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A root-finding technique to compute eigenfrequencies for elastic beams

Miguel Angel Moreles[a,*], Salvador Botello[a], Rogelio Salinas[b]

[a]*CIMAT, A.P. 402, Callejón Jalisco S/N, Valenciana, Guanajuato, GTO 36240, Mexico*
[b]*Universidad Autónoma de Aguascalientes (UAA), Av. Universidad #940 C.P. 20100, Aguascalientes, Ags., Mexico*

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Abstract

In this manuscript a method to compute eigenfrequencies for elastic beams is presented. For beam modeling, three fundamental effects are considered: bending, rotary inertia and shear deformation. The method consists on enclosing each eigenfrequency in an interval where the characteristic function is monotonic. Then, a root finding technique is used to compute the eigenfrequency to any desired accuracy. The method is applied successfully to equations involving bending and either rotary inertia or shear deformation.

1. Introduction

Flexural motion of elastic beams is a problem of interest in structural engineering. In particular, engineers need to calculate the natural frequencies, or eigenfrequencies of beam elements. The reason is that another part of the system may force it to vibrate at a frequency near one of its natural frequencies. If so, resonance brings about a large amplification of the forcing amplitude with potentially disastrous consequences.

The most realistic and accurate approach for computing eigenfrequencies is to model the elastic beam based on the fundamentals of elasticity theory, then compute eigenfrequencies by means of

*Corresponding author. Tel.: +52 473 732 7155x49568; fax: +52 473 732 5749.
E-mail addresses: moreles@cimat.mx (M.A. Moreles), botello@cimat.mx (S. Botello), rsalinas@correo.uaa.mx (R. Salinas).

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the finite element method (FEM). The model is three dimensional and consequently, the computational cost is high.

In applications, one-dimensional models are preferred. Three fundamental effects are considered; bending, rotary inertia and shear deformation. All effects are considered in the *Timoshenko equation* (*TE*)

$$\rho A\,\frac{\partial^2 Y}{\partial t^2} - \rho I\,\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\,\frac{\partial^4 Y}{\partial x^4} + \frac{\rho I}{KG}\left(\rho\,\frac{\partial^4 Y}{\partial t^4} - E\,\frac{\partial^4 Y}{\partial t^2 \partial x^2}\right) = 0.$$

Here $Y(x,t)$ represents the vertical displacement of the *elastic axis* of the beam. The physical constants in the model are: ρ, density; A, cross-sectional area; E, Young's modulus; G, shear modulus; I, second moment of area and K, shear coefficient. A physical derivation of this equation is presented in Ref. [1]. The modeling aspects are also presented in Refs. [2,3].

In this equation $-\rho I \partial^4 Y/\partial t^2 \partial x^2$ is the contribution of rotary inertia and the term due to shear deformation is $\rho I/KG(\rho \partial^4 Y/\partial t^4 - E\,\partial^4 Y/\partial t^2 \partial x^2)$. If both effects are neglected the well known Euler–Bernoulli (E–B) equation is obtained:

$$\rho A\,\frac{\partial^2 Y}{\partial t^2} + EI\,\frac{\partial^4 Y}{\partial x^4} = 0.$$

For the E–B equation, Chen and Coleman [4] apply the wave propagation method (WPM) to estimate high-order eigenfrequencies. By means of a formal perturbation approach the estimates are improved to include all low order eigenfrequencies. An alternative is presented here. It will be shown that each eigenfrequency is contained in an interval where the characteristic function associated with the time-reduced form of the equation is monotonic. Consequently, eigenfrequencies can be found by a simple iterative method to any desired accuracy. An advantage of this approach is that it generalizes to more general beam equations. In particular, to quasi-TEs, that is, equations which involve bending and either rotary inertia or shear deformation. These equations are, respectively,

$$\rho A\,\frac{\partial^2 Y}{\partial t^2} - \rho I\,\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\,\frac{\partial^4 Y}{\partial x^4} = 0 \tag{1}$$

and

$$\rho A\,\frac{\partial^2 Y}{\partial t^2} - \frac{\rho EI}{KG}\,\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\,\frac{\partial^4 Y}{\partial x^4} = 0. \tag{2}$$

Eq. (1) is also known as the Rayleigh equation. Eq. (2) shall be referred as the B + S equation.

