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# Structural modification. Part 2: assignment of natural frequencies and antiresonances by an added beam

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**Abstract**

In this paper the inverse structural modification problem is solved in order to determine the dimensions of the cross-section of a beam that when added to an original structure will assign natural frequencies or antiresonances as specified. In order for this to be accomplished rotational receptances must be measured as presented in the companion paper. When the modification is cast as an additional forcing term on the original structure a system of multivariate polynomials in the parameters of the beam cross-section are revealed. The solution of this system yields the beam parameters that assign the specified vibration behaviour.


## 1. Introduction

Structural modification is about the effect of mass, stiffness and damping changes on the dynamic behaviour of mechanical systems. The direct structural modification problem is concerned with determining the changes in eigenvalues and eigenvectors brought about by a designated physical modification. Conversely, the inverse structural modification problem occurs when changes in mass, stiffness and damping are sought in order that the modified system shall possess certain desirable dynamic characteristics usually in the form of natural frequencies, mode shapes or antiresonances.

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The Rayleigh quotient may be thought of as the first seed of direct dynamic modification, though according to Temple and Bickley [1] the stationary property of the Rayleigh quotient was known to Lagrange. Rayleigh showed that the smallest natural frequency is the global minimum and the largest natural frequency the global maximum of the quotient. A consequence of this minimal property is that any stiffness increase or mass decrease will generally result in an increase the system natural frequencies, except when a mass or stiffness is added at a vibration node when there is no change in that particular natural frequency. This result may of course be found in Rayleigh's famous treatise "The Theory of Sound" [2].

Wittrick [3] concluded that any small change to an eigenvalue should be attributed to small parameter changes only and not to any small changes to the modes shape, because of the stationarity of the Rayleigh quotient. Fox and Kapoor [4] showed that expressions of both eigenvalue and eigenvector rate of change may be written in terms of only the corresponding unmodified eigenvector and eigenvalue. The importance of knowing the rates of change of eigenvectors and eigenvalues with respect to structural changes is that they can be used to obtain a first-order approximation of the actual modified eigenvalues and eigenvectors. Structural modification techniques based on the Rayleigh Quotient, and in general on techniques that rely on the estimation of rates of change of eigenvalues and eigenvectors with respect to structural parameters, are suitable only for infinitesimal modifications.

Weissenberg [5,6] treated lumped mass and stiffness modifications as a symmetric unit rank perturbation on the eigenvalue problem of the unmodified structure. For example, for a point mass modification this perturbation is given by

$$(\mathbf{K} - \bar{\omega}_i^2\mathbf{M} - \bar{\omega}_i^2\delta m\mathbf{u}\mathbf{u}^{\mathrm{T}})\bar{\varphi}_i = 0, \tag{1}$$

where $\delta m$ is the mass modification and $\mathbf{u}$ a unit vector that indicates the position of the structural modification. He obtained the expressions

$$\frac{1}{\delta m} = \sum_{j=1}^{n} \frac{\omega^2\eta_j^2}{\omega_j^2 - \omega^2}, \tag{2}$$

$$z_{ji} = \beta_i \frac{\eta_j}{\omega_j^2 - \bar{\omega}_i^2}, \tag{3}$$

where $\omega_j$ and $\bar{\omega}_i$ are, respectively, the $j$th natural frequency of the original system and the $i$th natural frequency of the modified system. The latter is one of several frequencies $\omega = \bar{\omega}_i$ that satisfies Eq. (2). $z_{ji}$ the $j$th component of $\mathbf{z}_i = \mathbf{M}\boldsymbol{\Phi}^{\mathrm{T}}\bar{\varphi}_i$ where $\boldsymbol{\Phi}$ is the modal matrix having the eigenvectors of the original system in its columns. $\mathbf{M}$ is the mass matrix and $\bar{\varphi}_i$ is the $i$th eigenvector of the modified system. $\eta_j$ is the $j$th component of the vector $\eta = \boldsymbol{\Phi}\mathbf{u}$ and $\beta_i$ is a scaling constant. Pomazal and Snyder [7] extended this methodology to the case of damped structures.

One may see a duality between the direct and inverse modification problem from Eq. (2). In the direct problem, the designated modification $\delta m$ is obviously independent of frequency $\omega$ whereas the right-hand side of Eq. (2) is a function of $\omega$, say $g(\omega)$. Therefore from a plot of $g(\omega)$ versus $\omega$ and for a given $\delta m$ the modified natural frequencies may be read directly from the intersections of the horizontal straight line $1/\delta m$ with $g(\omega)$. In the case of inverse problem, by setting a target natural frequency $\bar{\omega}_i$, the value at the intersection of the vertical line $\bar{\omega}_i$ and $g(\omega)$ is the desired

$1/\delta m$. In the case of non-intersection in the range of practically feasible modifications it must be concluded that the assignment of the desired natural frequency $\bar{\omega}_i$ cannot be achieved.

Ram and Braun [8] considered the problem of assigning mode shapes. The modified eigenvectors were constrained to the linear span of the measured original ones. Bucher and Braun [9] used left eigenvectors to prove that under certain conditions on the modification values $\delta m$ and $\delta k$ the modified modal vectors can be expressed as the superposition of the unmodified modal vectors. Mottershead et al. [10] assigned the nodes of the mode shapes.

Antiresonances, or zeros, are as important as the resonances since they are the frequencies at different steady-state response locations at which the vibration disappears to zero, or to low levels when damping is present. It is therefore useful to be able to extend the methodologies of structural modification to the case of assigning antiresonances at certain locations as well be able to determine the effect of structural changes on the zeros. Mottershead [11] demonstrated that the sensitivities of the zeros are a linear superposition of the sensitivities of eigenvalues and eigenvectors. Mottershead and Lallement [12] used this knowledge together with the theory of unit rank modification [5,6] to assign natural frequencies and antiresonances at the same values and thereby create vibration nodes. Mathematically, the anti-resonance frequencies are the eigenvalues of the adjoint system; the system obtained by deleting a row and a column from the original dynamic stiffness matrix. Mottershead [13] applied force constraints at the modification coordinates in order to derive the frequency response function matrix of the adjoint system from the original frequency response matrix. He then used the adjoint frequency response function matrix to assign zeros at predetermined frequency values.

