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Averaging method for quasi-integrable Hamiltonian systems

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Abstract

A deterministic averaging method for quasi-integrable Hamiltonian systems is proposed to predict the approximate response of many degree-of-freedom autonomous or non-autonomous strongly non-linear systems. The form and dimension of the averaged equations depend on the number of degree-of-freedom and the number of resonant relations of the associated Hamiltonian systems. In non-resonant case, the averaged equations for n action variables or n independent integrals of motion are derived. In resonant case, the averaged equations for n action variables and α combinations of angle variables are derived. Two examples are given to illustrate the application of the proposed method. It is shown that the analytical results agree well with those from numerical solution even for systems with very strong nonlinearity.

1. Introduction

Many techniques have been developed to predict the approximate response of single-degree-of- freedom (sdof) or multi-degree-of-freedom (mdof) quasi-linear systems [1]. The most well-known ones are the perturbation method [2], the method of multiple scales [3] and the averaging method [4,5]. The standard Krylov–Bogoliubov averaging method has been extensively used to predict the response and determine the stability and bifurcation of quasi-linear

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systems with or without internal and/or external resonances. Yuste and Bejarano [6] proposed an improved K–B averaging method using Jacobi elliptic functions to predict the approximate response of strongly nonlinear autonomous Duffing oscillator. Xu and Cheung [7,8] developed an averaging method for strongly nonlinear oscillators using generalized harmonic functions. Huang et al. [9,10] extended this method to predict stochastic jump and bifurcation of sdof strongly nonlinear Duffing oscillators under combined harmonic and white noises excitations and under bounded noise excitation, respectively. Cveticanin [11] obtained the approximate solution of two coupled pure cubic nonlinear oscillators using Jacobi elliptic functions.

On the other hand, a stochastic averaging method for quasi-integrable Hamiltonian systems has been developed [12–14] and successfully applied to predict the response and to determine the stochastic stability [15]. The Hamiltonian formulation of mdof strongly nonlinear systems provides better understanding of the interaction among various degrees of freedom (dof) of the system [16].

In the present paper, a deterministic averaging method for quasi-integrable Hamiltonian systems is developed. It is shown that the form and dimension of averaged equations depend on the number of dof and the number of internal and/or external resonant relations. Two examples are worked out to illustrate the proposed method.

2. Quasi-integrable Hamiltonian systems

Consider a mdof strongly nonlinear non-autonomous system. The equations of motion of the system are of the form

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i},\\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon c_{ij}(\mathbf{q},\mathbf{p})\frac{\partial H}{\partial p_j} + \varepsilon h_{ik}(\mathbf{q},\mathbf{p})\cos(\Omega_k t + \gamma_k),\\
& i,j = 1,\ldots,n, \quad k = 1,2,\ldots,m,
\end{aligned} \tag{1}
$$

where $\mathbf{q} = [q_1, q_2, \ldots, q_n]^{\mathrm{T}}$, $\mathbf{p} = [p_1, p_2, \ldots, p_n]^{\mathrm{T}}$ and q_i, p_i are generalized displacements and momenta, respectively; $H = H(\mathbf{q},\mathbf{p})$ is Hamiltonian; c_{ij} are coefficients of quasi-linear dampings; h_{ik} are amplitudes of harmonic excitations; ε is a small positive parameter; Ω_k and γ_k are the frequencies and initial phase angles of excitations.

Suppose that the Hamiltonian system associated system (1) with $\varepsilon = 0$ is completely integrable, i.e., there exists a set of canonical transformations

$$
\begin{aligned}
I_i &= I_i(\mathbf{q},\mathbf{p}),\\
\theta_i &= \theta_i(\mathbf{q},\mathbf{p}),\\
i &= 1,\ldots,n
\end{aligned} \tag{2}
$$

(the specific form of the transformations depends on the structure of the Hamiltonian) such that the new Hamilton equations are of the following form:

$$\dot{I}_i = -\frac{\partial H(\mathbf{I})}{\partial \theta_i} = 0,$$
$$\dot{\theta}_i = \frac{\partial H(\mathbf{I})}{\partial I_i} = \omega_i(\mathbf{I}),$$
$$i = 1, \ldots, n, \tag{3}$$

where $\mathbf{I} = [I_1, I_2, \ldots, I_n]^{\mathrm{T}}$; I_i and ω_i are action variables and frequencies, respectively; θ_i are the angle variable conjugated to I_i; and $H(\mathbf{I})$ is the transformed Hamiltonian independent of θ_i. I_i can be regarded as n independent integrals of motion which are in involution. The completely integrable Hamiltonian system associated system (1) with $\varepsilon = 0$ is resonant if there exist $\alpha\,(1 \leqslant \alpha \leqslant n-1)$ resonant relations

$$k_i^u \omega_i = 0, \quad u = 1, \ldots, \alpha, \quad \alpha \leqslant n-1, \tag{4}$$

where k_i^u are integers and not all zero for a fixed u, and α is the number of resonant relations.

The system governed by Eqs. (1) with $\varepsilon \neq 0$ is called quasi-integrable Hamiltonian system. Introducing canonical transformation (2), the differential equations for action and angle variables are of the form

$$\dot{I}_r = \varepsilon \left(c_{ij}(\mathbf{q}, \mathbf{p}) \frac{\partial H}{\partial p_j} + h_{ik}(\mathbf{q}, \mathbf{p}) \cos \Gamma_k \right) \frac{\partial I_r}{\partial p_i}, \tag{5a}$$

$$\dot{\theta}_r = \omega_r(\mathbf{I}) + \varepsilon \left(c_{ij}(\mathbf{q}, \mathbf{p}) \frac{\partial H}{\partial p_j} + h_{ik}(\mathbf{q}, \mathbf{p}) \cos \Gamma_k \right) \frac{\partial \theta_r}{\partial p_i},$$

$$r, i, j = 1, \ldots, n, \quad k = 1, \ldots, m, \tag{5b}$$

where $\Gamma_k = \Omega_k t + \gamma_k$. The number and form of the averaged equations of system (5) depend upon whether the associated Hamiltonian system is resonant or not. Three cases are considered in the following section.