Computing eigenfrequencies involves the solution of an eigenvalue problem for a differential operator. To make the problem well posed, boundary conditions need to be prescribed. Following Chen and Coleman [4] the following configurations are considered: clamped–clamped (C–C), clamped–simply supported (C–S), clamped–roller supported (C–R) and clamped–free (C–F). It will become apparent that the method applies to any other configuration. For cross-validation, eigenfrequencies are computed with FEM and with the method to be introduced.

An extensive comparative study of elastic beams, and computation of eigenfrequencies, is carried out in Ref. [5]. There, all numerical tables are presented for the different beam models and configurations below. The benchmark for comparison are the eigenfrequencies of 3-D specimens for a collection of materials and geometries computed with 3-D FEM.

The outline of this work is as follows.

The eigenvalue problem for the TE is the content of Section 2. There, the mathematical formulation of the problem is presented, and the quasi-TEs with corresponding eigenvalue problems are introduced. Equations are in dimensionless form for computation.

In Section 3, the method to compute eigenfrequencies based on a root-finding technique (RFT) is introduced. It is developed in the context of the E–B equation in the C–C configuration. For comparison, a simplified version of WPM is presented.

In Section 4, the same analysis is shown for the quasi-TEs. Frequencies are normalized, thus frequencies for an actual beam can be easily derived.

Extension of this work, as well as some problems for future research are part of the content of Section 5.

2. The eigenvalue problems

Recall the TE,

$$\rho A\,\frac{\partial^2 Y}{\partial t^2}-\rho I\,\frac{\partial^4 Y}{\partial t^2\partial x^2}+EI\,\frac{\partial^4 Y}{\partial x^4}+\frac{\rho I}{KG}\left(\rho\frac{\partial^4 Y}{\partial t^4}-E\,\frac{\partial^4 Y}{\partial t^2\partial x^2}\right)=0.$$

Under harmonic motion

$$Y(x,t)=y(x)\mathrm{e}^{-\mathrm{i}\omega t}.$$

It follows that

$$-\rho A\omega^2 y+\rho I\omega^2\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+EI\,\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+\frac{\rho I}{KG}\,\omega^2\left(\rho\omega^2 y+E\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0. \tag{3}$$

In dimensionless form $\xi=x/L$, $\eta=y/L$, $\phi^2=(\rho A\omega^2L^4)/EI$, $\alpha=EI/(KGAL^2)$ and $\beta=I/(AL^2)$. Eq. (3) then becomes

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}\xi^4}+\phi^2(\alpha+\beta)\,\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2}-\phi^2(1-\phi^2\alpha\beta)\eta=0. \tag{4}$$

The following boundary conditions are of interest: (A) displacement zero, $\eta=0$; (B) total slope zero, $\mathrm{d}\eta/\mathrm{d}\xi=0$; (C) moment zero, $\mathrm{d}^2\eta/\mathrm{d}\xi^2+\phi^2\alpha\eta=0$ and (D) shear zero, $\mathrm{d}^3\eta/\mathrm{d}\xi^3+\phi^2(\alpha+\beta)\mathrm{d}\eta/\mathrm{d}\xi=0$.

To make the eigenvalue problem well posed, two boundary conditions need to be prescribed at both ends. In reference to this, consider the following conditions for any end of the beam: clamped (C): A, B; simply supported (S): A, C; roller supported (R): B, D and free (F): C, D.

The eigenvalue problem consists of finding ϕ, such that there is a non-trivial solution η of Eq. (4) subject to appropriate boundary conditions. As mentioned above, the configurations to be considered are: C–C, C–S, C–R, C–F.

Remark. (1) If $\alpha = 0$ ($\beta = 0$) the eigenvalue problems for the Rayleigh (B+S) equation are obtained.

(2) A boundary condition also of interest, but not considered here, is slope due to bending only zero, $\alpha \mathrm{d}^3\eta/\mathrm{d}\xi^3 + (1 + \phi^2\alpha^2)\mathrm{d}\eta/\mathrm{d}\xi = 0$.