This paper treats the inverse problem of modifying an original structure by adding a beam to it. Such a modification yields couplings at more than a single coordinate and as a consequence the modification equations take the form of multivariate polynomials. An extra difficulty arises because the introduction of a beam requires that rotational receptances are measured at the modification coordinates, the subject matter of the companion paper [14].

## 2. Modification theory

The effect of structural modifications on the system receptances may be determined when the receptances of the original system at the modification coordinates are measured. The theory is well developed and involves compatibility conditions on forces and displacements at the modification coordinates as explained for example in Refs. [15–17]. These methodologies are intuitively appealing since they involve the application of mechanical vibration principles similar to electrical engineering network concepts, but are cumbersome to compute and cast in a simple form for fairly large structures. However, by treating the modification as a forcing term on the unmodified structure, as for example in Refs. [18,19], a succinct formulation amenable to direct implementation is obtained.

Assume that the dynamic stiffness of the original structure is given by $\mathbf{B}(\omega)$ to be modified by structural components of dynamic stiffness $\mathbf{\Delta B}(\omega)$. This modification can be viewed as a perturbation $\mathbf{B}(\omega) + \mathbf{\Delta B}(\omega)$ on $\mathbf{B}(\omega)$ by $\mathbf{\Delta B}(\omega)$. The frequency domain equation of the combined system can now be written as

$$(\mathbf{B}(\omega) + \mathbf{\Delta B}(\omega))\mathbf{x}(\omega) = \mathbf{f}(\omega), \tag{4}$$

where $\mathbf{x}(\omega)$ is the displacement response vector due to the externally applied force vector $\mathbf{f}(\omega)$. Rearranging Eq. (4) is such a way as to treat $\mathbf{\Delta B}(\omega)\mathbf{x}(\omega)$ as an extra forcing on the unmodified structure the following equation is obtained:

$$\mathbf{B}(\omega)\mathbf{x}(\omega) = \mathbf{f}(\omega) - \mathbf{\Delta B}(\omega)\mathbf{x}(\omega). \tag{5}$$

Since the inverse of $\mathbf{B}(\omega)$ is the unmodified receptance matrix $\mathbf{H}(\omega)$, Eq. (5) can be rearranged in the form

$$\mathbf{x}(\omega) = (\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega))^{-1}\mathbf{H}(\omega)\mathbf{f}(\omega), \tag{6}$$

where $\mathbf{I}$ is the unit matrix. The term $(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega))^{-1}\mathbf{H}(\omega)$ is the matrix of *modified* receptances.

The matrix Eq. (6) may now be used to assign natural frequencies and antiresonances. The vanishing of $\mathbf{det}(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega))$ occurs at the newly created resonant frequencies and when the $pq$th term of the matrix product $\mathbf{adj}(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega))\mathbf{H}(\omega)$ vanishes an anti-resonance of the $pq$th receptance of the modified system is defined. Further discussion is presented in the following subsections. In the companion paper [14] Eq. (5) is manipulated in a different way to obtain the original receptance matrix $\mathbf{H}(\omega)$ that includes the sought after rotational receptances.

## 3. The experimental set-up and model of the added beam

The experimental arrangement is shown schematically in Fig. 1 and a more detailed technical drawing can be found in the companion paper [14]. The coordinate system used in the analysis and the important dimensions are shown. The unmodified structure, indicated by the full lines, is the portal frame with the leg on the right-hand side missing. The beam used to modify the original structure is the missing leg shown in dashed outline in Fig. 1. The height of the structure denoted by $L_1$ is 1 m and the overspan, $L_2$, is 0.5 m. The breadth, $b$, depth, $d$, and thickness, $t$, of the cross-sections are the same for all the beams and are 0.1, 0.05 and 0.004 m, respectively. In order to utilise the modification theory presented in Section 2 a model of the dynamic stiffness $\mathbf{\Delta B}(\omega)$ of

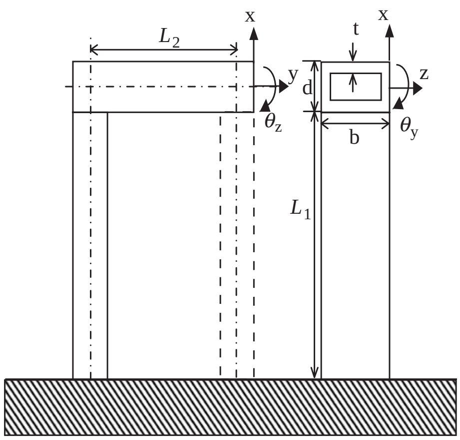


Fig. 1. Schematic diagram of the experimental rig.

the modification is required along with the measured receptance matrix $\mathbf{H}(\omega)$ at the connection point (the right-hand top corner) on the unmodified frame. In the analysis that follows only the equations in the $xy$-plane are given. The corresponding equations for the other two planes are of similar form and straightforward to derive.

