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Vehicle–bridge interaction dynamics and potential applications

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Abstract

The dynamic interaction between a moving vehicle and the sustaining bridge is studied. By the method of modal superposition, closed-form solutions are obtained for the vertical responses of both the bridge and moving vehicle, assuming the vehicle/bridge mass ratio to be small. For both the bridge and vehicle responses, it is confirmed that rather accurate solutions can be obtained by considering only the first mode. The displacement, velocity, and acceleration of the bridge are governed at different extents by two sets of frequencies, i.e., the driving frequency of the vehicle and natural frequencies of the bridge. From the spectrum for the bridge displacement, the vehicle speeds can be shown to be associated with some low-frequency pikes. On the other hand, the vehicle responses are governed by five distinct frequencies that appear as driving frequencies, vehicle frequency, and bridge frequencies with shift. From the vehicle's acceleration spectrum, the first bridge frequency (with shift) is shown to have rather high visibility and can be easily identified. The effects of damping of the vehicle and bridge are evaluated in the numerical studies. Potential applications of the present results, as well as further researches required, are also indicated in the paper.

1. Introduction

The problem of a vehicle traveling over a bridge is commonly encountered in the transportation facilities such as highway bridges, railroad bridges, aircraft/taxiway bridges in airports, and so on. When a vehicle passes over a bridge, certain impact or dynamic amplification effect will be

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induced on the bridge, which needs to be taken into account in the design of bridges. Attention paid to this subject dates back to the works of Willis [1] and Stokes [2] in the mid 19th century. It has long been observed that when a bridge is subjected to moving loads, the induced dynamic deflections and stresses can be significantly higher than those observed for the static case [3]. In this aspect, the majority of the literature has been devoted to investigation of the bridge vibrations using the so-called *moving load* [4–8], *moving mass* [9–15] and *moving sprung mass* models [6,16–19] for the vehicles.

The *moving load* is the simplest model that can be conceived for a vehicle in studying the bridge vibrations. With this model, the essential dynamic characteristics of the bridge caused by the vehicle's moving action can be captured with a sufficient degree of accuracy. However, it suffers from the drawback that the interaction between the bridge and moving vehicle is ignored. For this reason, the moving load model is strictly valid for the case where the mass of the vehicle is small relative to that of the bridge, and only when the vehicle response is *not* desired. The *moving mass* model represents an improvement over the moving load model in that the effect of inertia of the vehicle is taken into account. Nevertheless, it does *not* allow consideration of the bouncing action of the moving vehicle relative to the bridge. Such an effect is expected to be significant in the presence of pavement roughness or for vehicles moving at rather high speeds.

To account for the bouncing or suspension action of the moving vehicle, a spring can be attached to the moving mass to result in the *sprung mass* model shown in Fig. 1. This is the simplest model that can be used to study the *dynamic interaction* between the moving vehicle and the supporting bridge. It is true in the past two decades that researchers continue to develop vehicle models of various complexities to account for the dynamic properties of the vehicle, see, for instance, those listed in Refs. [18,20–25]. To the knowledge of the writers, however, a great majority of the previous related works have been focused on the *dynamic response of the bridge* in time domain, with little attention paid to the *dynamic behavior of the moving vehicle* or to the *frequency contents* of the interaction between the two subsystems. Perhaps, one exception in this regard is the series of works carried out by Yang and co-workers on the vehicle response, in which equal emphases have been placed on the bridge and vehicle responses, using the latter as an indicator for the riding comfort of passengers [17,18,24,26]. Recently, a preliminary study was

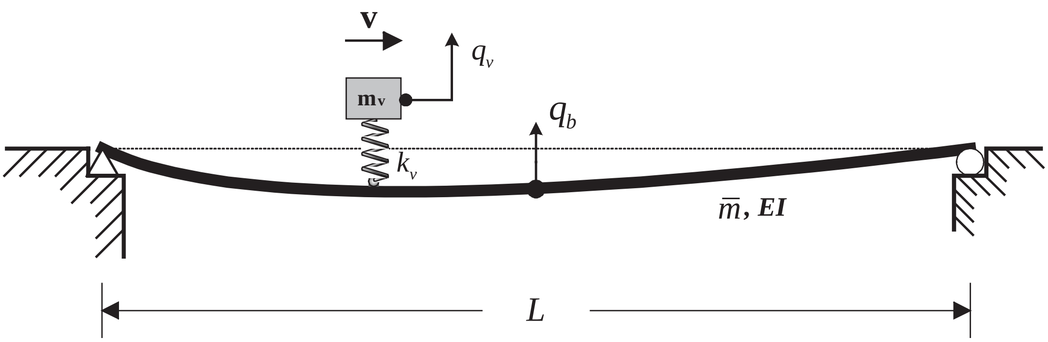

Fig. 1. Vehicle moving over a simply supported bridge.

conducted by the authors to extract the fundamental frequency of a bridge from the dynamic response of a passing vehicle [27].

Unlike most of the previous works, this paper is focused exclusively on the *dynamic interaction*, especially the frequency aspects of such an interaction, between the moving vehicle and the bridge, with equal emphases placed on the bridge and vehicle responses. To this end, two sets of second-order differential equations of motion will be written, one for the vehicle and the other for the bridge. It is the interaction or contact force existing between the two subsystems that makes the two sets of equations nonlinear and coupled, as the contact force moves (and thereby varies) from time to time, even though the two subsystems by themselves may be linear.

In order to highlight the effect of interaction, a simply supported beam subjected to a moving sprung mass as shown in Fig. 1 will be adopted. The mass of the vehicle is assumed to be small relative to that of the bridge. By the method of modal superposition, together with the use of convolution integrals, closed-form solutions will be obtained for both the bridge and the moving vehicle, which are approximate in the sense that iterations are not performed for updating the contact force existing between the two subsystems to consider the mutual interaction. The accuracies of the solutions obtained are compared with those obtained by independent finite element analyses. By transforming all the time-history responses into the frequency domain, some interesting phenomena can be observed. For example, from the displacement response of the bridge, one can identify the vehicle speeds as those appearing as low-frequency pikes. On the other hand, from the vehicle's acceleration spectrum, it is seen that the first bridge frequency can be identified with high visibility. Such implications are important and have their potential areas of applications. It is suggested that future research be carried out along the lines of interaction dynamics to investigate the practical aspects of the ideas presented herein.

2. Physical modelling and formulation

Consider a simply supported beam subjected to a vehicle moving at speed v, as shown in Fig. 1. The vehicle is represented as a concentrated mass m_v supported by a spring of stiffness k_v with the effect of damping of the suspension system neglected. The beam is assumed to be of the Bernoulli–Euler type with constant cross sections. No consideration will be made of the damping property or pavement irregularity of the bridge. The vehicle–bridge system as shown in Fig. 1 allows us to investigate various interaction phenomena involved, as will be presented below.

The equations of motion governing the transverse or vertical vibration of the bridge and moving vehicle are

$$\bar{m}\ddot{u} + EIu'''' = p(x,t), \tag{1}$$

$$m_v\ddot{q}_v + k_v q_v = k_v u|_{x=vt}, \tag{2}$$

where $\bar{m}$ denotes the mass per unit length, E the elastic modulus, and I the moment of inertia of the beam, m_v and k_v denote the mass of vehicle and stiffness of the suspension system, respectively, q_v is the vertical displacement of the vehicle, $u(x,t)$ is the vertical displacement of the bridge, and $p(x,t)$ is the applied force acting on the bridge through the contact point at position vt, which moves with the vehicle. The applied force $p(x,t)$ can be expressed as follows:

$$p(x,t) = f_c(t)\,\delta(x - vt), \tag{3}$$

where $\delta(x - vt)$ is the Dirac delta function evaluated at the contact point, $x = vt$, and the *contact force* $f_c(t)$ is equal to the sum of the vehicle weight and the elastic force of the suspension system, i.e.

$$f_c(t) = -m_v g + k_v\left(q_v - u|_{x=vt}\right) \tag{4}$$

with g denoting the gravity of acceleration.