(3) With the appropriate boundary conditions, the eigenvalue problem for any of the quasi-TEs has eigenvalues $0 < \phi_1 < \phi_2 < \cdots < \phi_n$, with $\phi_n \nearrow \infty$.

3. Computation of eigenfrequencies

Eigenfrequencies are the roots of transcendental equations. Roughly speaking, WPM approximates these transcendental equations, by equations that are solved in explicit form. In this section, a review of the method for the E–B equation in the C–C configuration is presented. For the same model, an RFT to compute eigenfrequencies is introduced.

3.1. The WPM

To illustrate the WPM consider the E–B equation

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}\xi^4} - \phi^2\eta = 0$$

subject to C–C conditions

$$\eta(0) = \eta'(0) = \eta(1) = \eta'(1) = 0. \tag{5}$$

For simplicity write

$$\eta^{(4)}(\xi) - k^4\eta(\xi) = 0, \quad 0 < x < 1, \tag{6}$$

where $k^2 = \phi$, $k > 0$.

The eigenvalue problem, therefore, consists of finding all values of k for which there is a non-trivial function η, solution of Eq. (6), subject to the boundary conditions given in Eq. (5).

It is well known that the eigenvalue problem does not have any closed-form solutions. A straightforward approach to determine k is as follows. For $k > 0$ the general solution of Eq. (6) is

$$\eta(\xi) = A\mathrm{e}^{\mathrm{i}k\xi} + B\mathrm{e}^{-\mathrm{i}k\xi} + C\mathrm{e}^{-k\xi} + D\mathrm{e}^{k(\xi-1)}. \tag{7}$$

Substituting this equation into the C–C boundary conditions in Eq. (5), one obtains

$$\begin{bmatrix} 1 & 1 & 1 & \mathrm{e}^{-k} \\ \mathrm{i}k & -\mathrm{i}k & -k & k\mathrm{e}^{-k} \\ \mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{-\mathrm{i}k} & \mathrm{e}^{-k} & 1 \\ \mathrm{i}k\mathrm{e}^{\mathrm{i}k} & -\mathrm{i}k\mathrm{e}^{-\mathrm{i}k} & -k\mathrm{e}^{-k} & k \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order to have a non-trivial solution, k satisfies the transcendental equation determined by the zero determinant condition

$$\begin{vmatrix} 1 & 1 & 1 & e^{-k} \\ ik & -ik & -k & ke^{-k} \\ e^{ik} & e^{-ik} & e^{-k} & 1 \\ ike^{ik} & -ike^{-ik} & -ke^{-k} & k \end{vmatrix} = 0 \tag{8}$$

or, after simplification

$$-2k^2\cos k + 4k^2 e^{-k} - 2k^2 \cos k e^{-2k} = 0.$$

Hence, the roots of the equation

$$-\cos k + 2e^{-k} - \cos k e^{-2k} = 0 \tag{9}$$

are needed.

An expression of k from Eq. (9) is not possible; an asymptotic approach to estimate the solution by means of the WPM is shown below.

Observe that in Eq. (7) for k large, the third term $e^{-k\xi}$ is negligible for $\xi = 1$, whereas the same is true for the fourth term $e^{-k(\xi-1)}$ if $\xi = 0$. Hence the function $\eta(\xi)$ behaves like $Ae^{ik\xi} + Be^{-ik\xi} + Ce^{-k\xi}$ for ξ near zero, and like $Ae^{ik\xi} + Be^{-ik\xi} + De^{k(\xi-1)}$ for ξ near one. This suggests to consider as zero the terms involving e^{-k} in the determinant equation (8). Thus, the determinant equation is

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ ik & -ik & -k & 0 \\ e^{ik} & e^{-ik} & 0 & 1 \\ ike^{ik} & -ike^{-ik} & 0 & k \end{vmatrix} = 0.$$

After some simplification, we are led to solve for k the equation

$$\cos k = 0.$$

Consequently, the eigenvalue problem

$$\eta^{(4)}(\xi) - k^4\eta(\xi) = 0, \quad 0<\xi<1,$$
$$\eta(0) = \eta'(0) = \eta(1) = \eta'(1) = 0$$

has a non-trivial solution η when

$$\phi^2 \approx k^4 = \left[(2n+1)\frac{\pi}{2}\right]^4, \quad n = 1, 2, \ldots$$

or

$$\phi \approx k^2 = \left[(2n+1)\frac{\pi}{2}\right]^2, \quad n = 1, 2, \ldots.$$

It can be seen from Table 1, that the frequencies in this expression are good estimates except for a few of the smallest eigenvalues.