The added beam is modelled by a single Euler–Bernoulli beam. After applying constraints at the built-in end its dynamic stiffness may be represented by

$$\mathbf{\Delta B}(\omega) = \begin{pmatrix} -\frac{1}{3}\,\omega^2\rho AL_1 + \frac{EA}{L_1} & 0 & 0 \\ 0 & -\frac{13}{35}\,\omega^2\rho AL_1 + 12\,\frac{EI_{zz}}{L_1^3} & -\frac{11}{210}\,\omega^2\rho AL_1 + 6\,\frac{EI_{zz}}{L_1^2} \\ 0 & -\frac{11}{210}\,\omega^2\rho AL_1 + 6\,\frac{EI_{zz}}{L_1^2} & -\frac{1}{105}\,\omega^2\rho AL_1 + 4\,\frac{EI_{zz}}{L_1} \end{pmatrix}, \tag{7}$$

where $A$ is the area, given by $A = bd - (b-2t)(d-2t)$, $I_{zz}$ the second moment of area about the $z$-axis, given by $I_{zz} = bd^3/12 - [(b-2t)(d-2t)^3/12]$, $E$ the Youngs' modulus and $\rho$ the density which are taken to be $210 \times 10^9\,\mathrm{N/m^2}$ and $7850\,\mathrm{kg/m^3}$, respectively, throughout.

With reference to the coordinates shown in Fig. 1 the unmodified frequency response matrix $\mathbf{H}(\omega)$ can be written as

$$\mathbf{H}(\omega) = \begin{pmatrix} h_{xx} & h_{xy} & h_{x\theta_z} \\ h_{yx} & h_{yy} & h_{y\theta_z} \\ h_{\theta_z x} & h_{\theta_z y} & h_{\theta_z\theta_z} \end{pmatrix}, \tag{8}$$

where $h_{pq}(\omega)$ is the steady-state displacement response at coordinate $p$ due to the application of a unit sinusoidal force of frequency $\omega$ at coordinate $q$

When the expressions for $A$ and $I_{zz}$ and Eqs. (7) and (8) are substituted into

$$\mathbf{det}(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega)) = 0, \tag{9}$$

the result is a multivariate polynomial in $b$, $d$, $t$ and $\omega$. The determinant is a function of the frequency response functions $h_{pq}$ which in turn are functions of the angular frequency $\omega$. The frequencies at which this polynomial becomes zero are the natural frequencies of the system modified by the added beam.

As described in Section 2 an antiresonance is assigned when a term of the matrix

$$\mathbf{adj}(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta B}(\omega))\mathbf{H}(\omega) \tag{10}$$

becomes vanishingly small. For example, the polynomial that assigns an antiresonance to $\bar{h}_{yy}$, where the overbar denotes the modified system, is given by

$$\begin{aligned} &\mathscr{A}(h_{yx} + h_{yx}h_{\theta_z y}\mathscr{C} + h_{yx}h_{\theta_z\theta_z}\mathscr{D} - h_{\theta_z x}h_{yy}\mathscr{C} - h_{\theta_z x}h_{y\theta_z}\mathscr{D})h_{xy} \\ &\quad + (1 + h_{\theta_z y}\mathscr{C} + h_{\theta_z\theta_z}\mathscr{D} + h_{xx}\mathscr{A} + h_{xx}\mathscr{A}h_{\theta_z y}\mathscr{C} \\ &\quad + h_{xx}\mathscr{A}h_{\theta_z\theta_z}\mathscr{D} - h_{\theta_z x}\mathscr{A}h_{xy}\mathscr{C} - h_{\theta_z x}\mathscr{A}h_{x\theta_z}\mathscr{D})h_{yy} \\ &\quad + (-h_{yy}\mathscr{C} - h_{y\theta_z}\mathscr{D} - h_{xx}\mathscr{A}h_{yy}\mathscr{C} - h_{xx}\mathscr{A}h_{y\theta_z}\mathscr{D} \\ &\quad + h_{yx}\mathscr{A}h_{xy}\mathscr{C} + h_{yx}\mathscr{A}h_{x\theta_z}\mathscr{D})h_{\theta_z y} = 0, \end{aligned} \tag{11}$$

where $\mathscr{A} = -\frac{1}{3}\omega^2\rho AL_1 + EA/L_1$, $\mathscr{B} = -\frac{13}{35}\omega^2\rho AL_1 + 12EI_{zz}/L_1^3$, $\mathscr{C} = -\frac{11}{210}\omega^2\rho AL_1 + 6EI_{zz}/L_1^2$ and $\mathscr{D} = -\frac{1}{105}\omega^2\rho AL_1 + 4EI_{zz}/L_1$.

## 4. Experimental procedure

The added right-hand leg of the portal frame was manufactured and receptances were acquired from experiments carried out with and without the added leg in place. The natural frequencies and antiresonances from the complete portal frame were then used as target values to be assigned using receptance data from the structure without the right-hand side leg. Application of the method described above was expected to result in the known dimensions of the added beam cross-section.

Fig. 2 shows the moduli of the translational receptances of the original system measured at the connection point, whereas Fig. 3 gives the moduli of the original-system rotational receptances obtained by using techniques described in the companion paper [14]. The natural frequencies of the original system, read from Fig. 2, are listed in Table 1 with the corresponding mode shapes depicted in Fig. 4. The first and third modes are entirely in the $x-y$ plane and the second mode involves mainly out of plane bending of the left-hand leg. The first mode at 20.85 Hz and the third at 62.90 Hz are present as clear sharp peaks on the receptance $h_{yy}(\omega)$. The second mode is very

