}$$

is the free-space Green's function in which $R = |Q - P|$, Q is any point on S, and P may be outside, inside, or on S, $k = \omega/c$ is the wavenumber, where ω is the angular frequency and c is the speed of sound, and n is the outward unit normal on S. The coefficient $C(P)$ can be calculated by

$$C(P) = 1, \quad P \text{ outside } S, \tag{3a}$$

$$C(P) = 0, \quad P \text{ inside } S \tag{3b}$$

and

$$C(P) = 1 - \int_S \frac{\cos\beta}{4\pi R^2} \, \mathrm{d}S(Q), \quad P \in S, \tag{3c}$$

where β is defined as the angle between the normal n and the vector R. Eq. (3c) includes the possibility that the surface may have a nonsmooth geometry such as when a surface has edges or corners [7]. On the boundary, the normal derivative of acoustic pressure is related to the normal velocity v_n through the momentum equation

$$\frac{\partial p}{\partial n} = -\mathrm{i}\omega\rho v_n, \tag{4}$$

where ρ is the density of the acoustic medium. The normal derivative of the Green's function appearing in Eq. (1) can be evaluated as

$$\frac{\partial G(Q,P)}{\partial n} = \frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}\left(\mathrm{i}k + \frac{1}{R}\right)\cos\beta. \tag{5}$$

Substitution of Eqs. (2)–(5) into the surface equation (1) yields

$$C(P)p(P) - \int_S p(Q)\frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}\left(\mathrm{i}k + \frac{1}{R}\right)\cos\beta\,\mathrm{d}S(Q) = \mathrm{i}\omega\rho\int_S v_n(Q)\frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}\,\mathrm{d}S(Q), \tag{6}$$

where P is on S.

A numerical solution to the boundary integral equation (6) can be achieved by discretizing the boundary surface S into a number of surface elements and nodes [7,8]. For each position of P, the boundary integral in Eq. (6) can be replaced by a sum of integrals over the surface elements (denoted by S_j, where $j = 1, \ldots,$ number of elements). The coordinates, pressures and normal velocities x_i, p, v_n at any points on a surface element are assumed to be related to the coordinates, pressures and normal velocities x_i^l, p^l, v_n^l ($l = 1, \ldots, L$, L is the number of nodes on the surface element) at nodal points on the element, by

$$x_i = \sum_{l=1}^{L} N_l(\xi,\eta)x_i^l, \quad p = \sum_{l=1}^{L} N_l(\xi,\eta)p^l, \quad v_n = \sum_{l=1}^{L} N_l(\xi,\eta)v_n^l, \tag{7}$$

where $N_l(\xi,\eta)$ are the shape functions of the local coordinates $-1 \leqslant \xi \leqslant 1$ and $-1 \leqslant \eta \leqslant 1$.

We then place P at each of nodal points on the surface successively. For each collocation point P, we substitute Eq. (7) into Eq. (6) and integrate the equation over the entire surface. The integration is actually done on an element-by-element basis. Each collocation point P and element S_j combination produces two "element coefficient vectors",

$$h^l = \int_{S_j} N_l(\xi,\eta)\left(\frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}\left(\mathrm{i}k + \frac{1}{R}\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{8}$$

$$g^l = \int_{S_j} N_l(\xi,\eta)\left(\mathrm{i}\omega\rho\frac{\mathrm{e}^{-\mathrm{i}kR}}{4\pi R}\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{9}$$

where $h^l = h_R^l + \mathrm{i}h_I^l$ has real part h_R^l and imaginary part h_I^l, as

$$h_R^l = \int_{S_j} N_l(\xi,\eta)\left(\left(\frac{-k\sin(-kR)}{4\pi R} + \frac{\cos(-kR)}{4\pi R^2}\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{10}$$

$$h_I^l = \int_{S_j} N_l(\xi,\eta)\left(\left(\frac{k\cos(-kR)}{4\pi R} + \frac{\sin(-kR)}{4\pi R^2}\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{11}$$

and $g^l = g^l_R + \mathrm{i}g^l_I$ has real part g^l_R and imaginary part g^l_I, as

$$g^l_R = \int_{S_j} N_l(\xi,\eta)\left(\frac{-\omega\rho\sin(-kR)}{4\pi R}\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{12}$$

$$g_I = \int_{S_j} N_l(\xi,\eta)\left(\frac{\omega\rho\cos(-kR)}{4\pi R}\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{13}$$

where $J(\xi,\eta)$ is the Jacobian of the transformation equation (7).