Table 1
Eigenfrequencies for the C–C beam, as estimated by the 1-D FEM and the WPM

Freq.	1-D FEM	WPM
1	22.37329	22.20660
2	61.67282	61.68503
3	120.90339	120.90265
4	199.85946	199.85949
5	298.55557	298.55553
6	416.99089	416.99079
7	555.16548	555.16525
8	713.07941	713.07892
9	890.73277	890.73180
10	1088.12565	1088.12389

Remark. The same conclusion holds for other boundary conditions. That is, the WPM fails only for a few low-order eigenfrequencies.

3.2. Eigenfrequencies for the E–B elastic beam by bracketing

In this paragraph, the approximation of the eigenvalues is improved by applying a simple iterative method. The technique is illustrated with the E–B equation in the C–C case.

Consider the real variable analog of Eq. (7), namely

$$\eta(\xi) = A\cos k\xi + B\sin k\xi + C\mathrm{e}^{-k\xi} + D\mathrm{e}^{k(\xi-1)}$$

and substitute the C–C boundary conditions to obtain

$$\begin{bmatrix} 1 & 0 & 1 & \mathrm{e}^{-k} \\ 0 & k & -k & k\mathrm{e}^{-k} \\ \cos k & \sin k & \mathrm{e}^{-k} & 1 \\ -k\sin k & k\cos k & -k\mathrm{e}^{-k} & k \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$f_d(k) = \det \begin{bmatrix} 1 & 0 & 1 & \mathrm{e}^{-k} \\ 0 & k & -k & k\mathrm{e}^{-k} \\ \cos k & \sin k & \mathrm{e}^{-k} & 1 \\ -k\sin k & k\cos k & -k\mathrm{e}^{-k} & k \end{bmatrix},$$

thus

$$f_d(k) = k^2 \det \begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & 1 & -1 & e^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -\sin k & \cos k & -e^{-k} & 1 \end{bmatrix}$$

or

$$f_d(k) = -2k^2[(1 + e^{-2k})\cos k - 2e^{-k}]. \tag{10}$$

From Eq. (10) it suffices to find zeros of the function

$$f(k) = (1 + e^{-2k})\cos k - 2e^{-k}. \tag{11}$$

It is readily seen that in the intervals

$$n\pi \leqslant k \leqslant (n+1)\pi, \quad n = 1, 2, \ldots, \tag{12}$$

$f(k)$ is strictly monotone and $f(n\pi)f((n+1)\pi) < 0$; hence, there is only one root, k_n, of $f(k)$ in such intervals.

Recall that $\phi = k^2$, hence ϕ is monotone on k. Thus, interval (12) provides asymptotic estimates for the eigenfrequencies

$$n^2\pi^2 \leqslant \phi_n \leqslant (n+1)^2\pi^2, \quad n = 1, 2, \ldots.$$

The roots of function (11) can be found by bisection to any desired accuracy. The method is denoted by RFT. See the results in Table 2 for the natural frequencies.

Remark. (1) By considering the full determinant function, the RFT is more accurate than the 1-D FEM. Starting with the fourth eigenfrequency, there is a slight difference in the estimates. This is

Table 2
Eigenfrequencies for the C–C beam, as estimated by the 1-D FEM and the RFT

Freq.	1-D FEM	RFT
1	22.37329	22.37329
2	61.67282	61.67282
3	120.90339	120.90339
4	199.85946	199.85945
5	298.55557	298.55554
6	416.99089	416.99079
7	555.16548	555.16525
8	713.07941	713.07892
9	890.73277	890.73180
10	1088.12565	1088.12389

due to the accumulation of error when solving the generalized eigenvalue problem arising from FEM.

