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Asymptotic analysis of nonlinear vibration of an elastic plate under heavy fluid loading

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Abstract

The objective of this paper is identification and analysis of excitation regimes, when nonlinear effects are pronouncedly developed in the stationary dynamics of an infinitely long uniform elastic plate under heavy fluid loading. The method of multiple scales is applied and the solutions of the amplitude modulation equations for two types of excitation are obtained in a closed analytical form. Results of the asymptotic analysis reported in this paper highlight several aspects of the nonlinear dynamics of such a plate, which have not previously been studied in detail. It is shown that for 'weak' excitation of a resonant wave the stationary response is controlled by the structure-originated nonlinearity, whereas for 'strong' sub-harmonic excitation the stationary response is controlled by the fluid-originated nonlinearity. In both these cases, a dependence of the amplitudes of directly and indirectly excited resonant waves on the amplitude of the driving force is determined.

1. Introduction

The nonlinear dynamics of an elastic plate under heavy fluid loading is of considerable practical concern both in 'classical' technical applications (for example, vibrations of naval structures in water) and in 'advanced' applications (for example, dynamics of electro-statically driven micro-

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electromechanical systems). Apart from this practical importance, analysis of forced resonant response of such a plate is of fundamental interest from the viewpoint of the theory of nonlinear dynamical systems.

This nonlinear problem has been studied in Refs. [1–6]. In Refs. [1–3], classical perturbation methods are used to deal with the case of so-called 'light fluid loading' of a nonlinear structure. In effect, this assumption implies that the resonant frequencies of a plate of finite dimensions are not altered due to the presence of fluid loading, which is not realistic in the case of, say, vibration of a steel plate in water. Vibration of an infinitely long plate with periodically spaced supports exposed to heavy fluid loading is considered in Ref. [4], where the cubic structural 'stretching due to bending' nonlinearity is considered together with the quadratic nonlinearity in the formulation of the contact pressure and in the continuity condition. It is shown that the structural nonlinearity controls the steady-state response in resonant excitation conditions (which are identified with linear fluid loading effects taken into account), whereas in the case of sub-harmonic excitation the 'fluid-generated' nonlinearity becomes dominant. The same model is used to deal with nonlinear vibrations of a baffled plate in Ref. [5]. Mean flow effects on the dynamics of a nonlinear plate with heavy fluid loading are studied in Ref. [6]. In the latter reference, a model of an incompressible fluid is used and the cubic nonlinear curvature of a plate is considered as well as the 'stretching due to bending' nonlinearity.

The present paper is aimed to extend the methodology suggested in Ref. [4] to the case, when 'stretching due to bending' structural nonlinearity does not exist inasmuch an infinitely long plate is not periodically supported, but this structure exhibits the nonlinear behaviour of another physical origin. In contrast to Ref. [6], the analysis involves fluid's compressibility in the absence of a mean flow.

2. Problem formulation

In the present paper, the theory suggested in Refs. [4–6] is used in the case when a fluid-loaded infinitely long plate does not have any periodic supports and is exposed to resonant excitation. The plane problem formulation is explored (see Fig. 1). The plate has thickness h, Young's modulus E and density ρ. It is loaded by a static axial tensile stresses σ (which gives a force resultant of σh) and is exposed to fluid loading generated by an acoustic medium which occupies

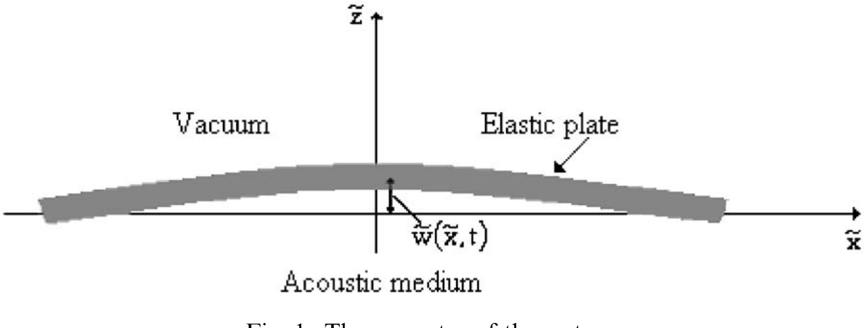


Fig. 1. The geometry of the system.

the lower half-space. The acoustical properties of this medium are sound speed c_{fl} and density ρ_{fl}. The plate is driven by a distributed lateral harmonic force $\tilde{q}(x,t)$, which produces flexural vibrations defined by the function $\tilde{w}(x,t)$.