Assemble the element coefficient vector h^l into a global matrix $[H']$, and g^l into a global matrix $[G]$. This produces

$$[C]\{p\} + [H']\{p\} = [G]\{v_n\}, \tag{14}$$

where $[C]$ is a diagonal matrix. Combining $[C]$ and $[H']$ into one single matrix $[H]$ yields

$$[H]\{p\} = [G]\{v_n\}, \tag{15}$$

where $[H]$ and $[G]$ are the final global coefficient matrices, $\{p\}$ and $\{v_n\}$ are the vectors consisting of the field values at the nodal locations of a grid defining the surface of the structure for the surface acoustic pressure and normal velocity. It should be noted that for an exterior problem the previous formulation based on the surface Helmholtz integral equation has nonuniqueness problem and may fail to yield a unique solution at certain characteristic eigenfrequencies associated with the corresponding interior Dirichlet problem. To avoid this difficulty, the combined Helmholtz integral equation formulation (CHIEF) method [9] can be employed.

With the solution for the pressure, the acoustic power can be computed by

$$W = \frac{1}{2}\int_S \mathrm{Re}(pv_n^*)\,\mathrm{d}S, \tag{16}$$

where the asterisk denotes the complex conjugate.

3. Multi-frequency calculation technique

From Eqs. (8) and (9), it can be seen that the element coefficient vectors h^l and g^l for each P and S_j combination contain integration of the Green's function and its normal derivative, respectively. Since the Green's function is frequency dependent, the integration has to be carried out for each different frequency in a multi-frequency run. This procedure is very time-consuming because the numerical integration is involved in setting up the coefficient matrices at every frequency.

To overcome this drawback, a new technique for multi-frequency calculation is proposed via the elimination of the frequency-dependent character of the BEM matrix coefficients. The idea is to take out the frequency term ω from the integrands in Eqs. (8)–(13). To do this, firstly use algebraic polynomials

$$P_n(-kR) = P_n\left(-\frac{\omega R}{c}\right) = a_0 + a_1\left(-\frac{\omega R}{c}\right) + \cdots + a_n\left(-\frac{\omega R}{c}\right)^n$$
$$= b_0 + b_1\omega + \cdots + b_n\omega^n = \sum_{i=0}^{n} b_i\omega^i \tag{17}$$

to approximate $\sin(-kR)$ and $\cos(-kR)$ in Eqs. (10)–(13), respectively, where n is a nonnegative integer and $a_0, \ldots, a_n$ are real constants. Then, substituting an algebraic polynomial $\sum_{i=0}^{n} b_{Si}\omega^i$ for $\sin(-kR)$ and an algebraic polynomial $\sum_{i=0}^{n} b_{Ci}\omega^i$ for $\cos(-kR)$ into Eqs. (10)–(13), respectively, yields

$$\begin{aligned} h_R^l &= \int_{S_j} N_l(\xi,\eta)\left(\left(\frac{-k\sin(-kR)}{4\pi R} + \frac{\cos(-kR)}{4\pi R^2}\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\approx \int_{S_j} N_l(\xi,\eta)\left(\left(\frac{-\omega}{4\pi Rc}\sum_{i=0}^{n} b_{Si}\omega^i + \frac{1}{4\pi R^2}\sum_{i=0}^{n} b_{Ci}\omega^i\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\approx \int_{S_j} N_l(\xi,\eta)\left(\left(\sum_{i=0}^{n+1} c_i(R)\omega^i\right)\cos\beta\right)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\approx \sum_{i=0}^{n+1}\omega^i\int_{S_j} c_i(R)N_l(\xi,\eta)(\cos\beta)J(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\approx \sum_{i=0}^{n+1}\omega^i h_{Ri}^l, \end{aligned} \tag{18}$$