3. Averaged equations

3.1. Non-resonant case

At first, consider the case where there is no internal and external resonance in systems (5). In this case, the terms containing $\cos \Gamma_k$ in Eqs. (5a,b) can be neglected in the first approximation. It can be seen from Eqs. (5a,b) that the action variables I_i vary slowly while the angle variables θ_i vary rapidly. Eq. (5a) can be rewritten as follows:

$$\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon U_r(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n), \quad r = 1, \ldots, n, \tag{6}$$

where

$$U_r = \left[c_{ij}(\mathbf{q},\mathbf{p}) \frac{\partial H}{\partial p_j} \frac{\partial I_r}{\partial p_i} \right] \Bigg|_{\substack{\mathbf{q}=\mathbf{q}(\mathbf{I},\boldsymbol{\theta}) \\ \mathbf{p}=\mathbf{p}(\mathbf{I},\boldsymbol{\theta})}}. \tag{7}$$

Note that since non-resonant integrable Hamiltonian system of n dof is ergodic on n-dimensional torus, the time averaging is equivalent to space averaging over the n-dimensional torus. Thus, the averaged equations can be derived by the averaging with respect to θ_i, i.e.,

$$\begin{aligned} &\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon \bar{U}_r(I_1, \ldots, I_n), \\ &r = 1, \ldots, n, \end{aligned} \tag{8}$$

where

$$\bar{U}_r(I_1, \ldots, I_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} U_r(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \tag{9}$$

3.2. Internal resonant case

Then consider the case where there are $\alpha\,(1 \leqslant \alpha \leqslant n-1)$ internal resonances and no external resonance in systems (5). In this case the terms containing $\cos \Gamma_k t$ in Eq. (5) can also be neglected in the first approximation. Suppose that the integrable Hamiltonian system governed by Eqs. (1) with $\varepsilon = 0$ is nearly resonant, i.e., there are $\alpha\,(1 \leqslant \alpha \leqslant n-1)$ weak resonant relations

$$\begin{aligned} &k_r^u \omega_r = \varepsilon \sigma_u \\ &u = 1, \ldots, \alpha; \quad r = 1, \ldots, n, \end{aligned} \tag{10}$$

where σ_u are detuning parameters. Introduce α combinations Φ_u of angle variables

$$\begin{aligned} &\Phi_u = k_r^u \theta_r \\ &u = 1, \ldots, \alpha; \quad r = 1, \ldots, n. \end{aligned} \tag{11}$$

The first α angle variables θ_i are replaced by α combinations Φ_u. The differential equations for $I_1, \ldots, I_n, \Phi_1, \ldots, \Phi_\alpha, \theta_{\alpha+1}, \ldots, \theta_n$ are obtained from Eq. (5) by using Eqs. (10) and (11) as follows:

$$\begin{aligned} &\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon U_r(\mathbf{I}, \boldsymbol{\Phi}, \theta_{\alpha+1}, \ldots, \theta_n), \\ &\frac{\mathrm{d}\Phi_u}{\mathrm{d}t} = \varepsilon \sigma_u + \varepsilon V_u(\mathbf{I}, \boldsymbol{\Phi}, \theta_{\alpha+1}, \ldots, \theta_n), \\ &\frac{\mathrm{d}\theta_v}{\mathrm{d}t} = \omega_v + \varepsilon W_v(\mathbf{I}, \boldsymbol{\Phi}, \theta_{\alpha+1}, \ldots, \theta_n), \\ &r = 1, \ldots, n, \quad u = 1, \ldots, \alpha, \quad v = \alpha + 1, \ldots, n, \end{aligned} \tag{12}$$

where

$$U_r = \left[c_{ij}(\mathbf{q},\mathbf{p}) \frac{\partial H}{\partial p_j} \frac{\partial I_r}{\partial p_i} \right] \Bigg|_{\substack{\mathbf{q}=\mathbf{q}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n) \\ \mathbf{p}=\mathbf{p}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n)}},$$

$$V_u = \left[k_r^u c_{ij}(\mathbf{q},\mathbf{p}) \frac{\partial H}{\partial p_j} \frac{\partial \theta_r}{\partial p_i} \right] \Bigg|_{\substack{\mathbf{q}=\mathbf{q}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n) \\ \mathbf{p}=\mathbf{p}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n)}},$$

$$W_v = \left[c_{lj}(\mathbf{q},\mathbf{p}) \frac{\partial H}{\partial p_j} \frac{\partial \theta_v}{\partial p_i} \right] \Bigg|_{\substack{\mathbf{q}=\mathbf{q}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n) \\ \mathbf{p}=\mathbf{p}(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n)}},$$

$$\boldsymbol{\Phi} = \{\Phi_1, \ldots, \Phi_\alpha\}^{\mathrm{T}} \tag{13}$$

It is seen from Eqs. (12) that n action variables I_r and α combinations Φ_u of angle variables vary slowly while the angle variables $\theta_{\alpha+1}, \ldots, \theta_n$ vary rapidly. The averaged equations for I_r, Φ_u can be obtained from Eq. (12) using space averaging with respect to $\theta_{\alpha+1}, \ldots, \theta_n$ as follows:

$$\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon \bar{U}_r(\mathbf{I},\boldsymbol{\Phi}),$$
$$\frac{\mathrm{d}\Phi_u}{\mathrm{d}t} = \varepsilon\sigma_u + \varepsilon \bar{V}_u(\mathbf{I},\boldsymbol{\Phi}),$$
$$r = 1,\ldots,n, \quad u = 1,\ldots,\alpha, \tag{14}$$

where

$$\bar{U}_r(\mathbf{I},\boldsymbol{\Phi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \cdots \int_0^{2\pi} U_r(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n)\,\mathrm{d}\theta_{\alpha+1}\cdots\mathrm{d}\theta_n,$$
$$\bar{V}_u(\mathbf{I},\boldsymbol{\Phi}) = \frac{1}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \cdots \int_0^{2\pi} V_u(\mathbf{I},\boldsymbol{\Phi},\theta_{\alpha+1},\ldots,\theta_n)\,\mathrm{d}\theta_{\alpha+1}\cdots\mathrm{d}\theta_n. \tag{15}$$

3.3. Both internal and external resonant case

Finally, consider the case where there are $\alpha\,(1 \leqslant \alpha \leqslant n-1)$ internal resonant relations and β external resonant relations $(1 \leqslant \beta \leqslant m)$ in system (5). i.e.,

$$k_r^u \omega_r = \varepsilon\sigma_u, \tag{16a}$$

$$L_r^v \omega_r + M_k^v \Omega_k = \varepsilon\delta_v, \tag{16b}$$

$$u = 1,\ldots,\alpha, \quad v = 1,\ldots,\beta, \quad r = 1,\ldots,n, \quad k = 1,\ldots,m,$$

where k_r^u, L_r^v, M_k^v are integers and σ_u, δ_v are detuning parameters. Introduce α combinations Φ_u of angle variables and β combinations Ψ_v of angle variables and phase angles of excitations.