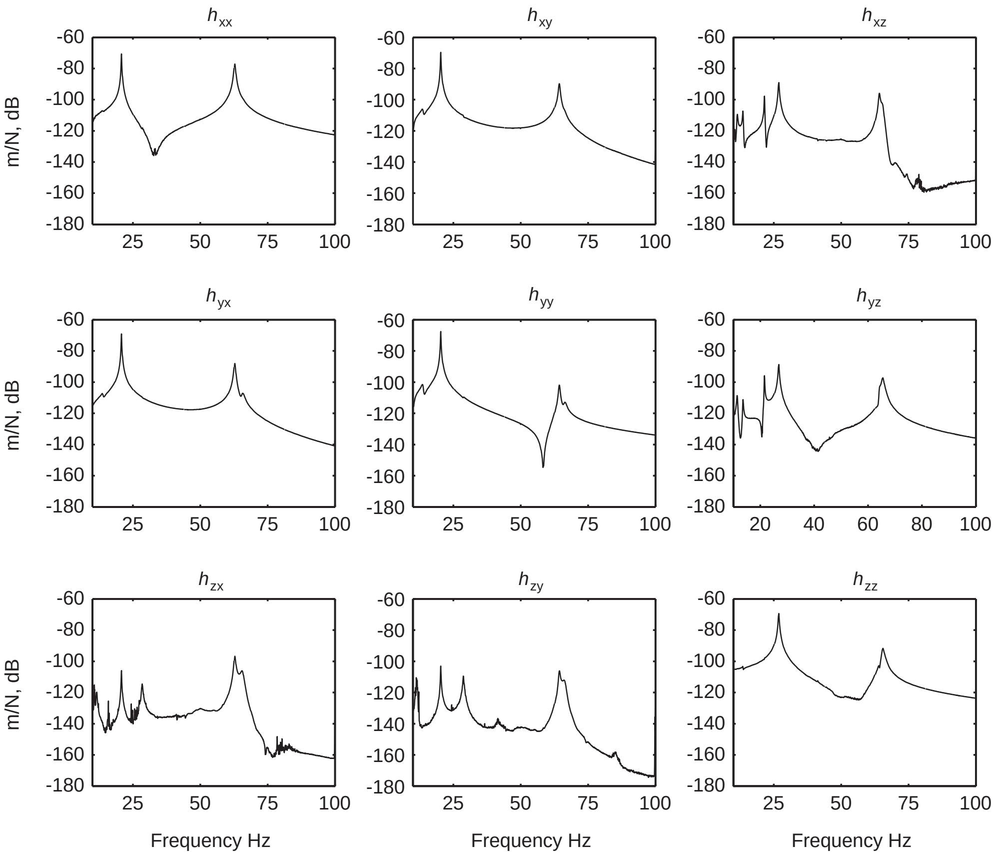


Fig. 2. Original translational receptances.

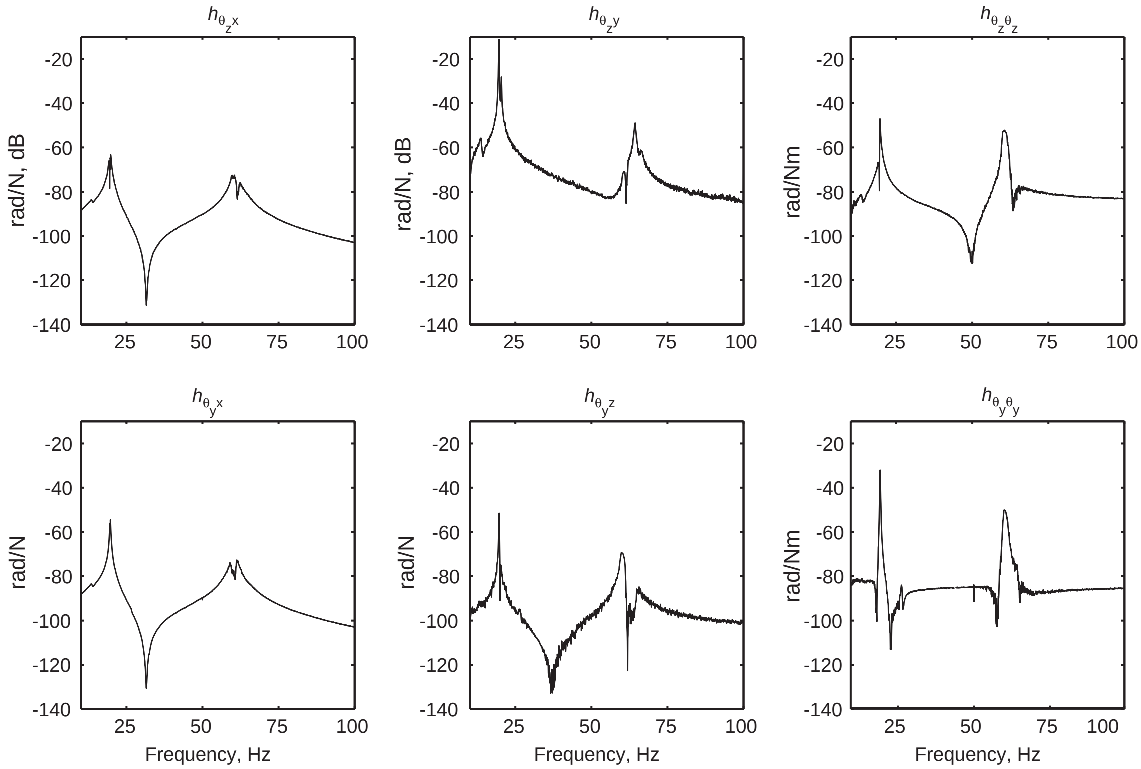


Fig. 3. Original rotational receptances.

Table 1
Original natural frequencies

| Mode | Natural frequencies, Hz |
|---|---|
| 1 | 20.85 |
| 2 | 26.85 |
| 3 | 62.90 |

clear in $h_{zz}(\omega)$ and the first mode appears clearly in the receptance $h_{\theta_z\theta_z}(\omega)$ determined from T-block measurements [14].

