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Short Communication

DSC analysis of free-edged beams by an iteratively matched boundary method

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Abstract

This communication introduces a novel scheme for the treatment of free edge supports in the analysis of beams by using the discrete singular convolution (DSC) algorithm. Accommodating free edges has been a challenging issue in the DSC analysis of beams, plates, and shells. An iteratively matched boundary (IMB) method is proposed to overcome the difficulty. Numerical experiments are carried out to demonstrate that the proposed IMB method works very well in dealing with arbitrary combinations of beam edge supports.

Recently, the discrete singular convolution (DSC) algorithm has emerged as a wavelet collocation scheme for the computer realization of singular convolutions [1,2]. The underlying mathematical structure of the DSC algorithm is the theory of distributions [3] and wavelet analysis. The DSC algorithm has the global methods' accuracy and the local methods' flexibility for handling complex geometry and boundary conditions in the analysis of fluid dynamics [4] and electrodynamics problems [5]. It is a simple and robust approach for structural analysis, and has found its success in structural analysis, including the vibration and buckling of beams [6], plate vibration under various edge and internal supports [7–14]. The DSC algorithm was extensively

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validated by analytical and exact solutions and carefully compared with other existing methods, including the series expansion [15], integral equation approach [16,17], finite strip and finite element methods [18–20], Galerkin methods [21], differential quadrature methods [22–24], and Rayleigh–Ritz variational methods [25–27]. The most distinguish feature of the DSC algorithm is its high level of accuracy and reliability. At present, it is the only available method that is able to accurately predict thousands of vibration modes of plates and beams without encountering numerical instability [11,12]. The DSC treatment of structures relies on the use of fictitious points (FPs) outside the computational domain. As such, it handles well the simply supported, clamped and transversely supported edges and their arbitrary combinations. However, the DSC method loses its high order of accuracy at the boundary when dealing with the free edge support. Consequently, difficulty of accommodating free edges has been a pressing issue in the DSC analysis of structures.

More recently, another fictitious grid scheme, called local adaptive differential quadrature methods (LaDQMs), was proposed for treating multiple boundary conditions raised in high-order differential equations [28]. The LaDQM admits the same number of FPs as the boundary conditions at an edge. It is capable of dealing with various boundary conditions, including free edges. However, the LaDQM loses the main feature of the original DSC, and can only be regarded as a differential quadrature method. In this communication, we propose a novel boundary approach, the iteratively matched boundary (IMB) method, to overcome the aforementioned difficulty in the DSC algorithm. The IMB method repeatedly utilizes the given boundary conditions to generate a sufficiently large number of FPs so that a translation invariant DSC kernel can be correctly implemented near the free edges. The present IMB method is reformulated from a recently developed interface scheme, the iteratively matched interface (IMI) [29], originally proposed for simulating electromagnetic wave scattering and propagation in inhomogeneous media. For simplicity, we utilize the Euler–Bernoulli beam as an example to illustrate the IMB method, albeit the proposed IMB can be easily extended to the DSC analysis of plates and shells with free edge supports.

For an Euler–Bernoulli beam of length L, the governing equation for free vibration is

$$\frac{\mathrm{d}^4 \bar{W}}{\mathrm{d}X^4} = \frac{\rho A \omega^2}{EI} \bar{W}, \tag{1}$$

where $\bar{W}$ is the transverse displacement of the beam, X is the Cartesian coordinate in the middle axis of the beam, ρ is the mass density of the beam, A is the cross-section area of the beam, ω is the angular frequency of the beam, E is Young's modulus and I is the second moment area about the neutral axis. For generality, the dimensionless governing equation is used,

$$\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} = \Omega^2 W, \tag{2}$$

where W is the dimensionless displacement ($W = \bar{W}/L$), Ω is the dimensionless frequency parameter ($\Omega = \omega(\rho A/EI)^{1/2}$), and x is the dimensionless coordinate along the X-direction ($x = X/L$). We choose the domain $[0, \pi]$.

In this communication, three types of boundary conditions are considered in terms of the dimensionless parameters and coordinates.

- Simply-supported edge (S):

$$W = 0, \quad \frac{d^2 W}{dx^2} = 0. \tag{3}$$

- Clamped edge (C):

$$W = 0, \quad \frac{dW}{dx} = 0. \tag{4}$$

- Free edge (F):

$$\frac{d^2 W}{dx^2} = 0, \quad \frac{d^3 W}{dx^3} = 0. \tag{5}$$

Only three combinations of these three edge supports, i.e., SF, CF, and FF, are considered in numerical experiments, as all other combinations pose no difficulty for the original DSC analysis.