It should be noticed that the vertical displacement of the vehicle is measured from its static equilibrium position. Therefore, there is no additional force acting on the vehicle except the elastic force resulting from the shortening or elongation of the supporting spring. The solution to Eq. (1) will be expressed in terms of the modal shapes, $\phi_n(x)$, and associated modal coordinates, $q_{bn}(t)$, as

$$u(x,t) = \sum_n \phi_n(x) q_{bn}(t). \tag{5}$$

For the simply supported beam considered in this study, the mode shapes of the bridge are known to be of the sinusoidal type. Therefore, Eq. (5) becomes

$$u(x,t) = \sum_n \left[\sin\frac{n\pi x}{L}\, q_{bn}(t)\right]. \tag{6}$$

Substituting Eq. (6) for the displacement $u(x,t)$ into Eq. (1), multiplying both sides of the equation by $\phi_m(x)$, and integrating with respect to x over the length L of the beam, one obtains

$$\int_0^L \bar{m}\phi_m \sum_n (\phi_n \ddot{q}_{bn})\,\mathrm{d}x + \int_0^L EI\,\phi_m \sum_n (\phi_n'''' q_{bn})\,\mathrm{d}x = \int_0^L f_c(t)\delta(x - vt)\phi_m\,\mathrm{d}x. \tag{7}$$

By use of the orthogonality conditions for the modal shapes and changing the subscript m into n, Eq. (7) reduces to

$$\ddot{q}_{bn} + \omega_{bn}^2 q_{bn} = \frac{f_c(t)\int_0^L \delta(x - vt)\phi_n(x)\,\mathrm{d}x}{\bar{m}\int_0^L \phi_n^2(x)\,\mathrm{d}x}, \tag{8}$$

where ω_{bn} is the frequency of vibration of the nth mode of the bridge,

$$\omega_{bn} = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\bar{m}}}. \tag{9}$$

Further, through manipulation of the right-hand side of Eq. (8), the nth modal equation of the bridge can be obtained as follows:

$$\ddot{q}_{bn} + \omega_{bn}^2 q_{bn} + \frac{2\omega_v^2 m_v}{\bar{m}L}\sin\frac{n\pi vt}{L}\sum_j\left[\sin\frac{j\pi vt}{L}\, q_{bj}\right] - \left[\frac{2\omega_v^2 m_v}{\bar{m}L}\sin\frac{n\pi vt}{L}\right] q_v = \frac{-2m_v g}{\bar{m}L}\sin\frac{n\pi vt}{L}. \tag{10}$$

In a similar way, Eq. (2) can be rewritten as

$$\ddot{q}_v + \omega_v^2 q_v = \omega_v^2 \sum_n \sin \frac{n\pi vt}{L} q_{bn}, \tag{11}$$

where ω_v is the frequency of vibration of the vehicle

$$\omega_v = \sqrt{\frac{k_v}{m_v}}. \tag{12}$$

By the assumption that the vehicle mass m_v is much less than the bridge mass $\bar{m}L$, i.e., $m_v/\bar{m}L \ll 1$, the governing equation in Eq. (10) can be approximated as follows:

$$\ddot{q}_{bn} + \omega_{bn}^2 q_{bn} = \frac{-2m_v g}{\bar{m}L} \sin \frac{n\pi vt}{L}, \tag{13}$$

which is identical to the one for a bridge subjected to a single moving load with constant speed v [27].

3. Dynamic responses of the bridge

Assuming zero initial conditions, the solution to the second-order linear differential equation of the bridge in Eq. (13) can be obtained as

$$q_{bn}(t) = \frac{\Delta_{\mathrm{st}n}}{1 - S_n^2} \left[\sin\left(\frac{n\pi vt}{L}\right) - S_n \sin\left(\omega_{bn} t\right) \right], \tag{14}$$

where $\Delta_{\mathrm{st}n}$ is the static deflection caused by the vehicle with respect to the nth mode,

$$\Delta_{\mathrm{st}n} = \frac{-2m_v g L^3}{n^4 \pi^4 EI} \tag{15}$$

and S_n is a non-dimensional speed parameter,

$$S_n = \frac{n\pi v}{L\omega_{bn}}. \tag{16}$$

Hence, the total displacement response of the bridge to a vehicle moving at speed v is

$$u(x,t) = \sum_n \frac{\Delta_{\mathrm{st}n}}{1 - S_n^2} \left\{ \sin \frac{n\pi x}{L} \left[\sin \frac{n\pi vt}{L} - S_n \sin \omega_{bn} t \right] \right\}. \tag{17}$$

Differentiating Eq. (17) with respect to time t, we obtain the velocity response of the bridge as follows:

$$\dot{u}(x,t) = \sum_n \frac{\Delta_{\mathrm{st}n}}{1 - S_n^2} \left\{ \sin \frac{n\pi x}{L} \left[\left(\frac{n\pi v}{L}\right) \cos \frac{n\pi vt}{L} - (\omega_{bn} S_n) \cos \omega_{bn} t \right] \right\} \tag{18}$$

which can further be differentiated to yield the acceleration response of the bridge as

$$\ddot{u}(x,t) = \sum_n \frac{\Delta_{\mathrm{st}n}}{1 - S_n^2} \left\{ \sin \frac{n\pi x}{L} \left[\left(\omega_{bn}^2 S_n\right) \sin \omega_{bn} t - \left(\frac{n\pi v}{L}\right)^2 \sin \frac{n\pi vt}{L} \right] \right\}. \tag{19}$$

As can be seen, each of the responses of the bridge given in Eqs. (17)–(19) can be divided into two components pertaining to the *driving frequencies* of the vehicle, $n\pi v/L$, and *natural frequencies* of the bridge, ω_{bn}. By substitution of $x = L/2$ into Eqs. (17)–(19), we obtain the mid-point responses of the bridge as

$$u\left(\frac{L}{2}, t\right) = \sum_n \frac{\Delta_{\mathrm{st}n}}{(1 - S_n^2)} \sin\frac{n\pi}{2}\left[\sin\frac{n\pi vt}{L} - S_n \sin\omega_{bn}t\right], \tag{20}$$

$$\dot{u}\left(\frac{L}{2}, t\right) = \sum_n \frac{\Delta_{\mathrm{st}n}}{(1 - S_n^2)} \sin\frac{n\pi}{2}\left[\frac{n\pi v}{L}\cos\frac{n\pi vt}{L} - \frac{\pi v}{L}\cos\omega_{bn}t\right], \tag{21}$$

$$\ddot{u}\left(\frac{L}{2}, t\right) = \sum_n \frac{\Delta_{\mathrm{st}n}}{(1 - S_n^2)} \sin\frac{n\pi}{2}\left[-\left(\frac{n\pi v}{L}\right)^2 \sin\frac{n\pi vt}{L} + \frac{\pi v\omega_{bn}}{L}\sin\omega_{bn}t\right]. \tag{22}$$

By relating $\Delta_{\mathrm{st}n}$ to $\Delta_{\mathrm{st}1}$ using the definition in Eq. (15), the coefficients of the two components mentioned above for the bridge responses pertaining to the driving frequencies and natural frequencies have been listed in Table 1. Because the midpoint of the beam happens to be a stationary point for the modal shapes of even order, the amplitudes corresponding to all the even-order modes are merely equal to zero. Thus, only the odd-order modes need to be considered in Table 1. The contributions of the two frequency components to the midpoint displacement, velocity and acceleration of the bridge were also plotted in Fig. 2(a)–(c) for the case of $S_1 = 0.1$, which means that the time for the vehicle to pass over the bridge is 10 times the fundamental period of the bridge. As can be seen, the amplitudes of each individual mode pertaining to the two frequency components decrease drastically as the order of the mode increases. For practical applications, it is concluded that *solutions of sufficient accuracy can be obtained for the bridge if only the first mode of vibration is considered*, as was pointed out by Biggs [16]. Besides, the displacement amplitude is dominated mainly by the driving frequency component. The velocity

Table 1
Coefficients of frequency terms in bridge responses

	Driving frequency $n\pi v/L$	Natural frequency ω_{bn}
Displacement	$\dfrac{\Delta_{\mathrm{st}n1}}{n^4(1 - S_1^2/n^2)}$	$\dfrac{\Delta_{\mathrm{st}1}S_1}{n^6(1 - S_1^2/n^2)}$
Velocity	$\dfrac{\Delta_{\mathrm{st}1}S_1}{n^3(1 - S_1^2/n^2)}$	$\dfrac{\Delta_{\mathrm{st}1}S_1}{n^4(1 - S_1^2/n^2)}$
Acceleration	$\dfrac{\Delta_{\mathrm{st}1}S_1^2}{n^2(1 - S_1^2/n^2)}$	$\dfrac{\Delta_{\mathrm{st}1}S_1}{n^2(1 - S_1^2/n^2)}$