(2) For faster convergence, the Newton method can be used to approximate the roots of function (11).

4. Quasi-TEs

The quasi-TEs are

$$\frac{d^4\eta}{d\xi^4} + \phi^2\beta\,\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0,$$
$$\frac{d^4\eta}{d\xi^4} + \phi^2\alpha\,\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0.$$

Both models have the form

$$\frac{d^4\eta}{d\xi^4} + \gamma\phi^2\,\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0. \tag{13}$$

Unlike the E–B equation, some work needs to be carried out to estimate intervals enclosing eigenfrequencies for Eq. (13). It is required to study the mode of vibration associated with this equation.

Consider the characteristic polynomial associated with Eq. (13), namely

$$P(r) = r^4 + \gamma\phi^2 r^2 - \phi^2.$$

It has four roots:

$$r_1 = -r_2 = -i\lambda, \quad r_3 = -r_4 = -\mu,$$

where

$$\lambda = \sqrt{\frac{1}{2}\gamma\phi^2 + \frac{1}{2}\sqrt{(\gamma^2\phi^4 + 4\phi^2)}}, \quad \mu = \sqrt{-\frac{1}{2}\gamma\phi^2 + \frac{1}{2}\sqrt{(\gamma^2\phi^4 + 4\phi^2)}}. \tag{14,15}$$

Thus, the mode of vibration $\eta(\xi)$ is given by

$$\eta(\xi) = A\cos\lambda\xi + B\sin\lambda\xi + Ce^{-\mu\xi} + De^{\mu(\xi-1)}.$$

Intervals for the eigenfrequencies will be given in terms of λ. Let us deduce some properties of λ and μ as functions of ϕ.

It can be seen that

$$\lambda^2 - \mu^2 = \gamma\phi^2, \quad \mu\lambda = \phi. \tag{16}$$

From Eq. (14) it is readily seen that $\lambda'(\phi) > 0$, hence λ is strictly increasing and unbounded. It can be inverted to obtain

$$\phi^2 = \frac{\lambda^4}{1 + \gamma\lambda^2}. \tag{17}$$

For μ we can write

$$\mu^2 = \frac{2}{\gamma + \sqrt{\gamma^2 + 4/\phi}}.$$

It follows that μ is also strictly increasing with respect to ϕ. Moreover,

$$\mu \to \frac{1}{\sqrt{\gamma}}, \quad \text{when } \phi \nearrow \infty. \tag{18}$$

Because of Eq. (18), the term $e^{-\mu}$ does not tend to zero with ϕ, unlike the corresponding term for the E–B beam. Nevertheless, for actual beams, γ is small, thus $e^{-\mu}$ is small and decreases to $e^{-1/\sqrt{\gamma}}$. Thanks to these properties, we will be able to consider $e^{-\mu}$ negligible.

Observe that for each frequency ϕ_n there is a unique $\lambda_n > 0$ obtained from Eq. (17). Consequently, when finding an interval for λ_n, a corresponding interval for ϕ_n follows.

Next, intervals for enclosing λ_n for Rayleigh equation in all configurations are provided. As before, f_d denotes the full determinant function of the beam in consideration, and f the function for root finding. We list f_d, f, and the intervals enclosing the eigenfrequencies. When necessary, following properties of λ and μ, additional details are provided.

4.1. The Rayleigh equation

The characteristic functions for the Rayleigh equation in all configurations involve the terms $\cos\lambda$, $\sin\lambda$ and $e^{-\mu}$. In essence, intervals enclosing the eigenfrequencies are determined by comparing the sign of $\cos\lambda$ and $\sin\lambda$. As remarked before, the term $e^{-\mu}$ is negligible.