$$h_I^l \approx \sum_{i=0}^{n+1}\omega^i h_{Ii}^l, \tag{19}$$

$$g_R^l \approx \sum_{i=1}^{n+1}\omega^i g_{Ri}^l, \tag{20}$$

$$g_I^l \approx \sum_{i=1}^{n+1}\omega^i g_{Ii}^l. \tag{21}$$

Let

$$h_i^l = h_{Ri}^l + \mathrm{i}h_{Ii}^l, \tag{22}$$

$$g_i^l = g_{Ri}^l + \mathrm{i}g_{Ii}^l. \tag{23}$$

It is apparent that the element coefficient vectors h_i^l and g_i^l are independent of the frequency ω. Assembling the element coefficient vectors h_i^l into global matrices $[H_i']$, and g_i^l into global matrices $[G_i]$, the matrices $[H_i']$ and $[G_i]$ are also independent of frequency ω and only need to be evaluated once for all the frequencies. Finally, for each different frequency ω in a multi-frequency run, the final global coefficient matrices $[H]$ and $[G]$ can be simply formed by

$$[H] = [C] + \sum_{i=0}^{n+1} \omega^i [H_i'], \tag{24}$$

$$[G] = \sum_{i=1}^{n+1} \omega^i [G_i]. \tag{25}$$

With this technique, it can be seen that the numerical integration is confined to a frequency-independent part and the boundary element global coefficient matrices $[H_i']$ and $[G_i]$ are frequency independent. The final global coefficient matrices at any frequency can be simply formed by a summation of the frequency-independent global matrices. Therefore, heavy numerical integration is not involved in setting up the coefficient matrices for every frequency. However, it is also clear that for the first frequency more integration time are required since the matrices $[H_i']$ $(i = 0, 1, \ldots, n+1)$ and $[G_i]$ $(i = 1, \ldots, n+1)$ need to be evaluated for the first frequency, but for the second or any subsequent frequency the computational effort spent on the integration time is saved. In addition, it should be noted that, as expressed in Eqs. (23) and (24), the computer memory capacity is required $n+2$ and $n+1$ times of that need by the conventional BEM to calculate the coefficient matrices $[H]$ and $[G]$, respectively. This considerably increases the memory requirements and may lead to a limitation for the large-scale problems.

Next, consider the problem of finding a polynomial of a specific degree to approximate the functions $\sin x$ and $\cos x$. Naturally, the power series of $\sin x$ and $\cos x$ can be used as the approximating algebraic polynomials, which is,

$$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots , \tag{26}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots . \tag{27}$$

In order to approximate $\sin x$ and $\cos x$ adequately by their power series, the truncated error $R_m(x)$ referring to the error involved in using a truncated summation of the first m terms of the above infinite series should be evaluated. The error term $R_m(x)$ is bounded by

$$|R_m(x)| \leqslant \frac{x^{2m+1}}{(2m+1)!} \quad \text{for } \sin x, \tag{28}$$

$$|R_m(x)| \leqslant \frac{x^{2m}}{(2m)!} \quad \text{for } \cos x. \tag{29}$$

For example, for low and medium frequency range of $kR \leqslant 5$, the error $R_{10}(x)$ using a truncated summation of the first 10 terms to approximate $\sin x$ and $\cos x$ is bounded by $|R_{10}(x)| \leqslant 9.3 \times 10^{-6}$ and $|R_{10}(x)| \leqslant 3.9 \times 10^{-5}$, respectively.

Alternatively, the least-squares approximating polynomial can be employed for $\sin x$ and $\cos x$ on an interval, for example, $kR \in [0, 5]$. The least-squares approximating polynomial of degree four on the interval $[0, 5]$ is

$$P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \tag{30}$$

where $a_0 = -0.05090119134439156$, $a_1 = 1.2664397327544807$, $a_2 = -0.2867434534435648$, $a_3 = -0.0932253274939894$, and $a_4 = 0.018647117481992697$ for $\sin x$, and $a_0 = 0.9390158755354185$, $a_1 = 0.3873333116334018$, $a_2 = -1.0747522082832677$, $a_3 = 0.31927601623158164$, and $a_4 = -0.02496192891542331$ for $\cos x$, respectively.