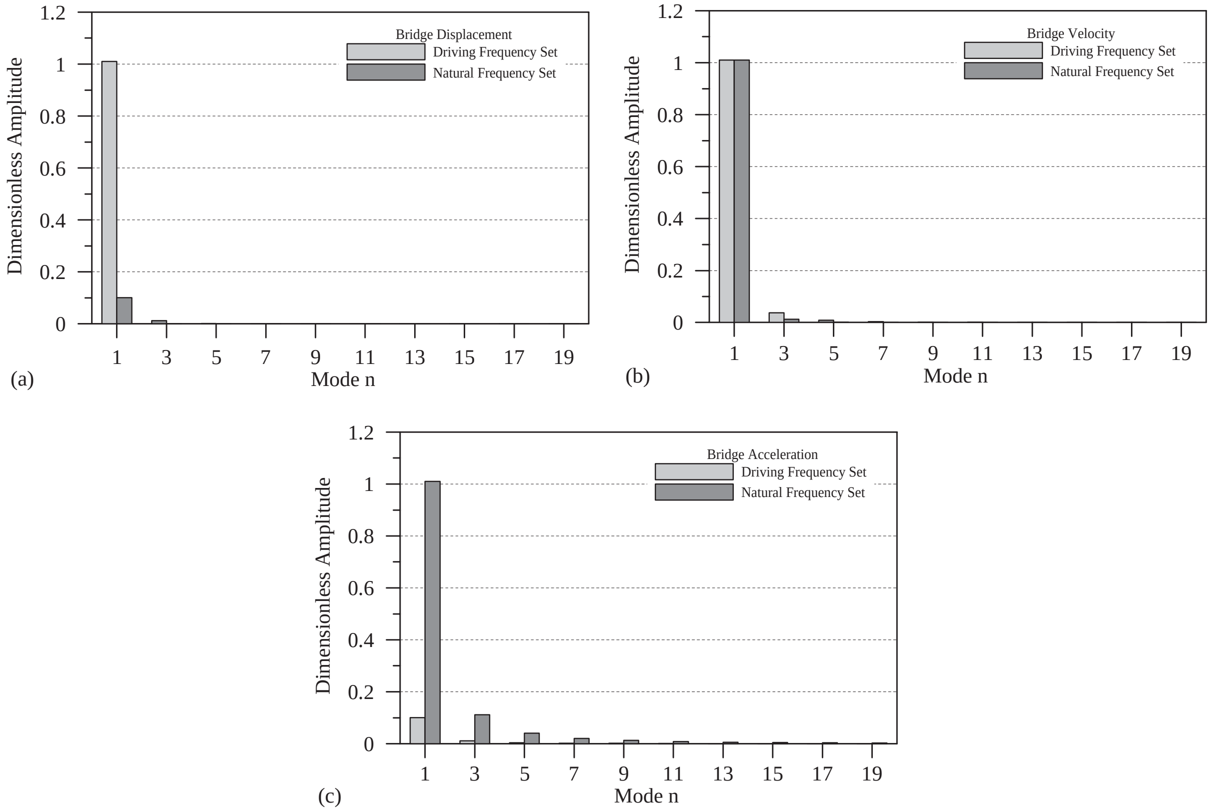

Fig. 2. Bridge response amplitude for two frequency components ($S_1 = 0.1$): (a) displacement, (b) velocity, and (c) acceleration.

amplitude, however, is nearly equally governed by both frequency components. On the contrary, the acceleration amplitude is dominated by the natural or bridge frequency component.

Another parameter that affects the amplitude of the individual frequency component is the speed parameter S. As defined in Eq. (16), the speed parameter represents the ratio of the driving frequency to the natural frequency of bridge, which is normally less than 0.5 in practice [28]. As can be seen from Figs. 3 and 4, the displacement amplitude of each mode, particularly of the first mode, increases as the speed parameter of the vehicle increases, due to the fact that the energy input to the bridge is higher as the vehicle moves over the bridge at a higher speed. On the other hand, a greater speed parameter also implies that a vehicle passes over a bridge with a lower natural frequency or a longer span with the same speed. A bridge with a longer span allows the vehicle to pass at a larger acting time, which in turn allows more energy to be accumulated on the bridge. This explains why the amplitude of the displacement of the bridge increases as the speed parameter of the vehicle increases. The same is also true for the velocity and acceleration responses of the bridge, which are not shown here for the sake of saving the paper length.

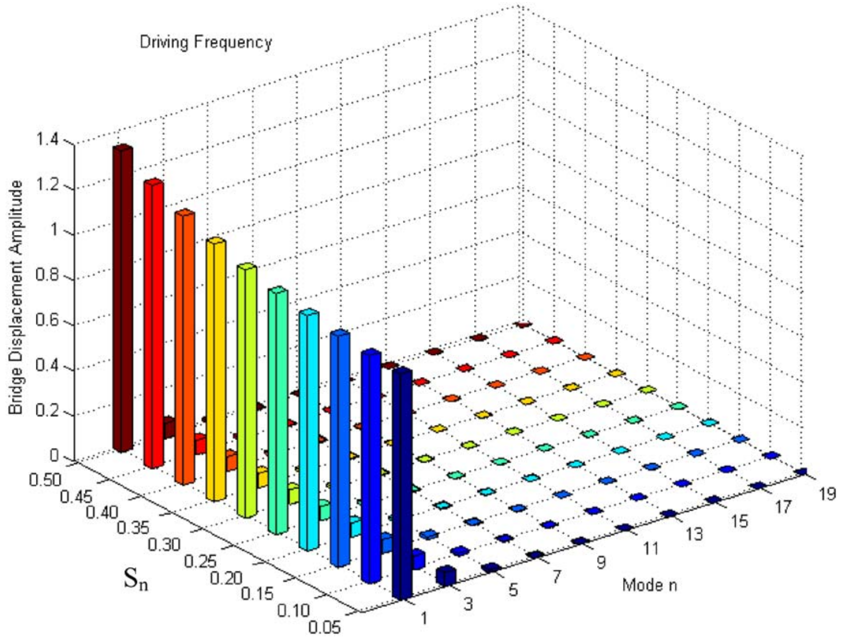

Fig. 3. Bridge displacement amplitude for driving frequency component.

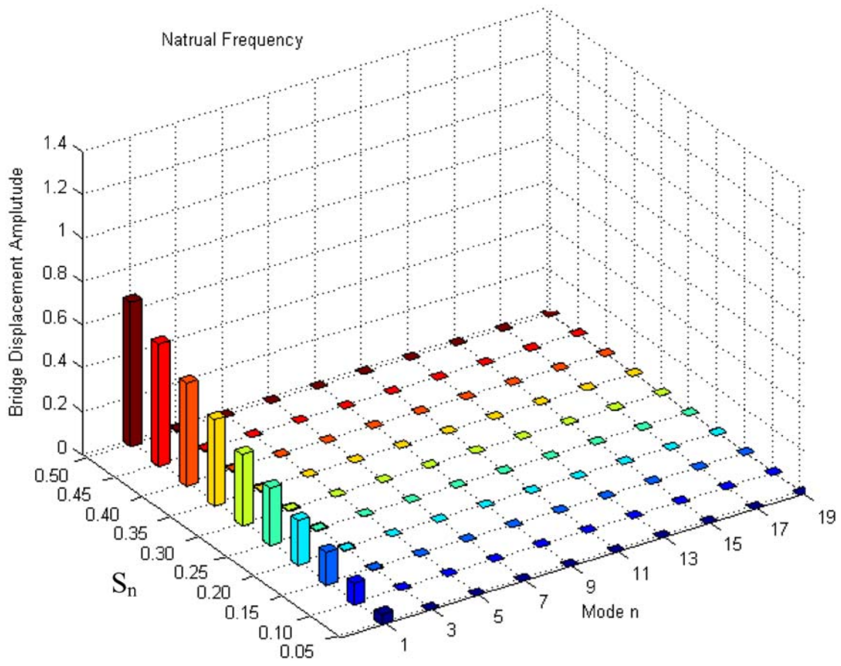

Fig. 4. Bridge displacement amplitude for natural frequency component.