The linear formulation of this classical problem in structural acoustics has been thoroughly explored by many authors (see, for example, Refs. [7,8]). However, in certain excitation conditions, which will be addressed in the paper, the behaviour of a driven plate may not be adequately described by a linear theory of fluid–structure interaction and nonlinear effects should be taken into account. As discussed in Ref. [4], there are several sources of nonlinearity, which are brought to light in excitation conditions generating a resonant wave. The 'structure-originated' nonlinearity is generated by the 'stretching due to bending' effect and by the effect of nonlinear curvature, which is produced by moderate amplitude of vibrations. As is known from the theory of nonlinear vibrations of beams and plates of finite length, the former effect is more pronounced than the latter. However, this 'stretching due to bending' effect exists in an infinitely long plate only if the plate is periodically supported with immobile stiffeners. In an infinitely long plate without these supports, the in-plane tension can be produced only by stresses σ acting at infinity (similar to the static uniform pre-stress considered in the theory of elasticity, e.g. in the theory of fracture). Thus, vibrations of an infinitely long fluid-loaded plate without intermediate supports, which are addressed in this paper, are governed by equation

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^2}{\partial\tilde{x}^2}\left\{\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}\left[1+\left(\frac{\partial\tilde{w}}{\partial\tilde{x}}\right)^2\right]^{-3/2}\right\}-\sigma h\,\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}+\rho h\,\frac{\partial^2\tilde{w}}{\partial\tilde{t}^2}=\tilde{q}+\tilde{p}. \tag{1}$$

The first term in this nonlinear equation, written in dimensional form, is related to the exact formulation of the bending moment in terms of curvature $\tilde{r}^{-1}(\tilde{x})$:

$$M(\tilde{x})=\frac{Eh^3}{12(1-\nu^2)}\tilde{r}^{-1}(\tilde{x})=\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}\left[1+\left(\frac{\partial\tilde{w}}{\partial\tilde{x}}\right)^2\right]^{-3/2}. \tag{2}$$

All other terms on the left-hand side of Eq. (1) are linear. The second term describes the influence of the axial tension σ, which is independent of flexural deflection. The third term presents the conventional formulation of the inertia of a plate.

A contact acoustic pressure $\tilde{p}$ exerted at the surface of the plate also contains the nonlinear components

$$\tilde{p}=-\rho_{\mathrm{fl}}\left[\frac{\partial\tilde{\varphi}}{\partial\tilde{t}}+\frac{1}{2}\left(\frac{\partial\tilde{\varphi}}{\partial\tilde{x}}\right)^2+\frac{1}{2}\left(\frac{\partial\tilde{\varphi}}{\partial\tilde{z}}\right)^2\right]. \tag{3}$$

Eq. (3) presents the nonlinearity in the exact formulation of Bernoulli's equation, which may be referred to as 'fluid-originated' nonlinearity.

The linear wave equation holds for an acoustic medium (see discussion in Ref. [4]):

$$\frac{\partial^2\tilde{\varphi}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{\varphi}}{\partial\tilde{z}^2}-\frac{1}{c_{\mathrm{fl}}^2}\frac{\partial^2\tilde{\varphi}}{\partial\tilde{t}^2}=0. \tag{4}$$

Finally, the following continuity condition is formulated at the surface of the plate, $\tilde{z} = 0$:

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{z}} = \frac{\partial \tilde{w}}{\partial \tilde{t}} + \frac{\partial \tilde{w}}{\partial \tilde{x}} \frac{\partial \tilde{\varphi}}{\partial \tilde{x}}. \tag{5}$$

This equation is formulated at the deformed surface of the plate with first-order terms retained, $\sin[\arctan(\partial\tilde{w}/\partial\tilde{x})] \approx \partial\tilde{w}/\partial\tilde{x}$, $\cos[\arctan(\partial\tilde{w}/\partial\tilde{x})] \approx 1$. Thus, formula (5) describes the nonlinearity in fluid–structure coupling.

Formulation (1)–(5) of the problem of the dynamics of an infinitely long elastic plate under heavy fluid loading contains three nonlinear components, identified as 'structure-originated' nonlinearity (2), 'fluid-originated' nonlinearity (3), and 'coupling-originated' nonlinearity (5). This problem formulation differs from the earlier ones [4–6]. Unlike the case treated in Refs. [4,5], here forced vibrations of an infinitely long homogeneous plate (i.e., a plate without periodic immobile supports) are addressed. The compressibility of the fluid is taken into account, unlike in the problem solved in Ref. [6].