When the right-hand leg was added the receptances (moduli) shown in Fig. 5 were obtained at the connection point. The newly created natural frequencies, read from Fig. 5 and set as targets for the inverse problem, are listed in Table 2 and the mode shapes are shown in Fig. 6. The first mode shape is the in-plane bending mode of the two legs and the second mode is the out-of-plane twisting mode about $x$ with significant out-of-plane bending of the two legs. The first mode at 44.25 Hz is very clear in the plot of $\bar{h}_{yy}(\omega)$ and the second mode is prominent in $\bar{h}_{zz}(\omega)$. It is important to appreciate that the modification achieved by the added leg is very considerable; the

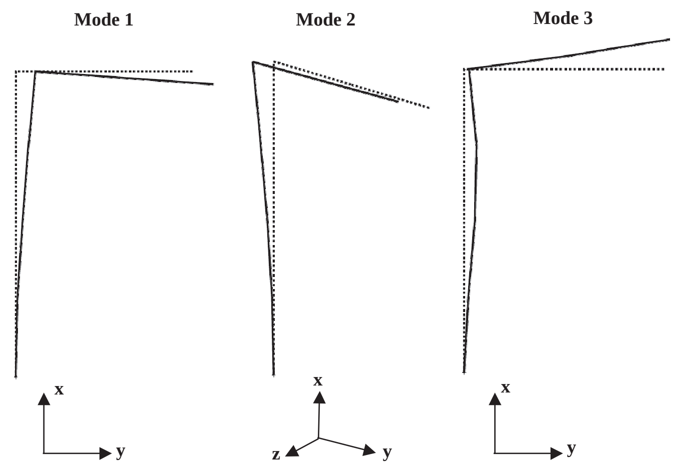


Fig. 4. Mode shapes of the original structure.

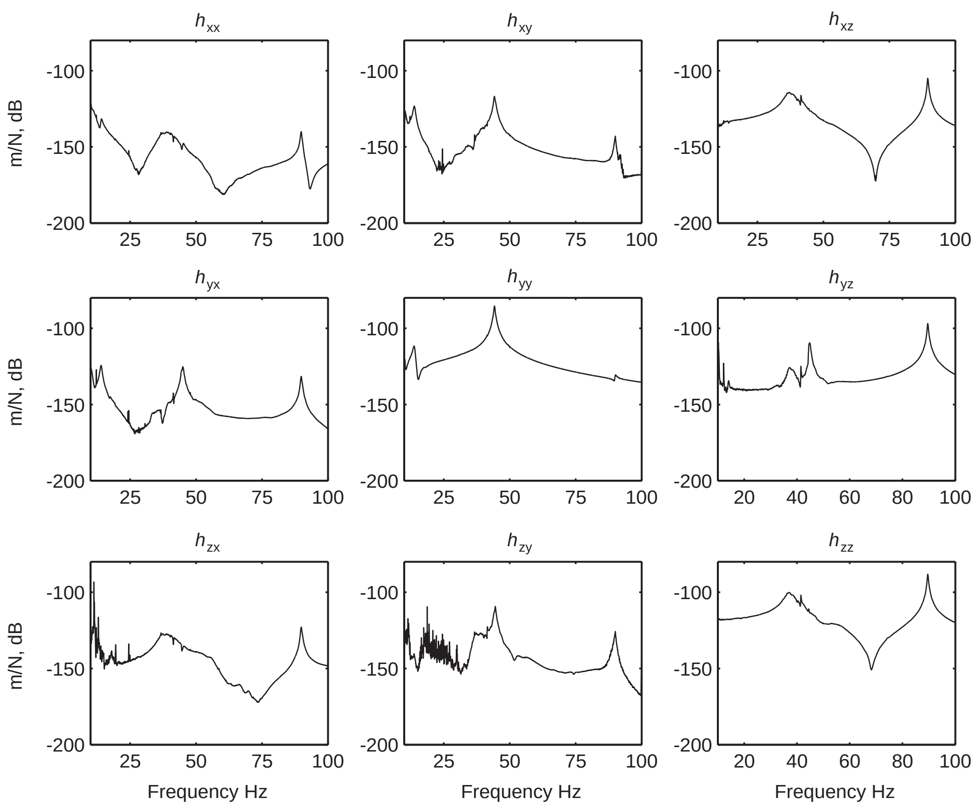


Fig. 5. Receptances of the modified structure.

Table 2
Modified natural frequencies

| Mode | Natural frequencies, Hz |
|---|---|
| 1 | 44.25 |
| 2 | 89.57 |

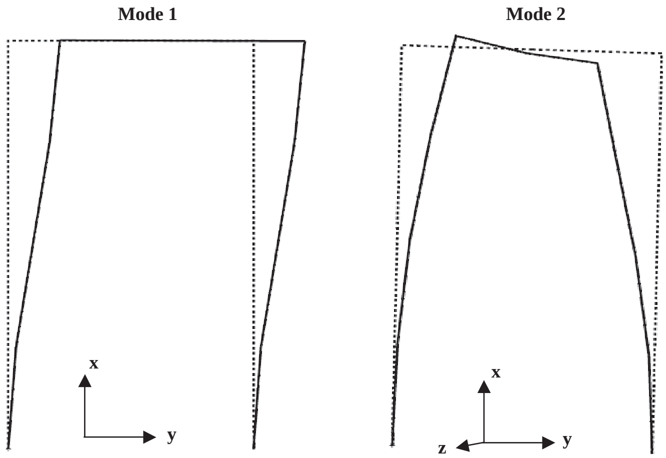


Fig. 6. Mode shapes of the modified structure.

modified natural frequencies are far away from the original natural frequencies and the mode shapes are also very different.

## 5. Assignment of a single natural frequency

The natural frequency assigned in this section is the modified natural frequency of 44.25 Hz. Substituting the original frequency response function values at this frequency, $\omega = 278\,\text{rad/s}$, into Eq. (9) a polynomial $f(b, d, t)$ is obtained. This polynomial may be considered to be a superposition of monomials $m_i = b^{n_b} d^{n_d} t^{n_t}$, where $n_b$, $n_d$ and $n_t$ are non-negative integer powers, with complex coefficients determined by the complex values of the original frequency response functions. If for each monomial a degree $d_{m_i} = n_b + n_d + n_t$ is defined then the degree of the multivariate polynomial is defined to be $\mathbf{max}(d_{m_i})$, $i = 1, 2, 3, \ldots$ . The complexity of the polynomial depends on the degree together with the number of monomials in the superposition. In this case the degree of the polynomial is 10 and involves all the monomials with degree 10 and less.