3.1. Bridge response to a single moving vehicle

Fig. 5 shows the midpoint acceleration response of a bridge to a vehicle of $m_v = 1200\,\text{kg}$ and $k_v = 500{,}000\,\text{N/m}$ moving at three different speeds, 5, 10 and 20 m/s. The following properties are adopted for the bridge: $L = 25\,\text{m}$, $\bar{m} = 4800\,\text{kg/m}$ and $EI = 3.3 \times 10^9\,\text{N}\,\text{m}^2$. As was mentioned above, the displacement response of a bridge is predominated by the driving frequency component, and the velocity response is equally governed by the driving frequency and bridge frequency components, but the acceleration response is predominated by the bridge frequency component. Such a fact is confirmed by the spectrum plotted in Figs. 6–8 for the displacement, velocity and acceleration of the midpoint displacement of the bridge. Clearly, the influence of the driving frequency component relative to the bridge frequency component *decreases* as we go from displacement, to velocity, and then to acceleration.

It is interesting to note that the three driving frequencies ($\pi v/L$) identified from Figs. 6–8 are 0.1, 0.2 and 0.4 Hz, which correspond exactly to the three speeds 5, 10, 20 m of the moving vehicle. The three driving frequencies all fall below 1 Hz, much less than the bridge frequency. Besides, higher response amplitudes are generated by vehicles moving at higher speeds, as indicated by the velocity and acceleration spectrum in Figs. 7 and 8, respectively. One implication herein is that if the dynamic response of a bridge under moving loads can be accurately monitored, the recorded data, when processed in real time, can be used for detecting the moving speeds of vehicles over the bridge, at least from the theoretical point of view. The feasibility of such an idea may be of interest to the traffic patrolpersons for chasing speeding vehicles. It is suggested that further research be conducted in this regard concerning the technical aspects.

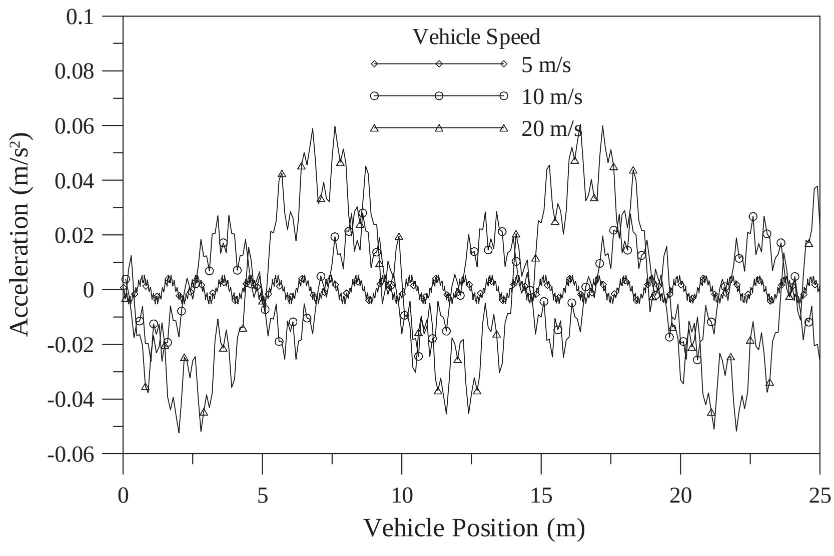

Fig. 5. Bridge midspan acceleration response.

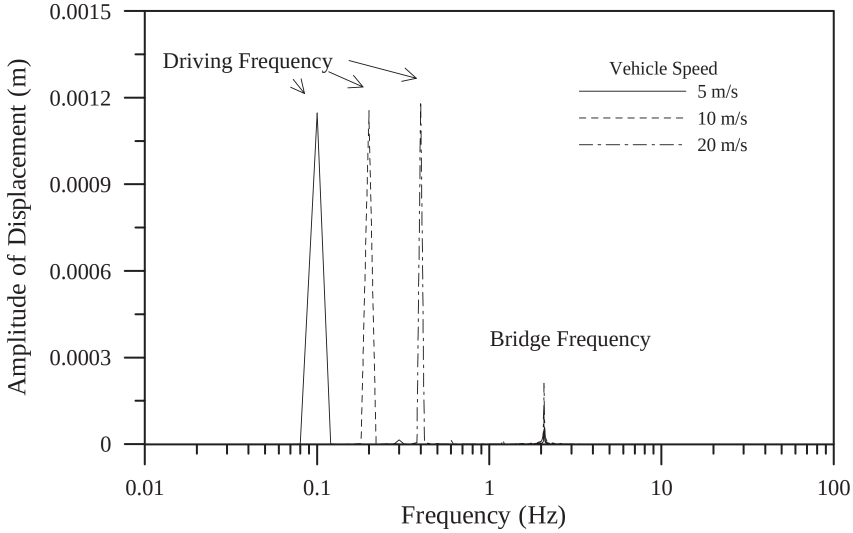

Fig. 6. Spectrum of bridge displacement response.

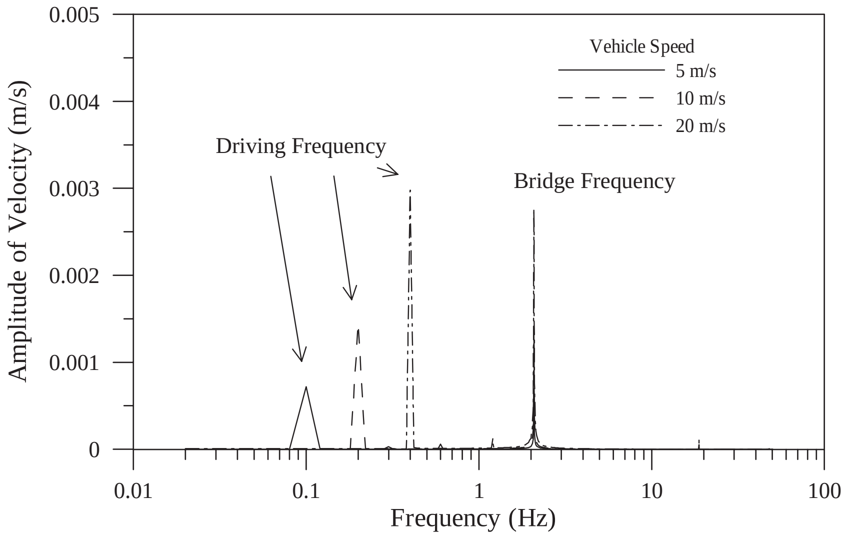

Fig. 7. Spectrum of bridge velocity response.

3.2. *Bridge response to five moving vehicles*

The same properties as those used in Section 3.1 are adopted for the bridge and vehicle. In order to examine the effect of multi-vehicle loadings, however, we shall consider an arbitrary case of five identical vehicles moving over the bridge at different speeds, i.e., 15, 8, 5, 12, 10 m/s, and at different entrance times. The solution to such a multi-vehicle case can be obtained by superposing

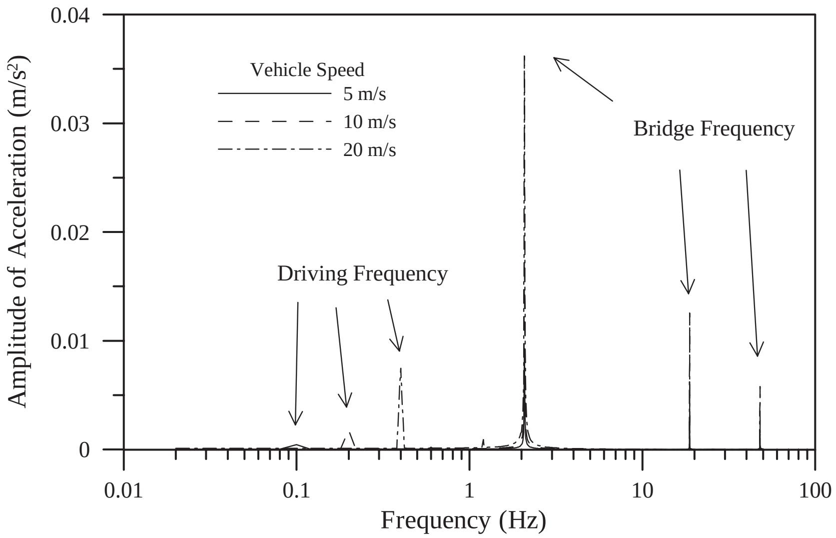

Fig. 8. Spectrum of bridge acceleration response.