In practice, it is not quite realistic to assume that the amplitude of stationary vibrations of a fluid-loaded plate may be much larger than its thickness, even in the resonant excitation conditions. So the finite but small amplitudes of displacements should be considered and the nonlinear term in Eq. (1) may be expanded in polynomial series with sufficient accuracy if only linear and cubic terms are retained, i.e.,

$$\frac{\partial^2}{\partial \tilde{x}^2}\left\{\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\left[1+\left(\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2\right]^{-3/2}\right\} = \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} - \frac{3}{2}\frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4}\left(\frac{\partial \tilde{w}}{\partial \tilde{x}}\right)^2 - 3\left(\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\right)^3 - 9\,\frac{\partial \tilde{w}}{\partial \tilde{x}}\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}\frac{\partial^3 \tilde{w}}{\partial \tilde{x}^3}. \tag{6}$$

Therefore problem (1)–(5) is treated here as a weakly nonlinear problem, which facilitates use of classical perturbation methods. Specifically, the method of multiple scales is applied as formulated in Ref. [9]. Then the system of governing equations is written as

$$\begin{aligned}&\frac{\partial^4 w}{\partial x^4} - \frac{12\sigma(1-\nu^2)}{E}\frac{\partial^2 w}{\partial x^2} + 12(1-\nu^2)\frac{\partial^2 w}{\partial t^2} - 12(1-\nu^2)p \\ &\quad = [\mu\delta_\mu + (1-\delta_\mu)]\,\frac{12(1-\nu^2)\tilde{q}}{E} + \mu\left[\frac{3}{2}\frac{\partial^4 w}{\partial x^4}\left(\frac{\partial w}{\partial x}\right)^2 + 3\left(\frac{\partial^2 w}{\partial x^2}\right)^3 + 9\,\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3}\right],\end{aligned} \tag{7}$$

$$p = -\frac{\rho_{\mathrm{fl}}}{\rho}\left[\frac{\partial \varphi}{\partial t} + \mu\left\{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right\}\right], \tag{8}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi}{\partial t^2} = 0, \tag{9}$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial w}{\partial t} + \mu\,\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x} \quad (z=0). \tag{10}$$

Here μ is a formal small parameter (see Ref. [9]), δ_μ is a Kronecker symbol and the non-dimensional variables are introduced as $\tilde{w} = wh$, $\tilde{\varphi} = \varphi ch$, $\tilde{x} = xh$, $\tilde{z} = zh$, $\tilde{t} = t(h/c)$, $c = \sqrt{E/\rho}$.

The first term in right-hand side of Eq. (7) should read as $\mu[(12(1-v^2)\tilde{q})/E]$ in the case, when the excitation frequency and the excitation wavenumber satisfy the dispersion equation for a linear fluid-loaded plate, i.e., at the resonant 'weak' excitation. This ranging of the amplitude of a driving force simply means that small force is able to produce large-amplitude response. Respectively, in the case, when the excitation frequency and the excitation wavenumber do not satisfy this dispersion equation, a larger force should be applied to provoke the similar response. These two different loading cases cast in Eq. (7) by means of parameter δ_μ. The resonant 'weak' excitation is recovered at $\delta_\mu = 1$, the non-resonant 'strong' excitation is recovered at $\delta_\mu = 0$.

The lateral displacement of the plate, $w(x,t)$, the contact acoustic pressure $p(x,z,t)$ and the velocity potential in the fluid, $\varphi(x,z,t)$, are sought in the form of regular asymptotic expansions in the formal small parameter μ and the first two terms are retained:

$$w(x,t) = w_0(x_0,T_0,x_1,T_1) + \mu w_1(x_0,T_0,x_1,T_1), \tag{11a}$$

$$p(x,z,t) = p_0(x_0,z_0,T_0,x_1,z_1,T_1) + \mu p_1(x_0,z_0,T_0,x_1,z_1,T_1), \tag{11b}$$

$$\varphi(x,z,t) = \varphi_0(x_0,z_0,T_0,x_1,z_1,T_1) + \mu \varphi_1(x_0,z_0,T_0,x_1,z_1,T_1). \tag{11c}$$

Here the 'fast' and the 'slow' coordinates are $x_0 = x,\ z_0 = z,\ T_0 = t,\ x_1 = \mu x,\ z_1 = \mu z,\ T_1 = \mu t$.