$$\Phi_u = k_r^u \theta_r, \tag{17a}$$

$$\Psi_v = L_r^v \theta_r + M_k^v \Gamma_k, \tag{17b}$$

$$u = 1,\ldots,\alpha, \quad v = 1,\ldots,\beta, \quad r = 1,\ldots,n, \quad k = 1,\ldots,m,$$

$\theta_1, \ldots, \theta_n, \Gamma_k, \ldots, \Gamma_m$ are replaced by $\Phi_1, \ldots, \Phi_\alpha$, $\theta_{\alpha+1}, \ldots, \theta_n$, $\Psi_1, \ldots, \Psi_\beta$, $\Gamma_{\beta+1}, \ldots, \Gamma_m$. The differential equations for $I_1, \ldots, I_n, \Phi_1, \ldots, \Phi_\alpha, \Psi_1, \ldots, \Psi_\beta, \theta_{\alpha+1}, \ldots, \theta_n$ are of the form

$$\begin{aligned}
&\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon U_r(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m),\\
&\frac{\mathrm{d}\Phi_u}{\mathrm{d}t} = \varepsilon\sigma_u + \varepsilon V_u(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m),\\
&\frac{\mathrm{d}\Psi_v}{\mathrm{d}t} = \varepsilon\delta_v + \varepsilon W_v(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m),\\
&\frac{\mathrm{d}\theta_s}{\mathrm{d}t} = \omega_s + \varepsilon X_s(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m),\\
&r = 1, \ldots, n, \quad u = 1, \ldots, \alpha, \quad v = 1, \ldots, \beta, \quad s = \alpha+1, \ldots, n,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&U_r = c_{ij}(\mathbf{q}, \mathbf{p})\frac{\partial H}{\partial p_j}\frac{\partial I_r}{\partial p_i} + h_{ik}(\mathbf{q}, \mathbf{p})\frac{\partial I_r}{\partial p_i}\cos\Gamma_k,\\
&V_u = k_r^u c_{ij}(\mathbf{q}, \mathbf{p})\frac{\partial H}{\partial p_j}\frac{\partial \theta_r}{\partial p_i} + k_r^u h_{ik}(\mathbf{q}, \mathbf{p})\frac{\partial \theta_r}{\partial p_i}\cos\Gamma_k,\\
&W_v = L_r^v\left(c_{ij}(\mathbf{q}, \mathbf{p})\frac{\partial H}{\partial p_j}\frac{\partial \theta_r}{\partial p_i} + h_{ik}(\mathbf{q}, \mathbf{p})\frac{\partial \theta_r}{\partial p_i}\cos\Gamma_k\right),\\
&X_s = c_{ij}(\mathbf{q}, \mathbf{p})\frac{\partial H}{\partial p_j}\frac{\partial \theta_s}{\partial p_i} + h_{ik}(\mathbf{q}, \mathbf{p})\frac{\partial \theta_s}{\partial p_i}\cos\Gamma_k,\\
&\boldsymbol{\Psi} = \{\Psi_1, \ldots, \Psi_\beta\}^{\mathrm{T}}.
\end{aligned} \tag{19}$$

It can be seen from Eq. (18) that n action variables I_r, α combinations Φ_u of angle variables and β combinations Ψ_v of angle variables and phase angles of excitations vary slowly while angle variables $\theta_{\alpha+1}, \ldots, \theta_n$ and phase angles of excitations $\Gamma_{\beta+1}, \ldots, \Gamma_m$ vary rapidly. The averaged equations for $I_1, \ldots, I_n$, $\Phi_1, \ldots, \Phi_\alpha$, $\Psi_1, \ldots, \Psi_\beta$ can be obtained from Eq. (18) using the space averaging with respect to $\theta_{\alpha+1}, \ldots, \theta_n$, $\Gamma_{\beta+1}, \ldots, \Gamma_m$ as follows:

$$\begin{aligned}
&\frac{\mathrm{d}I_r}{\mathrm{d}t} = \varepsilon \bar{U}_r(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}),\\
&\frac{\mathrm{d}\Phi_u}{\mathrm{d}t} = \varepsilon\sigma_u + \varepsilon \bar{V}_u(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}),\\
&\frac{\mathrm{d}\Psi_v}{\mathrm{d}t} = \varepsilon\delta_v + \varepsilon \bar{W}_v(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}),\\
&r = 1, \ldots, n, \quad u = 1, \ldots, \alpha, \quad v = 1, \ldots, \beta,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\bar{U}_r = \frac{1}{(2\pi)^{n+m-\alpha-\beta}}\int_0^{2\pi}\cdots\int_0^{2\pi} & U_r(\mathbf{I}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n,\\
& \Gamma_{\beta+1}, \ldots, \Gamma_m)\,\mathrm{d}\theta_{\alpha+1}\cdots\mathrm{d}\theta_n\,\mathrm{d}\Gamma_{\beta+1}\cdots\mathrm{d}\Gamma_m
\end{aligned}$$

$$\bar{V}_u = \frac{1}{(2\pi)^{n+m-\alpha-\beta}} \int_0^{2\pi} \cdots \int_0^{2\pi} V_u(\mathbf{I}, \mathbf{\Phi}, \mathbf{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m)\, \mathrm{d}\theta_{\alpha+1} \cdots \mathrm{d}\theta_n\, \mathrm{d}\Gamma_{\beta+1} \cdots \mathrm{d}\Gamma_m$$

$$\bar{W}_v = \frac{1}{(2\pi)^{n+m-\alpha-\beta}} \int_0^{2\pi} \cdots \int_0^{2\pi} W_v(\mathbf{I}, \mathbf{\Phi}, \mathbf{\Psi}, \theta_{\alpha+1}, \ldots, \theta_n, \Gamma_{\beta+1}, \ldots, \Gamma_m)\, \mathrm{d}\theta_{\alpha-1} \cdots \mathrm{d}\theta_n\, \mathrm{d}\Gamma_{\beta+1} \cdots \mathrm{d}\Gamma_m. \tag{21}$$