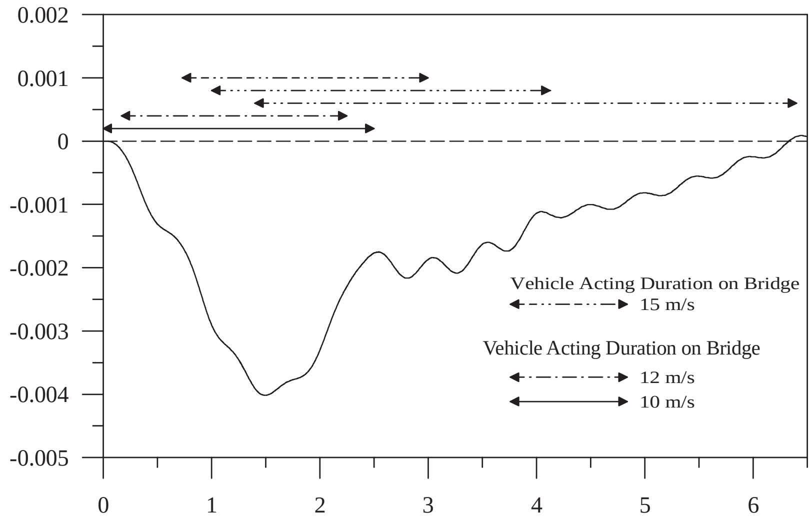

Fig. 9. Bridge midspan displacement response (five moving vehicles).

the results obtained for each individual vehicle, with due account taken of the effect of time lag and different initial conditions [8]. The displacement response obtained for the midpoint of the bridge has been plotted in Fig. 9, along with the acting duration of each vehicle on the bridge indicated.

The spectrum of the displacement, velocity and acceleration of the midpoint of the bridge has been shown in Figs. 10–12. From these figures, it is observed that the five driving frequencies

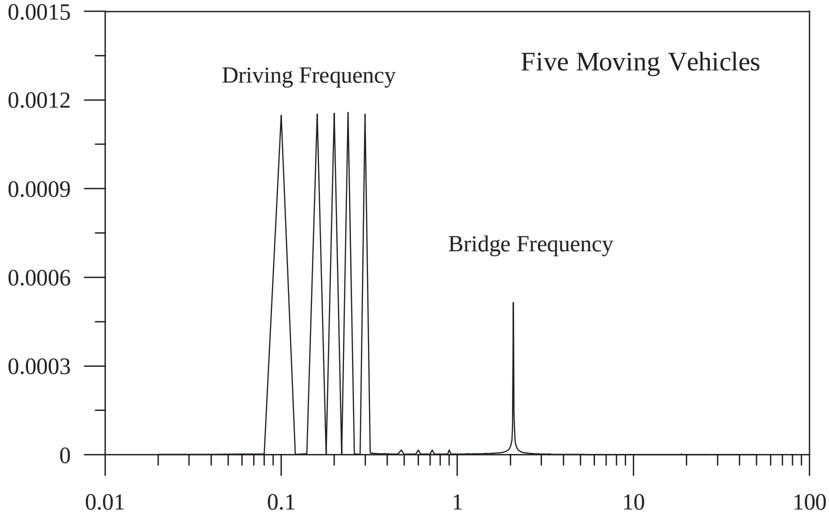

Fig. 10. Spectrum of bridge displacement response (five moving vehicles).

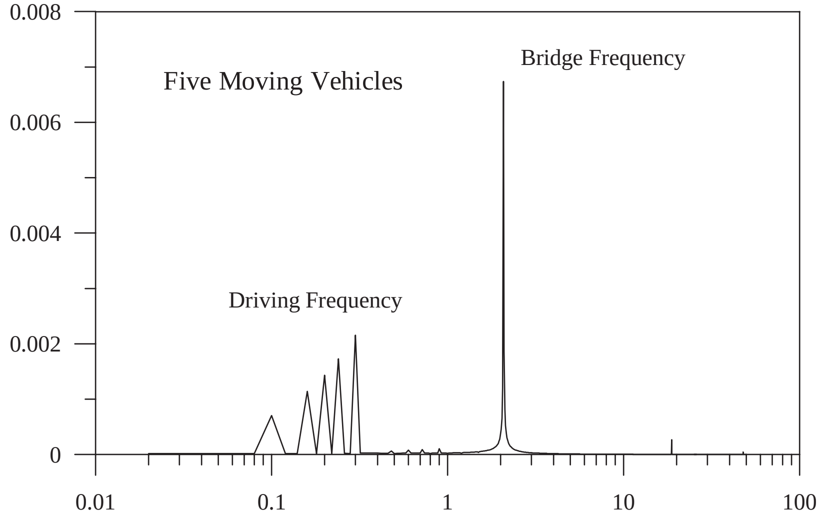

Fig. 11. Spectrum of bridge velocity response (five moving vehicles).

implied by the five vehicle speeds can be clearly identified from the displacement spectrum, not so clearly from the velocity spectrum, but almost vanish in the acceleration spectrum. Such an observation is consistent with the statement that the driving frequency component loses its influence as we go from displacement, to velocity and then to acceleration of the bridge. Therefore, if the speed of a moving vehicle is to be *detected* from the dynamic response of the supporting bridge, it is suggested that we work on the displacement or velocity response. In this regard, further works should be conducted to develop techniques that are suitable for monitoring the displacement or velocity, rather than acceleration response, of a bridge in the low-frequency

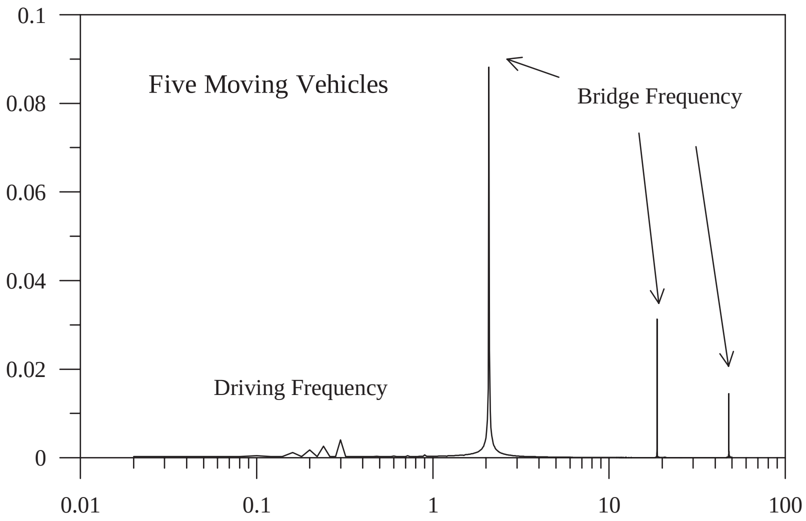

Fig. 12. Spectrum of bridge acceleration response (five moving vehicles).

range, as most vibration sensors available nowadays are of the acceleration or velocity type and are good for high-frequency vibrations.

4. Dynamic response of the vehicle

Substituting the expression in Eq. (14) for the bridge displacement $q_{bn}(t)$ into the right-hand side of Eq. (11), the equation of motion for the vehicle can be written as

$$\ddot{q}_v + \omega_v^2 q_v = \underbrace{\sum_{n=1}^{\infty} \frac{\Delta_{\mathrm{st}n}\omega_v^2}{1 - S_n^2} \left\{ \sin\left(\frac{n\pi vt}{L}\right) \left[\sin\left(\frac{n\pi vt}{L}\right) - S_n \sin(\omega_{bn} t) \right] \right\}}_{g(t)}. \tag{23}$$

The term denoted as $g(t)$ on the right-hand side above represents the interaction effect between the bridge and moving vehicle, which is a function of time. The solution to Eq. (23) can be obtained by the Duhamel integral as

$$q_v(t) = \frac{1}{\omega_v} \int_0^t g(\tau) \sin \omega_v(t - \tau) \mathrm{d}\tau. \tag{24}$$