In the DSC algorithm, a function and its nth order derivative are usually approximated via a discretized convolution

$$W^{(n)}(x) \approx \sum_{k=-M}^{M} \delta_{\alpha,\sigma}^{(n)}(x - x_k) W(x_k), \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $2M + 1$ is the computational bandwidth and $\delta_{\alpha,\sigma}(x - x_k)$ is a collective symbol for the (regularized) DSC kernels. The higher order derivative terms $\delta_{\alpha,\sigma}^{(n)}(x - x_k)$ in Eq. (6) are given by

$$\delta_{\alpha,\sigma}^{(n)}(x - x_k) = \left(\frac{d}{dx}\right)^n \delta_{\alpha,\sigma}(x - x_k). \tag{7}$$

Here, the differentiation can be carried out analytically. Numerical solution of differential equations can be easily implemented by a collocation scheme using Eq. (6).

Although many other DSC kernels can be similarly employed, the regularized Shannon's kernel (RSK) [1] is employed in the present study,

$$\delta_{h,\sigma}(x - x_k) = \frac{\sin(\pi/h)(x - x_k)}{(\pi/h)(x - x_k)} e^{-(x - x_k)^2/2\sigma^2}, \tag{8}$$

where $h = \pi/(N - 1)$ is the grid spacing and N is the number of grid points. Here the parameter σ determines the width of the Gaussian envelop and often varies in association with the grid spacing, i.e., $\sigma = rh$, where r is a parameter chosen in computation. It has been proved that the truncation error of the DSC algorithm using the RSK decays exponentially with respect to the increase in sampling points [30].

As the DSC kernel is symmetric, the DSC computation requires a total of M FPs outside each edge. More precisely, it requires function values on these FPs so that the derivative approximation (6) near the boundary can be carried out. For the simply supported edge, Eq. (3) and the clamped edge, Eq. (4), antisymmetric extension and symmetric extension are conducted, respectively, to obtain the function values on FPs from those inside the boundary. It has been shown [13,14] that Eqs. (3) and (4) are satisfied by these treatments.

For the free edge, function values on M FPs are to be determined from two boundary conditions in Eq. (5). This appears impossible as $M \geqslant 2$. Hereby, we introduce an iterative procedure to overcome this difficulty.

The IMB modeling introduces a fictitious domain outside the boundary (see Fig. 1), and matches boundary conditions across the boundary. At the first step, since only two boundary conditions are available, one can only determine two FPs. In order to achieve higher order accuracy for the boundary condition, one-sided finite difference (FD) approximations are considered, which involve L grid points on the inner side of the boundary; see Fig. 1. Thus, boundary conditions (5) are approximated by using an $L+3$ points FD scheme,

$$w_1^{(k)} f_2 + w_2^{(k)} f_1 + \sum_{i=3}^{L+3} w_i^{(k)} g_{i-2} = 0, \tag{9}$$

where f_1 and f_2 are unknown FP function values and g_i for $i = 1, \dots, L+1$ are the known function values on the right-hand side of the boundary. Here $w_i^{(k)}$ for $i = 1, \dots, L+3$, and $k = 2, 3$ are one-sided FD weights [29], with its differentiation at point 3. The superscript (k) represents the second- or third-order derivative approximation, and i is the grid index. The two FP values can be solved from Eq. (9). For free boundary conditions, $L \geqslant 2$ and $M \geqslant 2$ are required. While the length of L determines the level of accuracy, it can be either larger or smaller than M. For simplicity, L and M are considered as even integers.

To gain a sufficient number of function values at FPs, we use an iterative procedure as introduced in the IMI method [29]. By treating the previous calculated FPs as knowns, we seek two more FPs as shown in Fig. 2. Numerically, this is accomplished by discretizing the same two boundary conditions again, but with two new FPs,

$$\sum_{i=1}^{4} w_i^{(k)} f_{5-i} + \sum_{i=5}^{L+5} w_i^{(k)} g_{i-4} = 0, \tag{10}$$

where f_1 and f_2 have already been determined from Eq. (9), while f_3 and f_4 can be solved. The one-sided FD approximations used can be similarly formulated, as in Eq. (9). Through such an iterative procedure, the requested M FPs can be efficiently determined. Here, we note the flexibility of choosing the total number of terms used by varying L in the FD approximation at each step of the iteration. In order to apply the IMB method to an eigenvalue problem, a fundamental representation is essential for an implicit formulation [29],

$$f_i = \sum_{j=1}^{L+1} r_{i,j} g_j \quad \text{for } i = 1, 2, \dots, M, \tag{11}$$

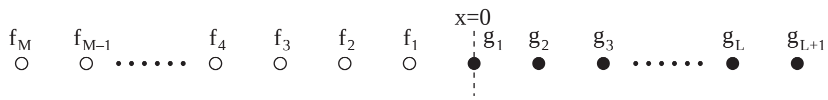

Fig. 1. Illustration of FPs near the left boundary.