Another form of the least-squares approximating polynomials for $\sin x$ and $\cos x$ on the interval $[0, 5]$ is

$$P_s(x) = a_0 x + a_1 x^3 + a_2 x^5 + a_3 x^7 \quad \text{for } \sin x, \tag{31}$$

where $a_0 = 0.983652676098689$, $a_1 = -0.1567147943100234$, $a_2 = 0.0066827929787795564$, and $a_3 = -0.00009216350206032148$, and

$$P_c(x) = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 \quad \text{for } \cos x, \tag{32}$$

where $a_0 = 0.9710932877480939$, $a_1 = -0.4565802435114196$, $a_2 = 0.0314245577791617$, and $a_3 = -0.0005761784011113857$. Fig. 1 shows these least-squares approximating polynomials.

4. Numerical results and discussion

The sound radiation from a pulsating sphere is given to test the present technique for multi-frequency acoustic analysis.

The exact analytical solution for the acoustic power radiated by the sphere of radius r_0 pulsating with uniform radial velocity v in a full-space is

$$\Pi = \frac{2\pi\rho c v^2 k^2 r_0^4}{1 + k^2 r_0^2}. \tag{33}$$

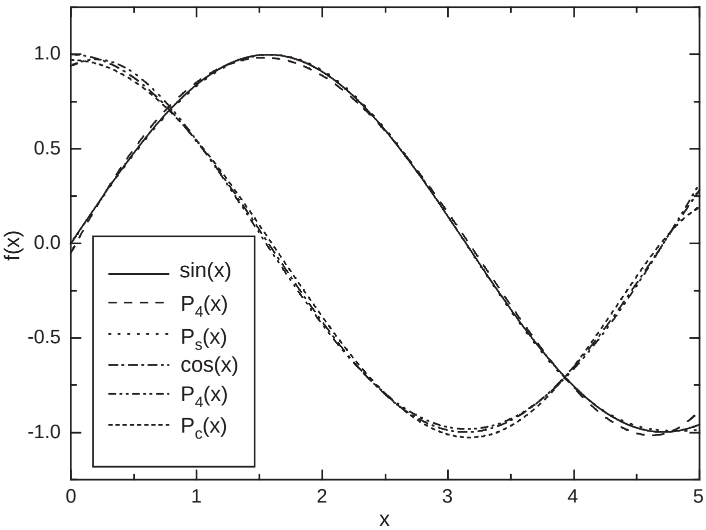

Fig. 1. Graphs of the least-squares approximating polynomials.

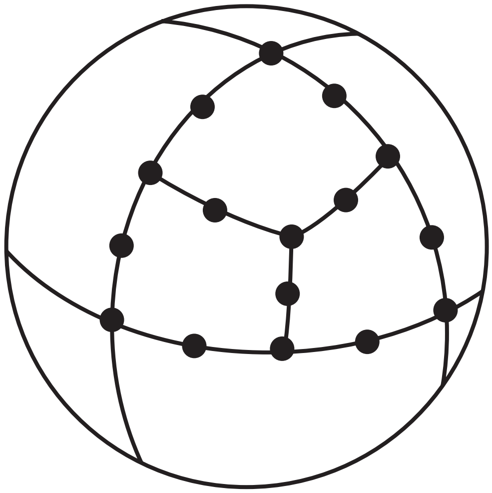

Fig. 2. Discretization of a sphere (one octant of the sphere showing nodes and element configuration).

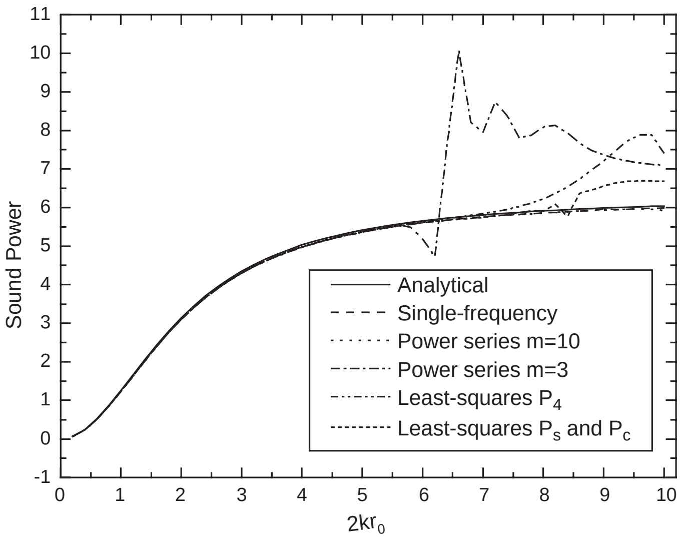

Fig. 3. Comparison of computed sound power of the present method, the conventional method, and the analytical result.