Using the expression for $g(t)$ in Eq. (23), the displacement response of the vehicle can be integrated as

$$\begin{aligned} q_v(t) = \sum_{n=1}^{\infty} \bigg[& A_{1n} \cos\left(\frac{(n-1)\pi v}{L} t\right) + A_{2n} \cos\left(\frac{(n+1)\pi v}{L} t\right) + A_{3n} \cos(\omega_v t) \\ & + A_{4n} \cos\left(\omega_{bn} - \frac{n\pi v}{L}\right) t + A_{5n} \left(\omega_{bn} + \frac{n\pi v}{L}\right) t \bigg], \end{aligned} \tag{25}$$

where the coefficients in the brackets are as follows

$$A_{1n} = \frac{\Delta_{\mathrm{st}n}\,\omega_v^2}{2(1-S_n^2)\big(\omega_v + (n-1)\pi v/L\big)\big(\omega_v - (n-1)\pi v/L\big)}, \tag{26}$$

$$A_{2n} = \frac{-\Delta_{\mathrm{st}n}\,\omega_v^2}{2(1-S_n^2)\big(\omega_v + (n+1)\pi v/L\big)\big(\omega_v - (n+1)\pi v/L\big)}, \tag{27}$$

$$\begin{aligned} A_{3n} = {} & \frac{2\Delta_{\mathrm{st}n}\,\omega_v^2\left(\frac{\pi v}{L}\right)^2 n}{\left(1-S_n^2\right)\left(\omega_v + (n-1)\pi v/L\right)\left(\omega_v - (n-1)\pi v/L\right)\left(\omega_v + (n+1)\pi v/L\right)\left(\omega_v - (n+1)\pi v/L\right)} \\ & - \frac{2\Delta_{\mathrm{st}n}\,S_n\omega_v^2\big(n\pi v/L\big)\omega_{bn}}{\left(\omega_v - \omega_{bn} + n\pi v/L\right)\left(\omega_v + \omega_{bn} - n\pi v/L\right)\left(\omega_v + \omega_{bn} + n\pi v/L\right)\left(\omega_v - \omega_{bn} - n\pi v/L\right)}, \end{aligned} \tag{28}$$

$$A_{4n} = \frac{-S_n\,\Delta_{\mathrm{st}n}\,\omega_v^2}{2(1-S_n^2)\left(\omega_v - \omega_{bn} + n\pi v/L\right)\left(\omega_v + \omega_{bn} - n\pi v/L\right)}, \tag{29}$$

$$A_{5n} = \frac{S_n\,\Delta_{\mathrm{st}n}\,\omega_v^2}{2(1-S_n^2)\left(\omega_v + \omega_{bn} + n\pi v/L\right)\left(\omega_v - \omega_{bn} - n\pi v/L\right)}. \tag{30}$$

The solution as derived here for the response of the moving vehicle is *approximate* in the sense that the bridge displacement solved from Eq. (13) has been directly used in computing the acting force for the vehicle equation in Eq. (23). In fact, the vehicle response computed from Eq. (23) may affect again the acting force on the bridge, and therefore the response of the bridge. Obviously, the interaction between the bridge and moving vehicle is an issue of iterative nature, but in this study, only the first cycle of iteration was considered. In the numerical study, it will be demonstrated that even with such an approximation, the closed-form solutions derived appear to be quite accurate compared with the finite element analysis that takes into account the full effect of interaction. Compared with the numerical approaches, the present approximate approach has the advantage that the *key parameters* involved in each phenomenon can be clearly identified, while clear *physical meanings* can be easily appreciated. It should be added that the solution presented in Eq. (25) reduces to that previously derived [27], if only the first mode of vibration is considered.

As can be seen from Eq. (25), the number of terms involved in the vehicle response increases as the number of vibration modes considered for the bridge increases. There are five types of frequencies involved in the vehicle response, which can further be categorized into three groups as: (1) driving frequencies, including $(n-1)\pi v/L$ and $(n+1)\pi v/L$; (2) vehicle frequency ω_v; and (3) bridge-related frequencies, $\omega_{bn} - n\pi v/L$ and $\omega_{bn} + n\pi v/L$, where the index n relates to the vibration mode number of the bridge. In particular, the third group represents the fact that the bridge frequencies ω_{bn} are shifted by an amount equal to the vehicle speed $\pm n\pi v/L$.

In this study, we are particularly interested in the terms containing the bridge frequencies with shift, i.e., $\omega_{bn} - n\pi v/L$ and $\omega_{bn} + n\pi v/L$, in Eq. (25), as they provide the clues for *extracting the bridge frequencies* from the dynamic response of a passing vehicle, at least from the theoretical point of view [27]. Before this can be realized, however, we like to have some idea about the relative influence of the terms containing $\omega_{bn} - n\pi v/L$ and $\omega_{bn} + n\pi v/L$ in the vehicle response of Eq. (25) with respect to the remaining terms, and particularly to see if they are *practically visible* from the response spectrum. To this end, we shall use μ_n to denote the ratio of the vehicle frequency ω_v to the bridge frequency ω_{bn} of the nth mode, $\mu_n = \omega_v/\omega_{bn}$, and rewrite the coefficients in Eqs. (29) and (30) accordingly,

$$A_{4n} = \frac{-\Delta_{\mathrm{st}n} S_n}{2(1-S_n^2)\left[1-\mu_n^2(1-S_n)^2\right]}, \tag{31}$$

$$A_{5n} = \frac{\Delta_{\mathrm{st}n} S_n}{2(1-S_n^2)\left[1-\mu_n^2(1+S_n)^2\right]}. \tag{32}$$

In Table 2, the magnitudes of all the coefficients $A_{1n}, A_{2n}, \ldots, A_{5n}$ have been listed. It can be observed that the coefficient A_{5n}, i.e., the term associated with the frequency $\omega_{bn} + n\pi v/L$, remains generally the largest for the practical ranges of μ and S considered, i.e., for $0 \leqslant \mu \leqslant 5$ and for $0 \leqslant S \leqslant 0.5$. Such a fact offers us a theoretical means to *extract* the bridge frequency from the vehicle response, if due account is taken of the shifting effect. Furthermore, the magnitude for the second mode is about one quarter of that of the first mode, and even smaller for higher modes, indicating that modes higher than the first one can generally be neglected in computing the vehicle response, as will be demonstrated later on. By setting the denominator for the coefficient A_{5n} equal to 0 for $n=1$, one can obtain the condition for the *resonance* to occur, as indicated by the vehicle–bridge frequency ratio μ, which turns out to be equal to 1 for small S, but slightly less than 1 for large S. Under the condition of resonance, the coefficient A_{5n} reaches a local maximum. The

Table 2
Amplitude of vehicle acceleration response

Frequency	Amplitude	First term ($n=1$)	Second term ($n=2$)
$\frac{(n-1)\pi v}{L}$	$\frac{\Delta_{\mathrm{st}n}}{2(1-S_n^2)} \frac{\left((n-1)\pi v/L\right)^2}{1-\left[(n-1)\mu_n S_1\right]^2}$	1	$\frac{1}{1-(\mu_1 S_1)^2}$
$\frac{(n+1)\pi v}{L}$	$\frac{\Delta_{\mathrm{st}n}}{2(1-S_n^2)} \frac{\left((n+1)\pi v/L\right)^2}{1-\left[(n+1)\mu_n S_1\right]^2}$	$\frac{1}{1-(2\mu_1 S_1)^2}$	$\frac{1}{1-(3\mu_1 S_1)^2}$
$\omega_{bn} - \frac{n\pi v}{L}$	$\frac{\Delta_{\mathrm{st}n}}{2(1-S_n^2)} \frac{S_1\mu_n^2\left(1-S_1/n^2\right)^2}{n^2\left(1-n^2\mu_n\left(1-S_1/n^2\right)^2\right)}$	$\frac{S_1\mu_1^2(1-S_1^2)}{1-\mu_1^2(1-S_1)^2}$	$\frac{S_1\mu_1^2\left(1-S_1/4\right)^2}{4\left(1-4\mu_1^2\left(1-S_1/4\right)^2\right)}$
$\omega_{bn} + \frac{n\pi v}{L}$	$\frac{\Delta_{\mathrm{st}n}}{2(1-S_n^2)} \frac{S_1\mu_n^2\left(1+S_1/n^2\right)^2}{n^2\left(1-n^2\mu_n\left(1+S_1/n^2\right)^2\right)}$	$\frac{S_1\mu_1^2(1+S_1)^2}{1-\mu_1^2(1+S_1)^2}$	$\frac{S_1\mu_1^2\left(1+S_1/4\right)^2}{4\left(1-4\mu_1^2\left(1+S_1/4\right)^2\right)}$

same is also true for the coefficient A_{4n}, except that the vehicle–bridge frequency ratio μ is equal to 1 for small S, but slightly greater than 1 for large S.