3.4. Some remarks

In the practical application of the proposed averaging method, it is more convenient to replace n action variables $I_1, \ldots, I_n$ with n independent integrals of motion, $H_1, \ldots, H_n$, in involution because it is difficult to obtain the action variables I_i in most cases. The differential equations for $H_1, \ldots, H_n$ and $\theta_1, \ldots, \theta_n$ can be obtained from Eqs. (8), (14) and (20) by replacing I_r with H_r. For example, in the case of α internal resonances and β external resonances the averaged equations of the system can be obtained as follows:

$$\begin{aligned}
\frac{\mathrm{d}H_r}{\mathrm{d}t} &= \varepsilon \tilde{U}_r(H_1, \ldots, H_n, \Phi_1, \ldots \Phi_\alpha, \Psi_1, \ldots, \Psi_\beta),\\
\frac{\mathrm{d}\Phi_u}{\mathrm{d}t} &= \varepsilon\sigma_u + \varepsilon \tilde{V}_u(H_1, \ldots, H_n, \Phi_1, \ldots \Phi_\alpha, \Psi_1, \ldots, \Psi_\beta),\\
\frac{\mathrm{d}\Psi_v}{\mathrm{d}t} &= \varepsilon\delta_v + \varepsilon \tilde{W}_v(H_1, \ldots, H_n, \Phi_1, \ldots \Phi_\alpha, \Psi_1, \ldots, \Psi_\beta),\\
r &= 1, \ldots, n, \quad u = 1, \ldots, \alpha, \quad v = 1, \ldots, \beta,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\tilde{U}_r &= \frac{1}{(2\pi)^{n+m-\alpha-\beta}} \int_0^{2\pi} \cdots \int_0^{2\pi} \times \left[c_{ij} \frac{\partial H}{\partial p_j} \frac{\partial H_r}{\partial p_i} + h_{ik} \frac{\partial H_r}{\partial p_i} \cos \Gamma_k \right] \mathrm{d}\theta_{\alpha+1} \cdots \mathrm{d}\theta_n\, \mathrm{d}\Gamma_{\beta+1} \cdots \mathrm{d}\Gamma_m,\\
\tilde{V}_u &= \frac{1}{(2\pi)^{n+m-\alpha-\beta}} \int_0^{2\pi} \cdots \int_0^{2\pi} \times \left[k_r^u c_{ij} \frac{\partial H}{\partial p_j} \frac{\partial \theta_r}{\partial p_i} + k_r^u h_{ik} \frac{\partial \theta_r}{\partial p_i} \cos \Gamma_k \right] \mathrm{d}\theta_{\alpha+1} \cdots \mathrm{d}\theta_n\, \mathrm{d}\Gamma_{\beta+1} \cdots \mathrm{d}\Gamma_m,\\
\tilde{W}_v &= \frac{1}{(2\pi)^{n+m-\alpha-\beta}} \int_0^{2\pi} \cdots \int_0^{2\pi} \times \left[L_r^v c_{ij} \frac{\partial H}{\partial p_j} \frac{\partial \theta_r}{\partial p_i} + L_r^v h_{ik} \frac{\partial \theta_r}{\partial p_i} \cos \Gamma_k \right] \mathrm{d}\theta_{\alpha+1} \cdots \mathrm{d}\theta_n\, \mathrm{d}\Gamma_{\beta+1} \cdots \mathrm{d}\Gamma_m.
\end{aligned} \tag{23}$$

Obviously, the dimension of the averaged equations is usually less than that of the original equations. Besides, only slowly varying quantities are involved in the averaged equations while both slowly and rapidly varying quantities are involved in the original equations, If the original system is non-autonomous, the averaged system is autonomous. Thus, the averaged equations are simplified and easier to solve than the original equations. Since the internal and/or external resonances are considered in the derivation of the averaged equations, the essential characteristics of the original system are retained in the averaged equations. The functions of the proposed averaging method are illustrated in the following section.

4. Examples

4.1. Example 1

Consider a Duffing oscillator with hardening spring subject to additive harmonic excitation. The equation of motion is of the form

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = -\beta \dot{x} + \bar{E} \cos \Omega t, \tag{24}$$

where ω_0 is the frequency of degenerated linear oscillator; α is the intensity of nonlinearity; β is the coefficient of linear damping; $\bar{E}$ is the amplitude of harmonic excitation. β and $\bar{E}$ are of order of ε. This example has been studied by many authors [7–11]. The Hamiltonian associated with system (24) is

$$H = \frac{p^2}{2} + \frac{\omega_0^2}{2} q^2 + \frac{\alpha}{4} q^4, \tag{25}$$

where $q = x$ and $p = \dot{x}$. The expressions for action variable, instantaneous frequency and angle variable of the system are [17,18]

$$\begin{aligned} I(H) &= \frac{2}{\pi} \int_0^a \sqrt{2H - \omega_0^2 q^2 - \frac{\alpha}{2} q^4}\, \mathrm{d}q \\ &= \frac{2\omega_0^3}{3\pi\alpha} \sqrt[4]{1 + \frac{4\alpha H}{\omega_0^4}} \left[\left(\sqrt{1 + \frac{4\alpha H}{\omega_0^4}} + 1 \right) K(r) - 2E(r) \right], \end{aligned} \tag{26a}$$

$$\omega = \frac{\mathrm{d}I}{\mathrm{d}H} = \frac{\pi\sqrt{\alpha}}{2\sqrt{2}} \frac{\sqrt{a^2 + b^2}}{K(r)}, \tag{26b}$$

$$\theta = \omega \int_q^a \frac{\mathrm{d}q}{\sqrt{2H - \omega_0^2 q^2 - \frac{\alpha}{2} q^4}} = \frac{\pi}{2K(r)} F(\varphi, r), \tag{26c}$$

where $K(r)$ and $E(r)$ are the complete elliptic integrals of the first and second kind, respectively; $F(\varphi, r)$ is the elliptic integral of the first kind. $\varphi = \arccos(q/a)$, $r = a/\sqrt{a^2 + b^2}$, $b^2 = (\omega_0^2/\alpha)\left(\sqrt{1 + (4\alpha H/\omega_0^4)} + 1\right)$, $a^2 = (\omega_0^2/\alpha)\left(\sqrt{1 + (4\alpha H/\omega_0^4)} - 1\right)$.