The sphere is discretized into 24 quadratic quadrilateral isoparametric elements and 74 nodes. Fig. 2 shows the discretization. The numerical solutions of the present multi-frequency technique using different polynomials are compared to the analytical solution and a conventional BEM solution in Fig. 3. In the calculation, for different wavenumbers k, the parameters r_0, v, ρ and c are all considered equal to unity. One CHIEF point at the center of the sphere is added to overcome the non-uniqueness difficulty at the characteristic eigenfrequency corresponding to $k = \pi$. Good

Table 1
Comparison of CPU time (in seconds) for the sphere

Solution techniques	Conventional BEM	Power series $m = 10$	Power series $m = 3$	Least-squares P_4	Least-squares P_s and P_c
CPU time	101	15	9	11	10
Saving (%)	—	85	91	89	90

Table 2
Comparison of CPU time (in seconds) for different BEM meshes of the sphere

Number of elements	Number of nodes	Conventional BEM		Power series $m = 10$		Saving (%)
		First[a]	Second[b]	First[a]	Second[b]	Second[b]
61	185	4.40	3.88	16.42	1.33	66
116	350	16.75	16.13	45.89	8.61	47
182	548	67.16	66.38	119.66	43.45	35
386	1160	815.75	813.66	991.98	742.07	9

[a]The first frequency.
[b]The second or any subsequent frequency.

agreement is observed for all approximating polynomials for $2kr_0 \leqslant 5$, where $2r_0$ is the largest dimension of boundary surface S of the pulsating sphere. Excellent agreement is observed even at $2kr_0 = 10$ for using the power series of $m = 10$. In the case of using the least-squares approximating polynomial, the use of the set of functions $\{x^1, x^3, x^5, x^7\}$ and $\{x^0, x^2, x^4, x^6\}$ to approximate $\sin x$ and $\cos x$, respectively, is better than the use of the set of functions $\{x^0, x^1, x^2, x^3, x^4\}$.

Table 1 shows the CPU time used by the present technique and the conventional BEM. The number of frequency sweep is 100. All computations are performed here with a single-processor 1.5-GHz Pentium IV PC with 512 MB RAM. From Table 1, it can be seen the present technique saves a significant amount of CPU time (around 85–91%) for this multi-frequency run. It can also be seen that the CPU time saving decreases as the terms used in the approximating polynomials increase. This is obvious because more CPU time has to be spent in forming $[H'_i]$, $[G_i]$ and further $[H]$, $[G]$ as the terms increase.

Table 2 shows the CPU time comparison between the present and the conventional BEM's for different BEM meshes of the sphere using quadratic quadrilateral isoparametric elements. From Table 2, it can be seen that for the first frequency, the present technique is slower than the conventional BEM since more integration time and matrix assemblage time are required. For the second and subsequent frequencies the present technique saves a great deal of CPU time. However, the CPU time saving (in percentage) does decrease as the number of nodes increases. This is because as the number of nodes increases, the matrix solution time is by far more important than the integration and assemblage of the system matrices. When the integration time dominates the CPU time for a multi-frequency problem of small to medium size, the present technique does save a significant amount of CPU time. When the matrix solution time dominates the CPU time for the large-scale problems, the present technique may not save CPU time considerably.

5. Conclusions

A new technique for multi-frequency acoustic analysis by BEM is proposed. It is based upon the elimination of the frequency-dependent character of the boundary element matrix coefficients. With this technique the numerical integration is only confined to a frequency-independent part and the boundary element global matrices obtained are frequency independent. The final global coefficient matrices can be simply formed by a summation of the frequency-independent global matrices. This technique has been proved to be significantly faster than the conventional frequency sweep technique for a multi-frequency problem of small to medium size but to require more memory capacity. The technique therefore seems to be especially interesting while the integration time dominates the CPU time for a multi-frequency analysis. In conclusion, the technique presented in this paper turns out to be an efficient numerical tool to analyse multi-frequency acoustic problems.

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