The derivatives with respect to 'physical' variables are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \mu \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \mu \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \mu \frac{\partial}{\partial T_1}.$$

Then the problem to leading order $O(1)$ is linear and is,

in the case of 'weak' excitation:

$$\frac{\partial^4 w_0}{\partial x_0^4} - \frac{12\sigma(1-v^2)}{E}\frac{\partial^2 w_0}{\partial x_0^2} + 12(1-v^2)\frac{\partial^2 w_0}{\partial T_0^2} - 12(1-v^2)p_0 = 0, \tag{12a}$$

in the case of 'strong' excitation:

$$\frac{\partial^4 w_0}{\partial x_0^4} - \frac{12\sigma(1-v^2)}{E}\frac{\partial^2 w_0}{\partial x_0^2} + 12(1-v^2)\frac{\partial^2 w_0}{\partial T_0^2} - 12(1-v^2)p_0 = \frac{12(1-v^2)\tilde{q}}{E}, \tag{12b}$$

$$p_0 = -\frac{\rho_{\rm fl}}{\rho}\frac{\partial \varphi_0}{\partial T_0}, \tag{12c}$$

$$\frac{\partial^2 \varphi_0}{\partial x_0^2} + \frac{\partial^2 \varphi_0}{\partial z_0^2} - \frac{c^2}{c_{\rm fl}^2}\frac{\partial^2 \varphi_0}{\partial T_0^2} = 0, \tag{12d}$$

$$\frac{\partial \varphi_0}{\partial z_0} = \frac{\partial w_0}{\partial T_0} \quad (z_0 = 0). \tag{12e}$$

Eqs. (12c–e) are equally valid for the cases of 'weak' and 'strong' excitation. This problem is formulated as a problem for a plate under heavy fluid loading inasmuch as the leading-order equation (12a) or (12b) contains the coupling term $-12(1-v^2)p_0$. This system of equations in the absence of any external forcing ($\tilde{q} = 0$) is solved in the companion paper [10]. The solution is partly reproduced here in order to identify the resonant excitation regimes. The unknown functions are

$$w_0(x,t) = W_{01}(x_1,T_1)\exp(\mathrm{i}kx_0 - \mathrm{i}\omega T_0) + W_{02}(x_1,T_1)\exp(-\mathrm{i}kx_0 - \mathrm{i}\omega T_0) + W_{03}(x_1,T_1)\exp(\mathrm{i}kx_0 + \mathrm{i}\omega T_0) + W_{04}(x_1,T_1)\exp(-\mathrm{i}kx_0 + \mathrm{i}\omega T_0), \tag{13a}$$

$$\varphi_0(x,z,t) = \frac{\mathrm{i}\omega h/c}{\sqrt{k^2-(\omega h/c_{\mathrm{fl}})^2}}[-A_1(z_1,x_1,T_1)\exp(\mathrm{i}kx_0 - \mathrm{i}\omega T_0) - A_2(z_1,x_1,T_1)\exp(-\mathrm{i}kx_0 - \mathrm{i}\omega T_0) + A_3(z_1,x_1,T_1)\exp(\mathrm{i}kx_0 + \mathrm{i}\omega T_0) + A_4(z_1,x_1,T_1)\exp(-\mathrm{i}kx_0 + \mathrm{i}\omega T_0)]\exp\left(z_0\sqrt{k^2-(\omega h/c_{\mathrm{fl}})^2}\right). \tag{13b}$$

The dispersion equation is

$$k^4 + \left(\frac{c_M}{c_L}\right)^2 k^2 - \left(\frac{\omega h}{c_L}\right)^2 - \frac{\rho_{\mathrm{fl}}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{-1/2} = 0. \tag{14}$$

Here the following additional notation is used: $c_L = \sqrt{E/(12(1-\nu^2)\rho)}$, $c_M = \sqrt{\sigma/\rho}$.