Eq. (26c) can be rewritten as

$$q = a\,\mathrm{Cn}\left[\frac{2K(r)}{\pi}\,\theta\right] = a\sum_{n=1}^{\infty} C_n \cos(2n-1)\theta, \tag{27}$$

where Cn is cosine-amplitude and

$$C_n = \frac{2\pi}{rK(r)}\,\frac{\mathrm{e}^{-\left(n-\frac{1}{2}\right)K\left(\sqrt{1-r^2}\right)/K(r)}}{1+\mathrm{e}^{-(2n-1)K\left(\sqrt{1-r^2}\right)/K(r)}}. \tag{28}$$

Substituting Eq. (27) into Eq. (25), one obtains

$$\begin{aligned} p &= \sqrt{2H - \omega_0^2 q^2 - \frac{\alpha}{2}\,q^4} \\ &= a\sqrt{\omega_0^2 + \alpha a^2}\,\mathrm{Sn}\left(\frac{2K(r)\theta}{\pi}\right)\sqrt{1 - r^2\,\mathrm{Sn}^2\left(\frac{2K(r)\theta}{\pi}\right)} \\ &= a\sum_{n=1}^{\infty} P_n \sin(2n-1)\theta, \end{aligned} \tag{29}$$

where Sn is sine-amplitude and P_n are coefficients of Fourier expansion. The first three coefficients of Fourier expansion in Eqs. (27) and (29) are shown in Figs. 1(a) and (b), respectively. It is seen from Fig. 1 that only the first two or three terms need to be retained in the Fourier expansions.

Based on the averaging procedure described in Section 3, the differential equations for H and θ are of the form

$$\begin{aligned} \dot{H} &= p(-\beta p + \bar{E}\cos\Omega t), \\ \dot{\theta} &= \frac{\pi}{2K(r)}\left(\frac{\partial F}{\partial \varphi}\,\dot{\varphi} + \frac{\partial F}{\partial r}\,\frac{\mathrm{d}r}{\mathrm{d}H}\,\dot{H}\right) - \frac{\pi}{2K^2(r)}\,F(\varphi, r)\,\frac{\mathrm{d}K}{\mathrm{d}r}\,\frac{\mathrm{d}r}{\mathrm{d}H}\,\dot{H} \\ &= \omega - \left\{\omega q\left[\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}H} + \frac{r}{1-r^2}\left(1 - \frac{q^2}{a^2}\right)\frac{\mathrm{d}r}{\mathrm{d}H}\right] - \frac{\pi p[E(\varphi,r)K(r) - F(\varphi,r)E(r)]}{2K^2(r)r(1-r^2)}\,\frac{\mathrm{d}r}{\mathrm{d}H}\right\} \\ &\quad \times(-\beta p + \bar{E}\cos\Omega t), \end{aligned} \tag{30}$$

where $E(\varphi, r)$ is the elliptic integral of the second kind. Suppose that we are interested in primary resonance of the system, i.e.,

$$\frac{\Omega}{\omega} = 1 + \varepsilon\delta, \tag{31}$$

where δ is a detuning parameter. Introduce new variable

$$\Psi = \Omega t - \theta. \tag{32}$$

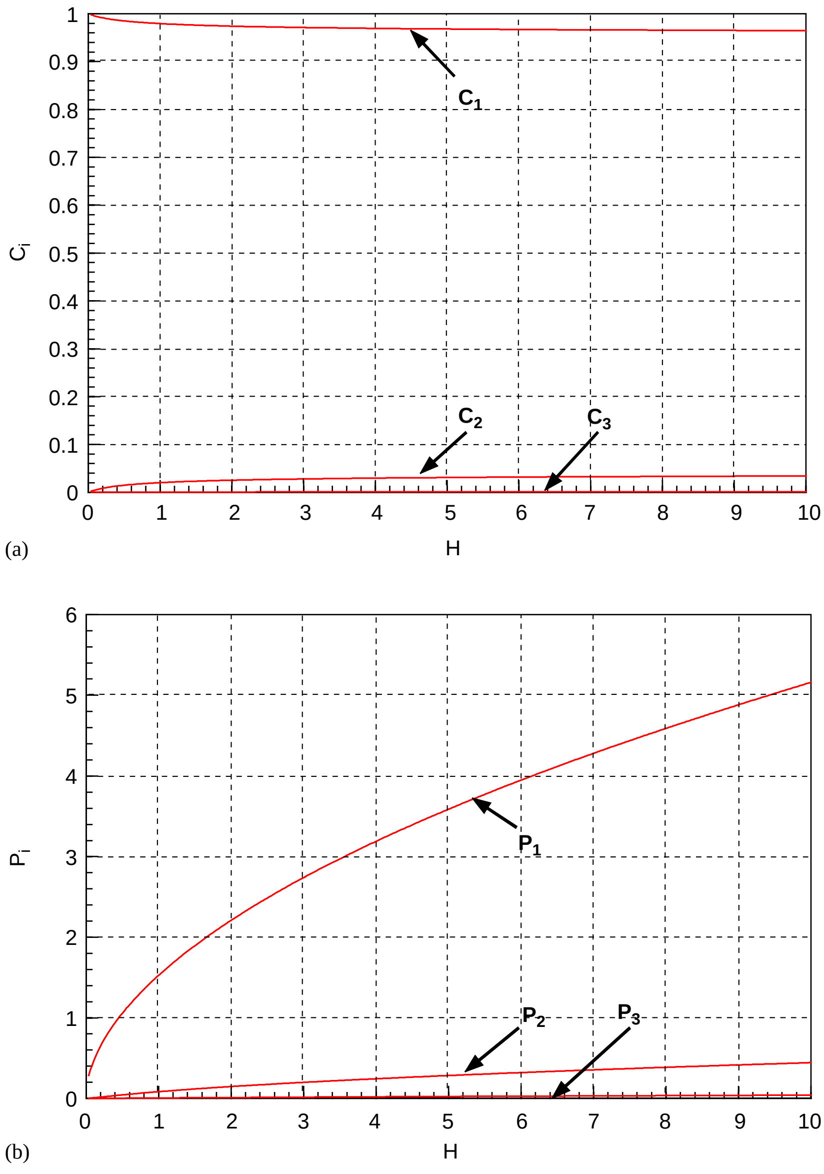

Fig. 1. Fourier coefficients of generalized displacement and momenta verse Hamiltonian H. $\omega_0 = 1, \beta = 0.1, \alpha = 1.0$: (a) generalized displacement; (b) generalized momentum.