In order to assign a single natural frequency two of the cross-section dimensions should be fixed and the third one determined by solving the resulting single-variable polynomial. The polynomial

Table 3
Assigning a natural frequency at 44.25 Hz

| Parameters | Fixed parameter values | Solution |
|---|---|---|
| $t$ | $(b, d) = (0.1, 0.05)$ | $0.00406 + 0.00627\mathrm{i}$ |
| $b$ | $(t, d) = (0.004, 0.05)$ | $0.10194 + 0.01321\mathrm{i}$ |
| $d$ | $(t, b) = (0.004, 0.1)$ | $0.05037 + 0.00205\mathrm{i}$ |

that assigns a natural frequency at 44.25 Hz and determines the thickness is obtained after entering the values $b = 0.1\,\mathrm{m}$ and $d = 0.05\,\mathrm{m}$ with the result that

$$\begin{aligned} g(t) = {} & (-0.136 \times 10^{21} + 0.705 \times 10^{19}\mathrm{i})t^{10} + (0.443 \times 10^{20} - 0.229 \times 10^{19}\mathrm{i})t^9 \\ & + (-0.140 \times 10^{19} + 0.728 \times 10^{17}\mathrm{i})t^8 + (-0.8272 \times 10^{18} + 0.427 \times 10^{17}\mathrm{i})t^7 \\ & + (0.118 \times 10^{18} - 0.614 \times 10^{16}\mathrm{i})t^6 + (-0.687 \times 10^{16} + 0.3613 \times 10^{15}\mathrm{i})t^5 \\ & + (0.193 \times 10^{15} - 0.102 \times 10^{14}\mathrm{i})t^4 + (-0.242 \times 10^{13} + 0.110 \times 10^{12}\mathrm{i})t^3 \\ & + (0.712 \times 10^{10} + 0.542 \times 10^{9}\mathrm{i})t^2 + (0.162809 \times 10^{6} + 0.603 \times 10^{4}\mathrm{i})t + 1, \end{aligned} \tag{12}$$

where $\mathrm{i} = \sqrt{-1}$. This polynomial is solved numerically in Maple and gives the following solution set:

$$\begin{aligned} & -0.159 + 0.128 \times 10^{-4}\mathrm{i}, \quad -0.114 \times 10^{-4} - 0.2780 \times 10^{-5}\mathrm{i}, \\ & -0.112 \times 10^{-4} + 0.3643 \times 10^{-5}\mathrm{i}, \ \mathbf{0.406 \times 10^{-2} + 0.728 \times 10^{-3}i}, \\ & 0.302 \times 10^{-1} - 0.128 \times 10^{-1}\mathrm{i}, \quad 0.339 \times 10^{-1} + 0.130 \times 10^{-1}\mathrm{i}, \\ & 0.562 \times 10^{-1} - 0.997 \times 10^{-3}\mathrm{i}, \quad 0.749 \times 10^{-1} - 0.237 \times 10^{-6}\mathrm{i}, \\ & 0.765 \times 10^{-1} + 0.593 \times 10^{-4}\mathrm{i}, \quad 0.208 - 0.175 \times 10^{-4}\mathrm{i}. \end{aligned} \tag{13}$$

The bold faced value in expression (13) is the actual value of the thickness. Its real value is 0.004 m, exactly the value it should be. Its imaginary value reflects the damping of the structure and since the model does not include damping, a damping modification term gets attributed to the thickness, giving in this way the positive imaginary value necessary to assign the natural frequency to the imaginary axis of the complex plane. Polynomials $g(b)$ and $g(d)$ with complex coefficients are obtained by fixing $(t, d)$ and $(t, b)$ to $(0.004, 0.05)$ and $(0.003, 0.1)$, respectively. As with the case of $g(t)$ these polynomials give a multitude of solutions amongst which lie the desired ones. Table 3, lists the experimental values of $t$, $b$ and $d$ obtained by solving the inverse problem as described. The real part of these values is very close to the actual of values of $b$, $d$ and $t$. The imaginary part amounts to the fact that damping in the physical structure is solely attributed to these parameters in each case.

## 6. Assignment of two natural frequencies

In this section, the inverse problem of assigning the two modified natural frequencies at 44.25 Hz (in-plane) and 89.57 Hz (out of plane) is presented. In order to obtain the modification

Table 4
Assigning natural frequencies at 44.25 Hz and 89.57 Hz

| Parameters | Fixed parameter value | Solution |
|---|---|---|
| $(t, b)$ | $d = 0.05$ | $0.0046 + 0.00128\mathrm{i}, 0.0890 - 0.0047\mathrm{i}$ |
| $(t, d)$ | $b = 0.1$ | $0.0026 + 0.00018\mathrm{i}, 0.055 - 0.00402\mathrm{i}$ |
| $(b, d)$ | $t = 0.004$ | $0.0924 - 0.0008\mathrm{i}, 0.0516 + 0.0022\mathrm{i}$ |

polynomial for the 89.57 Hz natural frequency the method described in Sections 2 and 3 is followed but with $b$ and $d$ interchanged, because of the out-of-plane bending, and using the $xz$-plane receptances. Specifically, in Eq. (7) $I_{zz}$ changes to $I_{yy}$, given by $I_{yy} = (db^3 - (d-2t)(b-2t)^3)/12$, and $\mathbf{H}(\omega)$ becomes

$$\mathbf{H}(\omega) = \begin{pmatrix} h_{xx} & h_{xz} & h_{x\theta_y} \\ h_{zx} & h_{zz} & h_{z\theta_y} \\ h_{\theta_y x} & h_{\theta_y z} & h_{\theta_y \theta_y} \end{pmatrix}. \tag{14}$$