It is convenient to introduce the scaling

$$\Omega = \frac{\omega h/c_L}{(c_{\mathrm{fl}}/c_L)^2}, \quad K = \frac{kh}{(c_{\mathrm{fl}}/c_L)}, \quad \delta = c_M/c_{\mathrm{fl}}, \quad \varepsilon = (\rho_{\mathrm{fl}}/\rho)/(c_{\mathrm{fl}}/c_L)$$

and to re-write Eq. (14) as

$$K^4 + \delta^2K^2 - \Omega^2 - \frac{\varepsilon\Omega^2}{(K^2-\Omega^2)^{1/2}} = 0. \tag{15}$$

The analysis of various regimes of wave motion in a fluid-loaded plate is performed in the companion paper [10] and asymptotic expansions for all roots of this equation are obtained there in terms of fluid loading parameter ε. In particular, the purely real root of this dispersion equation in the low-frequency regime is found to be

$$\Omega \cong \frac{K^{3/2}(K^2+\delta^2)^{1/2}}{(\varepsilon+K)^{1/2}} - \frac{\varepsilon}{4}\frac{K^{5/2}(K^2+\delta^2)^{3/2}}{(\varepsilon+K)^{5/2}}. \tag{16}$$

This formula defines several excitation regimes, which are controlled by nonlinear effects. Specifically, the following two loading cases should be considered in detail:

1. ‘Weak’ excitation at the resonant frequency and the resonant wavenumber (those, which satisfy linear dispersion equation (16)).
2. ‘Strong’ sub-resonant excitation.

It is appropriate to notice here that—although an infinitely long wave-guide is considered—it is entirely relevant to identify the forced response of a linear system in the case, when the excitation frequency and the excitation wavenumber satisfy the dispersion equation, as a resonant one inasmuch the linear theory predicts the unbounded amplitude of forced vibrations.

3. Asymptotic analysis of a steady-state response

To determine the dependence of the functions $w_0(x_0, T_0, x_1, T_1)$, $p_0(x_0, z_0, T_0, x_1, z_1, T_1)$ and $\varphi_0(x_0, z_0, T_0, x_1, z_1, T_1)$ on the 'slow' variables x_1, z_1, T_1, it is necessary to address the first-order problem. Its solution is different for the two regimes, which are identified in the previous section.

3.1. 'Weak' resonant excitation

Consider 'weak' resonant excitation ($\delta_\mu = 1$) of an infinitely long plate with heavy fluid loading at frequency $\omega_q = \omega$ and at wavenumber $k_q = k$. The frequency and wavenumber satisfy the dispersion equation (14) and are easily found from Eq. (16) as $\omega h/c_L = \Omega(c_{\mathrm{fl}}/c_L)^2$, $kh = K(c_{\mathrm{fl}}/c_L)$. To determine the dependence of the functions w_0 and φ_0 on the 'slow' coordinates (x_1, z_1, T_1) in formulae (13a,b), it is necessary to set up the problem to order $O(\mu)$.

The equation of structural dynamics to this order is

$$\frac{\partial^4 w_1}{\partial x_0^4} - \frac{12\sigma(1-\nu^2)}{E}\frac{\partial^2 w_1}{\partial x_0^2} + 12(1-\nu^2)\frac{\partial^2 w_1}{\partial T_0^2} - 12(1-\nu^2)p_1$$
$$= \frac{12(1-\nu^2)\tilde{q}}{E} + \frac{3}{2}\frac{\partial^4 w_0}{\partial x_0^4}\left(\frac{\partial w_0}{\partial x_0}\right)^2 + 3\left(\frac{\partial^2 w_0}{\partial x_0^2}\right)^3 + 9\frac{\partial w_0}{\partial x_0}\frac{\partial^2 w_0}{\partial x_0^2}\frac{\partial^3 w_0}{\partial x_0^3}$$
$$- 4\frac{\partial^4 w_0}{\partial x_0^3 \partial x_1} + \frac{24\sigma(1-\nu^2)}{E}\frac{\partial^2 w_0}{\partial x_0 \partial x_1} + 24(1-\nu^2)\frac{\partial^2 w_0}{\partial T_0 \partial T_1}. \tag{17}$$

The contact pressure to order $O(\mu)$ at the surface of a plate is defined as

$$p_1 = -\frac{\rho_{\mathrm{fl}}}{\rho}\left[\frac{\partial \varphi_1}{\partial T_0} + \frac{\partial \varphi_0}{\partial T_1} + \frac{1}{2}\left\{\left(\frac{\partial \varphi_0}{\partial x_0}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z_0}\right)^2\right\}\right]. \tag{18}$$