The differential equations for H and Ψ are of the form

$$\begin{aligned} \dot{H} &= -\beta p^2 + \bar{E}p\cos(\Psi + \theta), \\ \dot{\Psi} &= \Omega - \omega + \left\{ \omega q \left[\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}H} + \frac{r}{1-r^2}\left(1 - \frac{q^2}{a^2}\right)\frac{\mathrm{d}r}{\mathrm{d}H} \right] \right. \\ &\quad \left. - \frac{\pi p[E(\varphi, r)K(r) - F(\varphi, r)E(r)]}{2K^2(r)r(1-r^2)} \frac{\mathrm{d}r}{\mathrm{d}H} \right\} (-\beta p + \bar{E}\cos(\Psi + \theta)). \end{aligned} \tag{33}$$

It can be seen that variables $H(t)$ and $\Psi(t)$ in Eq. (33) vary slowly while $\theta(t)$ in Eq. (30) varies rapidly. Following the procedure in Section 3.3, one obtains the following averaged equations for H and Ψ:

$$\dot{H} = -\frac{\beta a^2}{2}\sum_{n=1}^{\infty} P_n^2 - \frac{\bar{E} a P_1}{2}\sin\Psi,$$
$$\dot{\Psi} = \Omega - \omega + \bar{E} S \cos\Psi, \tag{34}$$

where

$$S = \frac{1}{2\pi}\int_0^{2\pi}\left\{\omega q\left[\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}H} + \frac{r}{1-r^2}\left(1-\frac{q^2}{a^2}\right)\frac{\mathrm{d}r}{\mathrm{d}H}\right] - \frac{\pi p[E(\varphi,r)K(r) - F(\varphi,r)E(r)]}{2K^2(r)r(1-r^2)}\frac{\mathrm{d}r}{\mathrm{d}H}\right\}\cos\theta\,\mathrm{d}\theta \tag{35}$$

The steady-state response of the averaged system (34) can be obtained by letting $\dot{H} = \dot{\Psi} = 0$ as follows:

$$\left[\frac{\beta a}{P_1}\sum_{n=1}^{\infty} P_n^2\right]^2 + \left[\frac{\Omega-\omega}{S}\right]^2 = \bar{E}^2. \tag{36}$$

The approximate steady-state amplitude response curves of Duffing oscillator with hardening spring under additive harmonic excitation obtained by using the proposed averaging method are shown in Fig. 2 using solid and dash lines. They agree well with those (denoted by •▲) from numerical solution of original system (24). It is noted that $\alpha = 1,\ 2$ in Fig. 2 represent very strong nonlinearity of the system. Further investigation has shown that the proposed averaging method can be successfully applied to predict the amplitude response of system (24) with α up to 100 if the first five terms of Fourier expansions in Eqs. (27) and (29) are retained. To the authors' knowledge, no such method exits so far.

4.2. Example 2

Consider the nonlinearly coupled Duffing–van der Pol oscillators governed by the equations of motion

$$\ddot{x}_1 + \omega_{10}^2 x_1 + \alpha_1 x_1^3 = -\dot{x}_1(\beta_{11} + \beta_{12}x_1^2 + \beta_{13}x_2^2),$$
$$\ddot{x}_2 + \omega_{20}^2 x_2 + \alpha_2 x_2^3 = -\dot{x}_2(\beta_{21} + \beta_{22}x_1^2 + \beta_{23}x_2^2), \tag{37}$$

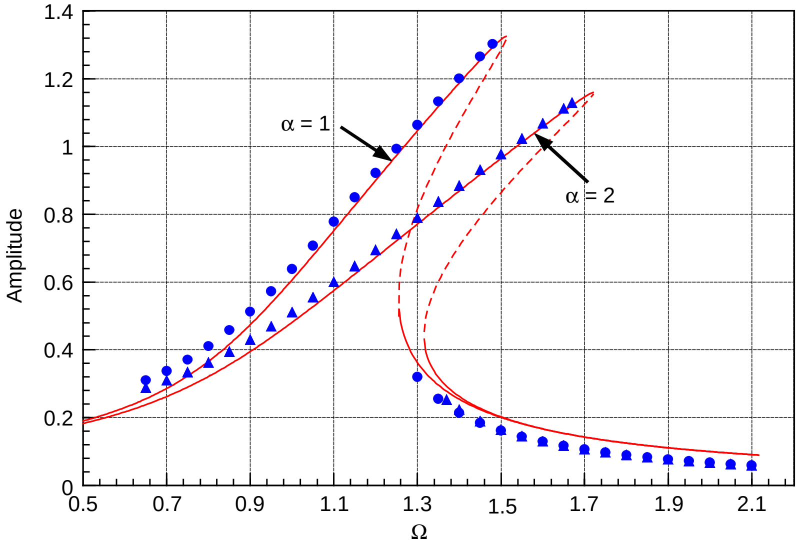

Fig. 2. Amplitude response curves of Duffing oscillator under additive harmonic excitation, (24). $\omega_0 = 1, \beta = 0.1, \bar{E} = 0.2$:—by using the proposed averaging method; •▲ from numerical solution of original system.

where ω_{10} and ω_{20} are the frequencies of two degenerated linear oscillators; α_1 and α_2 are the intensities of nonlinearity of the two oscillators; β_{ij} are coefficients of linear or nonlinear dampings of order of ε. System (37) has been studied by many authors [19–22]. Only the case of $\beta_{i1} < 0, \beta_{i2} > 0, \beta_{i3} > 0\,(i = 1, 2)$ is considered in the present paper.

The expressions for Hamiltonian, action variable, instantaneous frequency, angle variable, generalized displacement and generalized momenta of system (37) without damping are of the form [17,18]

$$H_i = \frac{p_i^2}{2} + \frac{\omega_{i0}^2}{2}\,q_i^2 + \frac{\alpha_i}{4}\,q_i^4,$$

$$\begin{aligned} I_i &= \frac{2}{\pi}\int_0^{a_i}\sqrt{2H_i - \omega_{i0}^2 q_i^2 - \frac{\alpha_i}{2}\,q_i^4}\,\mathrm{d}q_i \\ &= \frac{2\omega_{i0}^3}{3\pi\alpha_i}\sqrt[4]{1 + \frac{4\alpha_i H_i}{\omega_{i0}^4}}\left[\left(\sqrt{1 + \frac{4\alpha_i H_i}{\omega_{i0}^4}} + 1\right)K(r_i) - 2E(r_i)\right], \end{aligned}$$

$$\omega_i = \frac{\mathrm{d}I_i}{\mathrm{d}H_i} = \frac{\pi\sqrt{\alpha_i}}{2\sqrt{2}}\,\frac{\sqrt{a_i^2 + b_i^2}}{K(r_i)}, \tag{38}$$

$$\theta_i = \omega_i \int_{q_i}^{a_i} \frac{\mathrm{d}q_i}{\sqrt{2H_i - \omega_{i0}^2 q_i^2 - \frac{\alpha_i}{2} q_i^4}} = \frac{\pi}{2K(r_i)}\,F(\varphi_i, r_i),$$