4.1.1. The C–C case

$$f_d = \phi[-\beta\phi(1 - e^{-2\mu})\sin\lambda - 2(1 + e^{-2\mu})\cos\lambda + 4e^{-\mu}],$$

$$f(\lambda) = \phi\beta(1 - e^{-2\mu})\sin\lambda + 2(1 + e^{-2\mu})\cos\lambda - 4e^{-\mu},$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda_n < (1 + n)\pi, \quad n = 1, 2, 3, \ldots.$$

4.1.2. The C–S case

$$f_d(\lambda) = (\mu^2 + \lambda^2)[(1 + e^{-2\mu})\mu\sin\lambda - (1 - e^{-2\mu})\lambda\cos\lambda],$$

$$f(\lambda) = (1 + e^{-2\mu})\mu\sin\lambda - (1 - e^{-2\mu})\lambda\cos\lambda,$$

$$n\pi < \lambda_n < \left(\frac{1}{2} + n\right)\pi, \quad n = 1, 2, 3, \ldots.$$

4.1.3. The C–R case

$$f_d(\lambda) = \lambda\mu(\mu^2 + \lambda^2)(\mu\cos\lambda(1 - e^{-2\mu}) + \lambda\sin\lambda(1 + e^{-2\mu})),$$

$$f(\lambda) = (\mu\cos\lambda(1 - e^{-2\mu}) + \lambda\sin\lambda(1 + e^{-2\mu})),$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda_n < (1 + n)\pi, \quad n = 0, 1, 2, \ldots .$$

4.1.4. The C–F case

$$f_d(\lambda) = \mu\lambda[(\mu^4 + \lambda^4)(1 + e^{-2\mu})\cos\lambda - \lambda\mu(\lambda^2 - \mu^2)(1 - e^{-2\mu})\sin\lambda + 4\lambda^2\mu^2 e^{-\mu}],$$

but from Eq. (16),

$$f_d(\lambda) = \mu\lambda\phi^2[(2 + \beta\phi^2)(1 + e^{-2\mu})\cos\lambda - \phi\beta(1 - e^{-2\mu})\sin\lambda + 4e^{-\mu}],$$

$$f(\lambda) = (2 + \beta\phi^2)(1 + e^{-2\mu})\cos\lambda - \phi\beta(1 - e^{-2\mu})\sin\lambda + 4e^{-\mu},$$

$$\frac{\pi}{2} < \lambda_1 < \pi,$$

$$n\pi < \lambda_{n+1} < \left(\frac{1}{2} + n\right)\pi, \quad n = 1, 2, 3, \ldots .$$

4.2. The E–B and B+S equations

Intervals containing the eigenfrequencies for the E–B equation are found by letting $\beta = 0$ in the expressions above. In this case $\mu = \lambda = k$.

Notice that for the C–C and C–R configurations, the eigenvalue problems for the Rayleigh and B + S equations are identical. To find the intervals for the B + S equation in these cases, just substitute α instead of β in the corresponding expressions.

In practice there are other configurations of interest. Hopefully, the reader may adapt the method presented here to the configuration and beam equation of his (her) own choosing.

5. Concluding comments

We have introduced a method to compute eigenfrequencies in Section 4 and applied it successfully to the quasi-Timoshenko equations in Section 5. The method is simple, highly accurate and allows one to compute frequencies of any order at virtually no cost.

One of the problems currently under study is the eigenvalue problem for the Timoshenko equation,

$$\frac{d^4\eta}{d\xi^4} + \phi^2(\alpha + \beta)\frac{d^2\eta}{d\xi^2} - \phi^2(1 - \phi^2\alpha\beta)\eta = 0.$$

The extension of the method is by no means straightforward.

On the other hand, by denoting $\lambda = \phi^2$ we obtain a quadratic eigenvalue problem:

$$\frac{d^4\eta}{d\xi^4} + \lambda(\alpha + \beta)\frac{d^2\eta}{d\xi^2} - \lambda(1 - \lambda\alpha\beta)\eta = 0.$$

By using the FEM it can be reduced to a linear generalized eigenvalue problem. The matrices in this problem are unstructured, and due to the change of variable, spurious eigenvalues are found. An algorithm to solve this eigenvalue problem is of interest.

A related problem is to establish asymptotic estimates for eigenfrequencies of elastic beams, see Ref. [6]. In our case, rough estimates are provided by the intervals enclosing the roots. For the E–B beam, sharp estimates are easily obtained. We leave for future work the case of quasi-Timoshenko equations.

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