Substituting the frequency response function values at $\omega = 562.78$ rad/s (89.57 Hz) into the determinant leads to a polynomial in $b$, $d$, $t$ that should vanish simultaneously with the corresponding polynomial for the 44.25 Hz natural frequency. This amounts to the simultaneous solution of two multivariate polynomials in three unknowns, $b$, $d$ and $t$ with complex coefficients. To solve such a system one variable is fixed and a solution is sought for the other two. In what follows, the solution for $b$ and $d$ is sought whilst fixing $t = 0.004$. The results for the other cases are listed in Table 4. The polynomial obtained from in-plane motion at 44.25 Hz will be denoted by $g_1$ and the one obtained from out-of-plane motion at 89.57 Hz by $g_2$. The real, $\mathscr{R}$, and imaginary parts, $\mathscr{I}$, of these polynomials are given by

$$\begin{aligned} \mathscr{R}(g_1) = {} & \tfrac{85660693}{9} b^3 - 3847656680 d^6 + 449558583 b^2 d^2 + 68201010140 d^7 \\ & + \tfrac{80504103}{13} b^2 + \tfrac{167286692}{5} b d^2 + \tfrac{170868787}{15} d^3 + \tfrac{73852759}{12} d^2 \\ & + \tfrac{160511381}{13} b d + \tfrac{447618433}{3} d^4 + \tfrac{440158}{1197} + 367599583 b d^3 \\ & - \tfrac{6613039}{69} d - 47800278630 b d^4 + \tfrac{461742503}{2} d b^3 - 67084561440 b^2 d^3 \\ & + \tfrac{234085561}{8} b^2 d + 613809091200 b^3 d^4 - \tfrac{22365631}{233} b \\ & + 1023015152000 b^2 d^5 - 9820945459 b^3 d^3 - 28819622650 b^3 d^2 \\ & + 477407070800 b d^6 - 28081613150 b^2 d^4 - 19812291590 b d^5 - 9535339844 d^5, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathscr{I}(g_1) = {} & -\tfrac{34958842}{71} b^3 + 196157172 d^6 - \tfrac{334920112}{3} b^2 d^2 - 3528177472 d^7 \\ & + \tfrac{15430360}{31} b^2 + \tfrac{19603741}{53} b d^2 + \tfrac{12478919}{36} d^3 + \tfrac{16185428}{33} d^2 \\ & + \tfrac{7905765}{8} b d - \tfrac{185908697}{5} d^4 - \tfrac{273729223}{2} b d^3 + \tfrac{124929}{4166} - \tfrac{2418376}{315} d \\ & + 2473360860 b d^4 - \tfrac{107490970}{9} d b^3 + 3470837456 b^2 d^3 - \tfrac{23393174}{29} b^2 d \end{aligned}$$

$$- 31753597240b^3d^4 - 52922662060b^2d^5 + 508057556b^3d^3 + 1490897909b^3d^2 - 24697242290bd^6 + 1426709510b^2d^4 + 1007590514bd^5 + 493421313d^5 - \tfrac{3287467}{427}b, \quad (16)$$

$$\mathscr{R}(g_2) = -\tfrac{19615250}{3}b^3 - \tfrac{345334115}{2}b^2d^2 - \tfrac{48284201}{24}b^2 - \tfrac{227446592}{11}bd^2 - \tfrac{71035737}{10}d^3 - \tfrac{4022483}{2}d^2 - \tfrac{52300090}{13}bd - \tfrac{160264249}{4}bd^3 + \tfrac{8705494}{191}d - \tfrac{380199831}{2}db^3 + 5006597269b^2d^3 - \tfrac{141739039}{7}b^2d + 415127847b^3d^3 + 11673119580b^3d^2 + \tfrac{6789894}{149}b + 1665147515b^5 - 2882832273b^7 - \tfrac{899415}{3889} + 1319861976b^4d^2 + 926033300b^5d + 8331669833b^4d - 25945490460b^4d^3 - 43242484090b^5d^2 - 20179825900b^6d + \tfrac{354803083}{2}b^6 - \tfrac{287594507}{5}b^4, \quad (17)$$

$$\mathscr{I}(g_2) = -\tfrac{19685524}{67}b^3 - \tfrac{240090041}{37}b^2d^2 - \tfrac{8378014}{87}b^2 - \tfrac{3645981}{4}bd^2 - \tfrac{99590617}{321}d^3 - \tfrac{22730003}{236}d^2 - \tfrac{14060720}{73}bd - \tfrac{55996791}{32}bd^3 + \tfrac{40922}{23}d - \tfrac{20700365}{3}db^3 + \tfrac{437329883}{2}b^2d^3 - \tfrac{22517024}{25}b^2d + \tfrac{271962877}{15}b^3d^3 + 509903618b^3d^2 + \tfrac{686639}{386}b + \tfrac{145502443}{2}b^5 - \tfrac{251817479}{2}b^7 - \tfrac{82676}{10451} + \tfrac{211604329}{4}b^4d^2 + 37281928b^5d + 363989898b^4d - 1133178654b^4d^3 - 1888631090b^5d^2 - 881361175b^6d + \tfrac{86651095}{12}b^6 - \tfrac{15131929}{7}b^4. \quad (18)$$

In an earlier study [19] using simulated data natural frequencies and zeros were assigned by solving similar multivariate polynomials exactly by the theory of Groebner bases. A brief discussion of Grobner bases is given in Ref. [19] and further details can be found in the references therein. It was found that when using experimental data, as in the present study, exact solutions are not attainable and the solution of the system presented by Eqs. (15)–(18) could only be approximated through a numerical algorithm. A set of solutions were obtained by the Maple command *solve* and are shown in the third row of Table 4. The real part of $b$ is 0.0924 m whereas the actual value is 0.1 m. The real part of $d$ is 0.0516 m and the actual value is 0.05 m. The other two rows list the results obtained when determining $t$ and $b$ together and $t$ and $d$ together.