$$q_i = a_i \,\mathrm{Cn}\left(\frac{2K(r_i)}{\pi}\,\theta_i\right) = a_i \sum_{n=1}^{\infty} C_{in}\cos(2n-1)\theta_i$$

$$\begin{aligned} p_i &= \sqrt{2H_i - \omega_{i0}^2 q_i^2 - \frac{\alpha_i}{2}\,q_i^4} \\ &= a_i\sqrt{\omega_{i0}^2 + \alpha_i a_i^2}\,S_n\left(\frac{2K(r_i)\theta_i}{\pi}\right)\sqrt{1 - r_i^2\,\mathrm{Sn}^2\left(\frac{2\kappa(r_i)\theta_i}{\pi}\right)} \\ &= a_i \sum_{n=1}^{\infty} P_{in}\sin(2n-1)\theta_i, \end{aligned}$$

$$p_i q_i = a_i^2 \sum_{n=1}^{\infty} e_{in}\sin(2n)\theta_i,$$

$$b_i^2 = \frac{\omega_{i0}^2}{\alpha_i}\left(\sqrt{1 + \frac{4\alpha_i H_i}{\omega_{i0}^4}} + 1\right), \quad a_i^2 = \frac{\omega_{i0}^2}{\alpha_i}\left(\sqrt{1 + \frac{4\alpha_i H_i}{\omega_{i0}^4}} - 1\right),$$

$$r_i = a_i \Big/ \sqrt{a_i^2 + b_i^2},$$

$$\varphi_i = \arccos\frac{q_i}{a_i}, \quad C_{in} = \frac{2\pi}{r_i K(r_i)}\,\frac{\mathrm{e}^{-(n-1/2)K(\sqrt{1-r_i^2})/K(r_i)}}{1 + \mathrm{e}^{-(2n-1)K(\sqrt{1-r_i^2})/K(r_i)}},$$

$$i = 1, 2,$$

where $K(r_i)$ and $E(r_i)$ are the complete elliptic integrals of the first and second kinds, respectively, for the ith oscillator; $F(\varphi_i, r_i)$ and $E(\varphi_i, r_i)$ are the elliptic integrals of the first and second kinds, respectively, for the ith oscillator. The differential equations for H_i and θ_i are of the form

$$\dot{H}_i = -p_i^2(\beta_{i1} + \beta_{i2}q_1^2 + \beta_{i3}q_2^2), \tag{39a}$$

$$\begin{aligned} \dot{\theta}_i = \omega_i &+ \left\{\omega_i q_i p_i\left[\frac{1}{a_i}\frac{\mathrm{d}a_i}{\mathrm{d}H_i} + \frac{r_i}{1-r_i^2}\left(1 - \frac{q_i^2}{a_i^2}\right)\frac{\mathrm{d}r_i}{\mathrm{d}H_i}\right]\right. \\ &\left. - \frac{\pi p_i^2[E(\varphi_i, r_i)K(r_i) - F(\varphi_i, r_i)E(r_i)]}{2K^2(r_i)r_i(1-r_i^2)} \times \frac{\mathrm{d}r_i}{\mathrm{d}H_i}\right\}(\beta_{i1} + \beta_{i2}q_1^2 + \beta_{i3}q_2^2) \\ &= \omega_i + g_i(\mathbf{q}, \mathbf{p}), \quad i = 1, 2. \end{aligned} \tag{39b}$$

The form and dimension of averaged equations of the system depend on whether the system is in resonance or not. Two special cases are considered in the following.

Case 1: Non-resonant case. In this case only two independent first integrals H_1 and H_2 are slowly varying quantities. Following the proposed method described in Section 3.3, the averaged equations for H_1 and H_2 are the form

$$\begin{aligned}\dot{H}_1 &= -\frac{1}{2}\left[\beta_{11}a_1^2\sum_{n=1}^{\infty}P_{1n}^2+\beta_{12}a_1^4\sum_{n=1}^{\infty}e_{1n}^2+\frac{\beta_{13}}{2}a_1^2a_2^2\left(\sum_{n=1}^{\infty}P_{1n}^2\right)\left(\sum_{n=1}^{\infty}C_{2n}^2\right)\right]\\ &= m_1(H_1,H_2),\\ \dot{H}_2 &= -\frac{1}{2}\left[\beta_{21}a_2^2\sum_{n=1}^{\infty}P_{2n}^2+\beta_{23}a_2^4\sum_{n=1}^{\infty}e_{2n}^2+\frac{\beta_{22}}{2}a_1^2a_2^2\left(\sum_{n=1}^{\infty}P_{2n}^2\right)\left(\sum_{n=1}^{\infty}C_{1n}^2\right)\right]\\ &= m_2(H_1,H_2).\end{aligned} \tag{40}$$

The steady-state solutions of the averaged equation (40) can be obtained by letting $\dot{H}_1=\dot{H}_2=0$. Four possible steady-state solutions $(a_1,a_2)=(0,0),(a_1^*,a_2^*),(0,a_2^0),(a_1^0,0)$ can be obtained. Solution $(0,0)$ is always unstable. Solution $(0,a_2^0)$ is stable if $\beta_{13}>\beta_{13c}$. Solution $(a_1^0,0)$ is stable if $\beta_{22}>\beta_{22c}$. Solution (a_1^*,a_2^*) is stable if $\beta_{13}<\beta_{13c}$ and $\beta_{22}<\beta_{22c}$. Critical values β_{13c}, β_{22c} can be determined by using the linearized equation of system (40) in the vicinity of the steady-state solutions.

The approximate steady-state amplitude response of system (37) without resonance obtained by using the proposed averaging method is shown in Fig. 3(a) using solid lines. β_{13} and β_{22} are taken their critical values $\beta_{13c}\approx 0.0516$ and $\beta_{22c}\approx 0.0526$, respectively. It can be seen form Fig. 3(a) that the analytical results agree well with those (denoted by •▲) from numerical solution of original system (37).