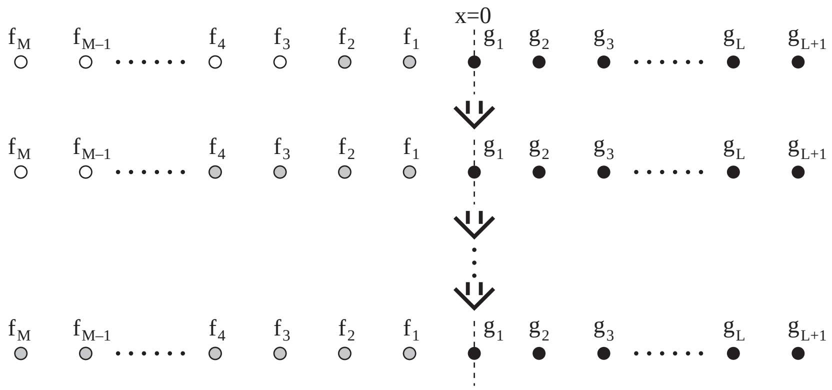

Fig. 2. Illustration of the iterative procedure.

Table 1
Comparison of exact and DSC solutions of the frequency parameters$\sqrt{\omega}$ for the SF, CF, and FF beams

Mode	SF			CF			FF		
	Exact	DSC		Exact	DSC		Exact	DSC	
		$N=21$	$N=31$		$N=21$	$N=31$		$N=21$	$N=31$
1	1.2499	1.2499	1.2499	0.5969	0.5969	0.5969	1.5056	1.5056	1.5056
2	2.2500	2.2500	2.2500	1.4942	1.4942	1.4942	2.4998	2.4998	2.4998
3	3.2500	3.2500	3.2500	2.5002	2.5004	2.5003	3.5000	3.5000	3.5000
4	4.2500	4.2500	4.2500	3.5000	3.5005	3.5001	4.5000	4.5001	4.5000
5	5.2500	5.2500	5.2500	4.5000	4.5011	4.5003	5.5000	5.5000	5.5000
6	6.2500	6.2497	6.2500	5.5000	5.5019	5.5005	6.5000	6.4988	6.5000
7	7.2500	7.2478	7.2500	6.5000	6.5025	6.5008	7.5000	7.4937	7.4999
8	8.2500	8.2426	8.2500	7.5000	7.5018	7.5013	8.5000	8.4810	8.5000
9	9.2500	9.2334	9.2501	8.5000	8.4979	8.5019	9.5000	9.4607	9.5005
10	10.2500	10.2255	10.2508	9.5000	9.4917	9.5029	10.5000	10.4447	10.5023

where $r_{i,j}$ are representation coefficients. Consequently, instead of finding f_i, one will implicitly determine the FP coefficients through the boundary modeling discussed above. By means of all obtained FP coefficients, the differentiation near the boundary can be carried out in an *implicit* manner. Note that in the IMB modeling, boundary conditions are enforced systematically so that it can achieve arbitrarily high orders in principle. A standard eigen value solver is employed to attain eigenvalues and eigenfunctions from the matrix formed by the DSC for discretization and by the present IMB for boundary treatment.

Numerical experiments are conducted with two grid sizes, $N = 21$ and 31, for the SF, CF, and FF beams. We set $M = 20$, $L = 8$ and $r = 2.65$ for $N = 21$, while $M = 24$, $L = 10$ and $r = 2.94$ for $N = 31$. Results are compared with the exact ones generated by solving the differential equation (2) directly. It is seen from Table 1 that while only a small number of grid points

($N = 21$) is used, the DSC-IMB solutions are very accurate and reliable up to the 10th mode. Moreover, when the grid size is slightly larger ($N = 31$), the numerical accuracies are further improved significantly. This indicates that the convergence rate of the DSC-IMB analysis is extremely high.

In summary, this communication addresses a long-standing difficulty in the DSC analysis of structures, i.e., the treatment of free boundary conditions. A novel high-order boundary scheme, the iteratively matched boundary (IMB) method, is proposed for the DSC analysis of beams with free edge supports. In the present method, the boundary conditions are repeatedly utilized to systematically determine a large number of function values at fictitious points (FPs). Consequently, translation invariant DSC kernels can be applied near the free edge. The typical high-order accuracy of the DSC approach is retained. We note that the IMB method proposed in this work can be easily employed for plate and shell analyses. Moreover, the present procedure can be generalized to deal with other complex boundary conditions, such as the Robin boundary condition and multiple boundary conditions occurring in high order differential equations.

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