The out-of-plane mode of the modified structure shown in Fig. 6 clearly involves twisting of the vertical legs as well as out-of-plane bending. There is in fact another out-of-plane mode at a lower frequency where the two legs bend together in-phase without twisting. However this mode is poorly excited by a force at one corner of the portal frame. The twisting mode at 89.57 Hz on the other hand is very well excited by the force at the connection point. It should be noted that the receptances involving the rotation $\theta_x$ were not included in the analysis, and despite this omission excellent estimates of the unknown variables $b$, $d$ and $t$ were achieved. To include all the receptances that should strictly be present would require the measurement of $h_{\theta_x\theta_x}$ which can only be achieved by using a $T^2$-block [14].

## 7. Assignment of an antiresonance

This section presents the results for assigning an antiresonance of $h_{yy}$ at 89.7 Hz, shown in Fig. 7 and a natural frequency at 44.25 Hz . Results obtained for assignment of the antiresonance alone and together with the natural frequency are given.

Following the procedure set out in Section 5 the values of the original frequency response functions at 89.7 Hz are substituted in Eq. (11) to give a multivariate polynomial in the cross-section characteristics $b$, $d$ and $t$. Since only one antiresonance is to be assigned two of the parameters $b$, $d$ and $t$ can be fixed and the third one estimated. The results obtained are summarised in Table 5.

A similar procedure to that described in Section 6 is adopted to simultaneously assign a natural frequency at 44.25 Hz and antiresonance 89.7 Hz. Two multivariate polynomials in $b$, $d$ and $t$, one for the assignment of the natural frequency and the other for the assignment of the antiresonance are obtained. By fixing one of the three parameters $b$, $d$ and $t$ the other two are determined by solving the resulting polynomial system. These results are shown in Table 6. The numerical solution for $t$ and $d$ with $b$ fixed failed to converge.

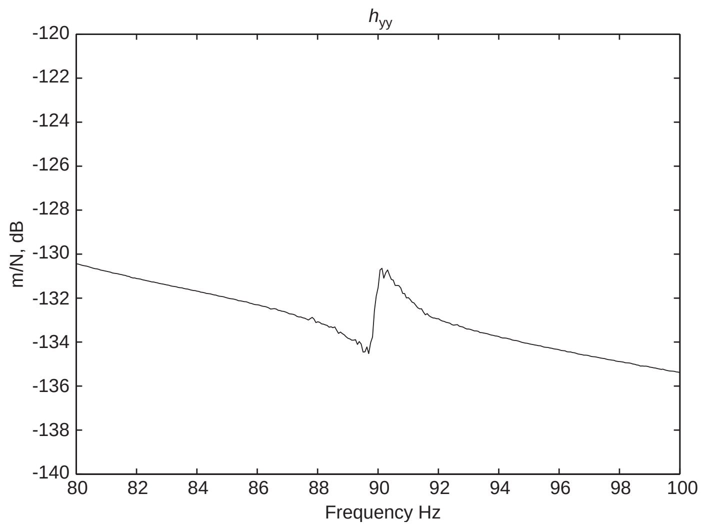


Fig. 7. Modified receptance $h_{yy}$ close to the antiresonance at 89.7 Hz.

Table 5
Assigning an antiresonance frequency at 89.97 Hz

| Parameters | Fixed parameter values | Solution |
|---|---|---|
| $t$ | $b=0.1$, $d=0.05$ | 0.0039−0.0018i |
| $d$ | $b=0.1$, $t=0.004$ | 0.0511−0.6636i |
| $b$ | $d=0.05$, $t=0.004$ | 0.101−0.0636i |

Table 6
Assigning a natural frequency at 44.25 Hz and an antiresonance at 89.97 Hz

| Parameters | Fixed parameter value | Solution |
|---|---|---|
| $(t, b)$ | $d = 0.05$ | $0.0046 + 0.00128\text{i}, 0.0890 - 0.0047\text{i}$ |
| $(t, d)$ | $b = 0.1$ | Failed to converge |
| $(b, d)$ | $t = 0.004$ | $0.0924 - 0.0008\text{i}, 0.0516 + 0.0022\text{i}$ |

## 8. Conclusions

In this article the inverse problem of assigning natural frequencies and antiresonances by a beam modification has been presented and experimentally implemented. The modification theory for linear systems has been presented from which a system of multivariate polynomials in the parameters of the beam cross-section has been developed. The solution of this polynomial gave the beam cross-section dimensions that assigned the desired natural frequencies and antiresonances.

Assigning natural frequencies is a narrow band solution to the problem of avoiding resonant excitation in structures. The methodology works well even for very significant modifications when the modified natural frequencies are far away from the natural frequencies of the original system and the mode shapes are very different. It gives good practical solutions when the natural frequencies and antiresonances are sensitive to the modifications. One would expect to experience difficulties of convergence or sensitivity to measurement noise when the derived natural frequencies and antiresonances are insensitive to a chosen parameter, such as when a modification is located close to a vibration node. This brings out issues of modification robustness. A future research suggestion is to study the measures, used in linear control theory, as given by $H_2$ and $H_\infty$ in order to design a robust modification.

A different approach would be to seek non-linear modifications that absorb energy from a linear structure at the frequency band where a lower level of response is required. Such studies on the theoretical level are available in the vibration engineering literature, see for example Refs. [20,21]. However, an experimental measure has not yet been developed that first quantifies the non-linear behaviour and second uses this information in the same way the frequency response function in this study is used to design a modification.

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