5. Numerical verification

In order to evaluate the effect of approximation made in deriving the solution to Eqs. (10) and (11), i.e., by assuming the mass of the vehicle to be small compared with that of the bridge, an independent finite element analysis [27] that does not rely on any such assumption will be conducted. The following properties are adopted for the bridge: cross-sectional area $A = 2.0\,\mathrm{m}^2$, moment of inertia $I = 0.12\,\mathrm{m}^4$, elastic modulus $E = 2.75 \times 10^{10}\,\mathrm{N/m}^2$, length $L = 25\,\mathrm{m}$, and mass per-unit-length $\bar{m} = 4800\,\mathrm{kg/m}$. The vehicle is assumed to have a mass $m_v = 1200\,\mathrm{kg}$ and a spring constant $k_v = 500{,}000\,\mathrm{N/m}$, which implies a vibration frequency of $\omega_v = 20.41\,\mathrm{rad/s}$ ($= 3.25\,\mathrm{Hz}$). And the beam is divided into 20 beam elements. The first three eigenvalues ω_b computed for the bridge are: $13.09\,\mathrm{rad/s}$ ($= 2.08\,\mathrm{Hz}$), $52.36\,\mathrm{rad/s}$ ($= 8.33\,\mathrm{Hz}$), $117.81\,\mathrm{rad/s}$ ($= 18.75\,\mathrm{Hz}$).

The time-history responses computed for the displacement, velocity and acceleration of the vehicle passing through the bridge at speed 10 m/s considering either a single mode or two modes have been plotted in Figs. 13–15, together with the solutions obtained from the finite element analysis. The first observation is that the solution obtained by considering only the first mode is very close to that obtained by considering two modes, indicating that the single-mode approach is reliable for simulating the vehicle response, and that the high modes can generally be neglected. The discrepancy between the present analytical solutions and the finite element solutions is owing to the fact that iteration was not performed in this study for updating the interaction forces existing between the moving vehicle and bridge, in addition to the assumption of a small mass

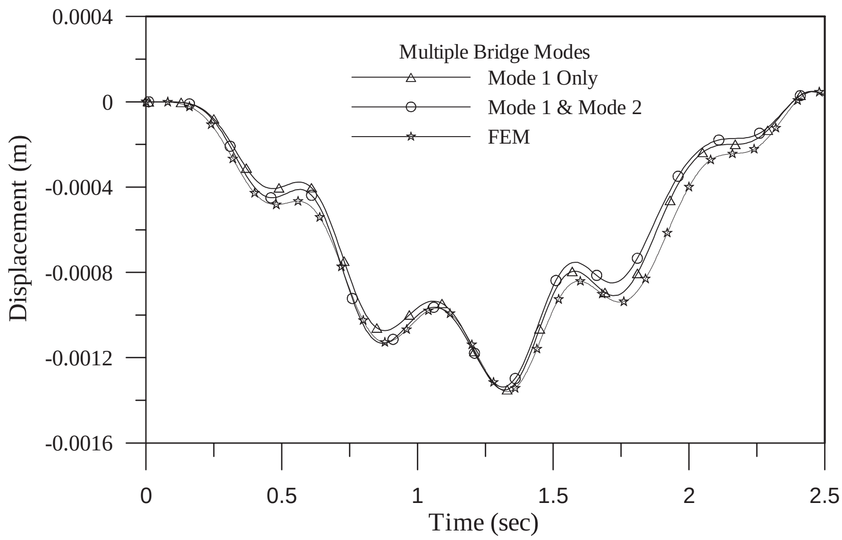

Fig. 13. Displacement response of vehicle.

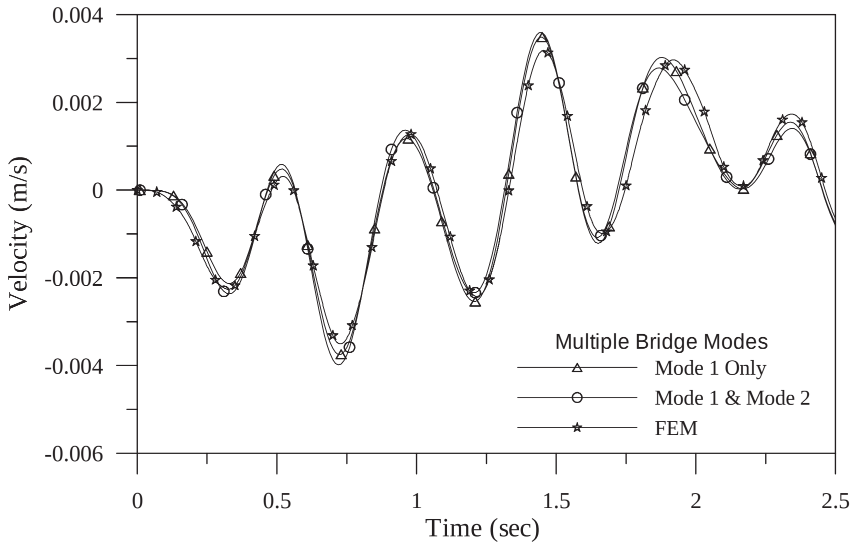

Fig. 14. Velocity response of vehicle.

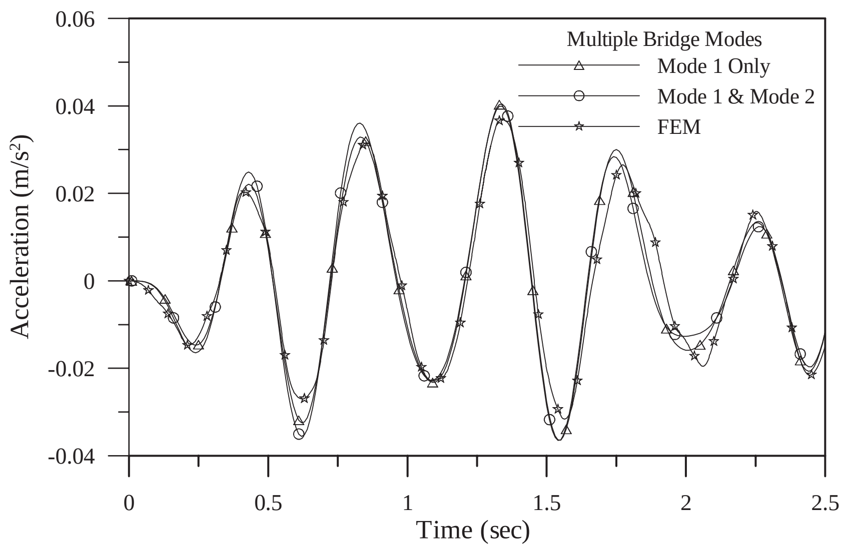

Fig. 15. Acceleration response of vehicle.

ratio for the vehicle. Such a discrepancy is generally negligible when observed from the frequency domain, as will be discussed below.

The Fourier transforms of the acceleration response of the vehicle obtained by the present analytical approach and the finite element method have been plotted in Figs. 16 and 17, respectively, in which several peaks can be clearly seen. An observation is that the frequencies

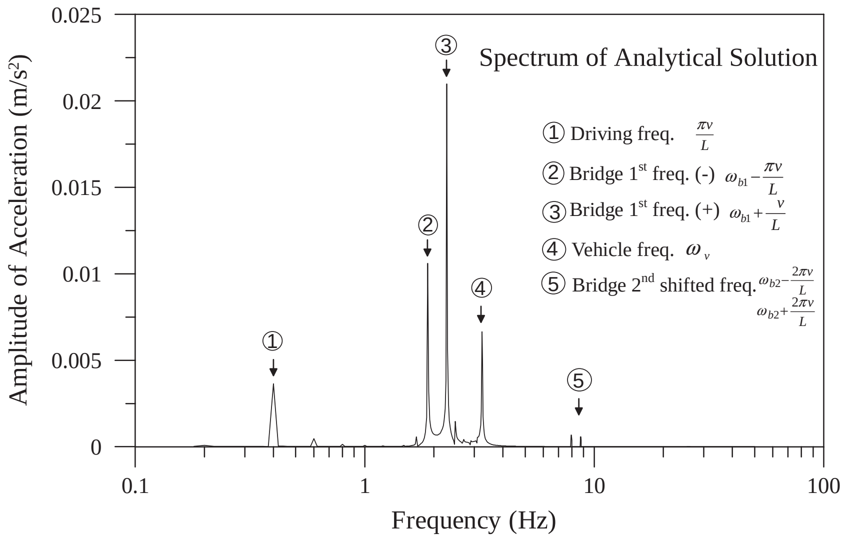

Fig. 16. Spectrum of vehicle acceleration response (analytical).