Case 2: Internal resonant case. Suppose that there exists primary internal resonant relation

$$\omega_1-\omega_2=\varepsilon\sigma, \tag{41}$$

where σ is a detuning parameter. Introduce combination of angle variables

$$\Phi=\theta_1-\theta_2. \tag{42}$$

The differential equation for Φ is of the form

$$\dot{\Phi}=\omega_1-\omega_2+g_1(\mathbf{q},\mathbf{p})-g_2(\mathbf{q},\mathbf{p}). \tag{43}$$

The averaged equation of H_1, H_2 and Φ can be obtained from Eqs. (39a) and (43) by averaging with respect to fast varying variable θ_2 as follows:

$$\begin{aligned}\dot{H}_1 &= m_1(H_1,H_2)+\sum_{n=1}^{\infty}\sigma_{1n}(H_1,H_2)\cos 2n\Phi,\\ \dot{H}_2 &= m_2(H_1,H_2)+\sum_{n=1}^{\infty}\sigma_{2n}(H_1,H_2)\cos 2n\Phi,\\ \dot{\Phi} &= \omega_1-\omega_2+\sum_{n=1}^{\infty}\sigma_{3n}(H_1,H_2)\sin 2n\Phi,\end{aligned} \tag{44}$$

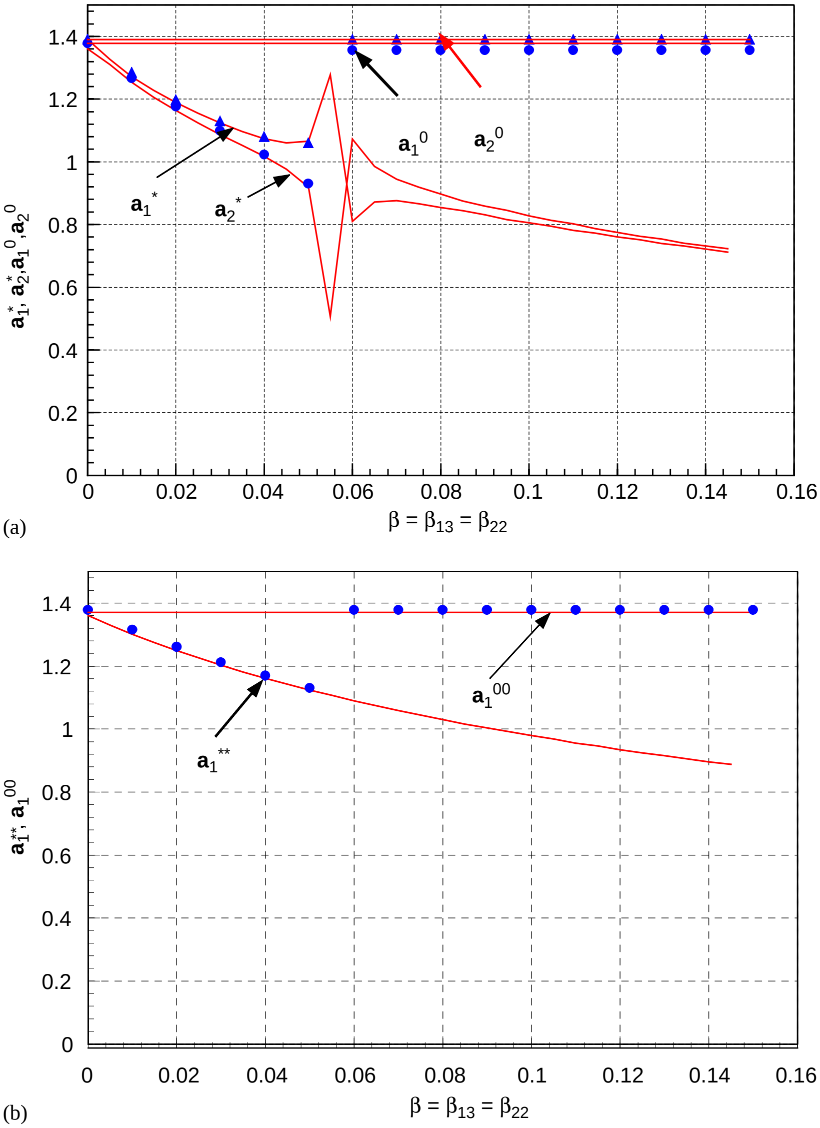

Fig. 3. Amplitude responses of nonlinearly coupled Duffing–van der Pol oscillators (37). $\omega_{01}=1$, $\alpha_1=\alpha_2=1$, $\beta_{11}=\beta_{21}=-0.05$, $\beta_{12}=\beta_{23}=0.1$: (a) non-resonant case, $\omega_{02}=1.5$, $\beta=\beta_{13}=\beta_{22}$; (b) resonant case, $\omega_{02}=1$, $\beta=\beta_{13}=\beta_{22}$;—by using the proposed averaging method; ●▲ from numerical solution of original system.

where $m_i(H_1, H_2)$ are defined in Eq. (40); $\sigma_{in}(H_1, H_2)$ are obtained from Eqs. (39a) and (43), respectively, by averaging with respect to θ_2.

The steady-state solution of averaged system (44) is obtained by letting $\dot{H}_1=\dot{H}_2=\dot{\Phi}=0$. There are four possible solutions $(a_1, a_2, \Phi)=(0,0,\pi), (a_1^{**}, a_1^{**}, \pi), (a_1^{00}, 0, \pi), (0, a_1^{00}, \pi)$ when $\beta_{12}=\beta_{23}>0, \beta=\beta_{13}=\beta_{22}>0, \omega_{10}=\omega_{20}, \alpha_1=\alpha_2$. Solution $(0,0,\pi)$ is always unstable. Solutions

$(a_1^{00}, 0, \pi)$ and $(0, a_1^{00}, \pi)$ are stable if $\beta > \beta_c$ while solution $(a_1^{**}, a_1^{**}, \pi)$ is stable if $\beta < \beta_c$ The approximate steady-state amplitude response of system (37) in resonant case obtained by using the proposed averaging method is shown in Fig. 3(b) with solid lines. The critical values of β_{13} and β_{22} for this special case are $\beta_{13c} = \beta_{22c} = \beta_c \approx 0.0526$. It can be seen form Fig. 3(b) that the analytical results agree well with those from numerical solution of original system (37) denoted by •.

5. Concluding remarks

A deterministic averaging method for quasi-integrable Hamiltonian systems has been developed in the present paper. The method can be applied to predict the approximate response of mdof autonomous or non-autonomous strongly nonlinear systems. The form and dimension of the averaged equation depend on the number of dof and the number of resonant relations of the systems. The averaged equations of the systems have been constructed for both non-resonant and resonant cases. The proposed procedure has been applied to predict the approximate steady-state amplitude response of a Duffing oscillator with hardening spring under additive harmonic excitation. The analytical results agree well with those from the numerical solution of the original equation and good results can be obtained even for much larger nonlinearity intensity. The proposed averaging method has also been successfully applied to predict the approximate steady-state responses of nonlinearly coupled Duffing–van der Pol oscillators with or without resonance. It has been shown using these examples that the proposed averaging method works well for the systems with very strong nonlinearity.

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