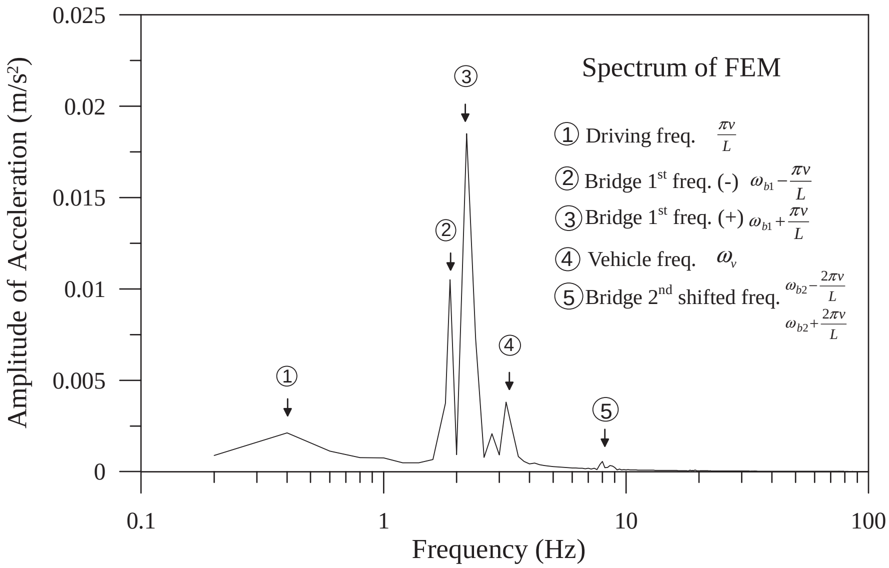

Fig. 17. Spectrum of vehicle acceleration response (FEM).

corresponding to all the peaks in the two figures appear to be in good agreement, indicating that the use of the single-mode approach can yield sufficiently accurate results, particularly when the frequency distribution is of concern.

In each figure, the first peak (counted from the left hand side) relates to the driving frequency, $\pi v/L = 0.2\,\text{Hz}$. The frequencies associated with the second and third peaks relate to the

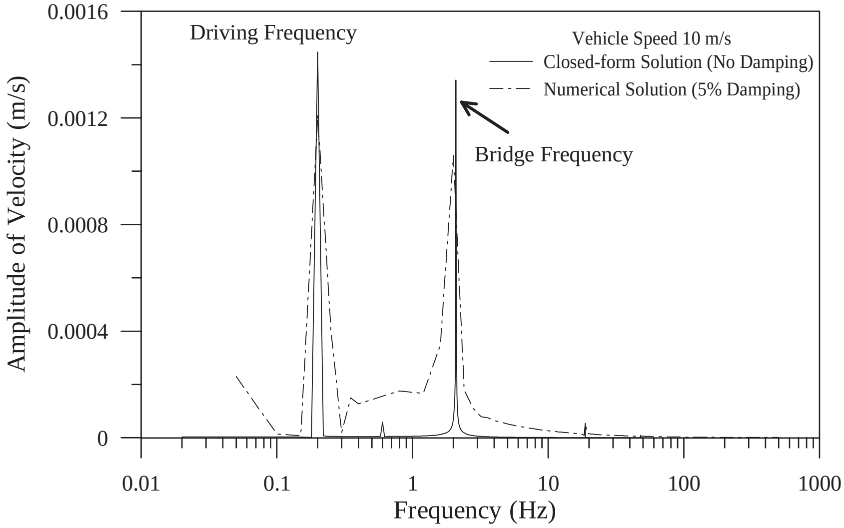

Fig. 18. Effect of bridge damping on bridge velocity spectrum.

fundamental frequency of the bridge with shift, i.e., $\omega_{b1} - \pi v/L = 1.88\,\mathrm{Hz}$ and $\omega_{b1} + \pi v/L = 2.28\,\mathrm{Hz}$. By removing the shifting effect, the fundamental frequency of the bridge can be recovered as $\omega_{b1} = 2.08\,\mathrm{Hz}$. The frequency identified from the figures for the fourth peak is the vehicle frequency, i.e., $\omega_v = 3.25\,\mathrm{Hz}$. Of interest are the two small peaks on the right-hand side of the figures, which are associated with the second frequency of the bridge with shift.

To evaluate the effect of damping, we assume the bridge damping to be of the Rayleigh type and assign a 5% damping ratio to each of the first two modes of the bridge. The midpoint velocity spectrum of the bridge solved by the finite element method has been compared with the closed-form solution for the zero damping case in Fig. 18. As can be seen, the amplitude of the bridge frequency decreases slightly due to the existence of damping, while basically no change can be observed on the magnitude of the vehicle frequency. It is concluded that both the driving and bridge frequencies can be clearly identified regardless of the presence of damping on the bridge.

As for the vehicle damping, a 10% damping ratio is assigned to the vehicle. Fig. 19 shows the vehicle acceleration spectrum computed for this case by the finite element method, along with the closed-form solution obtained for the case of zero vehicle damping. Although the existence of vehicle damping has caused a significant drop in the vehicle frequency component, its influence on the bridge frequencies is generally small. Clearly, the bridge frequency can be extracted with no difficulty from the vehicle response, notwithstanding the existence of the vehicle damping.

It should be noted that all the above results have been based on the assumption of smooth pavement surface for the bridge. Should this factor be taken into account, it is likely that some new peaks will be generated by irregularities in the pavement. Under such a situation, it is suspected that the third peak associated with the fundamental frequency, $\omega_{b1} + \pi v/L = 2.28\,\mathrm{Hz}$, can still be observed. Before closing this section, we like to say that further research is required in this regard, since it presents lots of advantages if the *bridge* frequency can be extracted indirectly from the acceleration response of a passing *vehicle*, compared with the conventional approaches of measuring the bridge frequencies directly from the bridge response.

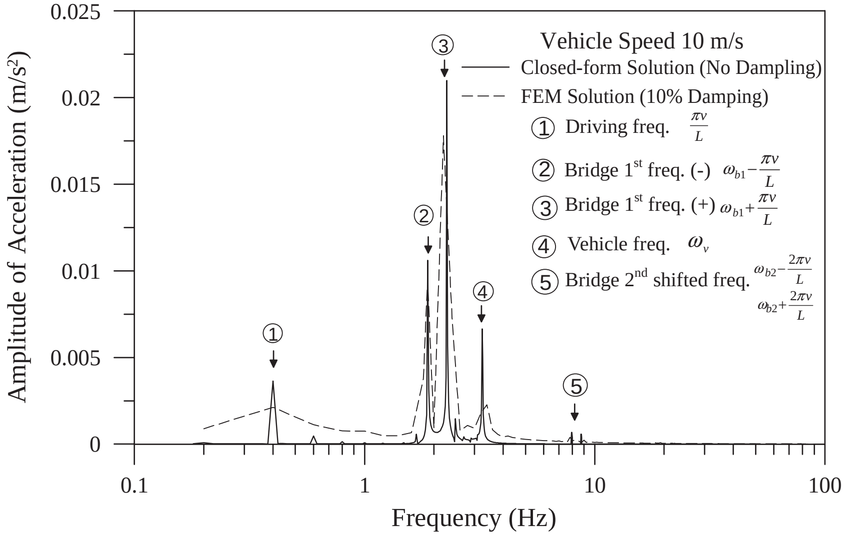

Fig. 19. Effect of vehicle damping on vehicle acceleration spectrum.

5. Concluding remarks

In this study, a modal superposition approach was adopted to derive the solutions for a vehicle–bridge system, with the vehicle simulated as a sprung mass and the bridge as a simply supported beam. Such an approach is approximate in that no iteration was conducting for updating the interaction force existing between the two subsystems. The solutions obtained, however, were shown to be quite accurate compared with those obtained by the finite element method. For both the bridge and vehicle responses, it is concluded that sufficiently accurate solutions can be obtained by considering only the first mode of vibration of the bridge. As for the bridge, the displacement response is predominated by the driving frequency component; the velocity is equally governed by the driving frequency and bridge frequency components; and the acceleration predominated by the bridge frequency component. There exists a possibility, at least theoretically, that the vehicle speed can be identified from the displacement or velocity spectrum of the bridge as it appears as a low-frequency peak. Further investigation is required in this regard. As for the vehicle response, there are three groups of frequencies involved, i.e., the driving frequency, vehicle frequency, and bridge frequency with shift. For the case where zero damping and smooth pavement surface are assumed for the bridge, all the three groups of frequencies can be identified from the vehicle response. Of interest is the fact that the shifted fundamental frequency of the bridge appears as the highest peak in the spectrum. If the effects of damping and irregular pavement are taken into account, it is suspected that the fundamental frequency of the bridge can still be identified from the vehicle response. Both further theoretical and experimental studies are required in this regard.

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