The velocity potential is governed by the inhomogeneous wave equation,

$$\frac{\partial^2 \varphi_1}{\partial x_0^2} + \frac{\partial^2 \varphi_1}{\partial z_0^2} - \frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi_1}{\partial T_0^2} = -2\frac{\partial^2 \varphi_0}{\partial x_0 \partial x_1} - 2\frac{\partial^2 \varphi_0}{\partial z_0 \partial z_1} + 2\frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi_0}{\partial T_0 \partial T_1}. \tag{19}$$

Finally, the boundary condition to order $O(\mu)$ is

$$\frac{\partial \varphi_1}{\partial z_0} + \frac{\partial \varphi_0}{\partial z_1} = \frac{\partial w_1}{\partial T_0} + \frac{\partial w_0}{\partial T_1} + \frac{\partial w_0}{\partial x_0}\frac{\partial \varphi_0}{\partial x_0}. \tag{20}$$

Problem (17)–(20) is linear for the functions $w_1(x_0, T_0, x_1, T_1)$, $\varphi_1(x_0, z_0, T_0, x_1, z_1, T_1)$ and is formulated only in the 'fast' variables x_0, z_0, T_0 with respect to these functions. Thus, its solution may be sought by use of a superposition principle. Problem (17)–(20) also defines the dependence of the functions $w_0(x_0, T_0, x_1, T_1)$ and $\varphi_0(x_0, z_0, T_0, x_1, z_1, T_1)$ on 'slow' variables x_1, z_1, T_1 via solvability conditions, which should be formulated to ensure that the asymptotic expansions (11) are uniformly valid. More precisely, right-hand sides of the differential equations (17) and (19) with respect to the functions $w_1(x_0, T_0, x_1, T_1)$, $\varphi_1(x_0, z_0, T_0, x_1, z_1, T_1)$ should not contain secular terms in the 'fast' variables x_0, z_0, T_0.

To identify secular terms, it is convenient to re-write Eq. (17) as

$$\frac{\partial^4 w_1}{\partial x_0^4} - \frac{12\sigma(1-\nu^2)}{E}\frac{\partial^2 w_1}{\partial x_0^2} + 12(1-\nu^2)\frac{\partial^2 w_1}{\partial T_0^2} - 12(1-\nu^2)p_{10}$$
$$= \frac{12(1-\nu^2)\tilde{q}}{E} + \frac{3}{2}\frac{\partial^4 w_0}{\partial x_0^4}\left(\frac{\partial w_0}{\partial x_0}\right)^2 + 3\left(\frac{\partial^2 w_0}{\partial x_0^2}\right)^3 + 9\frac{\partial w_0}{\partial x_0}\frac{\partial^2 w_0}{\partial x_0^2}\frac{\partial^3 w_0}{\partial x_0^3}$$
$$-4\frac{\partial^4 w_0}{\partial x_0^3 \partial x_1} + \frac{24\sigma(1-\nu^2)}{E}\frac{\partial^2 w_0}{\partial x_0 \partial x_1} + 24(1-\nu^2)\frac{\partial^2 w_0}{\partial T_0 \partial T_1} + 12(1-\nu^2)p_{11}. \tag{21}$$

Here the contact acoustic pressure is decomposed as

$$p_1(x_0, z_0, T_0, x_1, z_1, T_1) = p_{10} + p_{11}. \tag{22a}$$

Respectively, the same decomposition is applied to the velocity potential:

$$\varphi_1(x_0, z_0, T_0, x_1, z_1, T_1) = \varphi_{10} + \varphi_{11}. \tag{22b}$$

The first component in Eq. (22a) is defined exactly as its counterpart (12c):

$$p_{10} = -\frac{\rho_{\mathrm{fl}}}{\rho}\frac{\partial \varphi_{10}}{\partial T_0}. \tag{23}$$

The first component in Eq. (22b) is also defined exactly as in the problem to leading order by the homogeneous wave equation in 'fast' variables:

$$\frac{\partial^2 \varphi_{10}}{\partial x_0^2} + \frac{\partial^2 \varphi_{10}}{\partial z_0^2} - \frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi_{10}}{\partial T_0^2} = 0. \tag{24}$$

The boundary condition for this equation is

$$\frac{\partial \varphi_{10}}{\partial z_0} = \frac{\partial w_1}{\partial T_0}. \tag{25}$$

The operator defined by the left-hand side of Eq. (21) is identical to the operator on the left-hand side of Eq. (12a). Therefore, secular terms must be removed from the right-hand side of this equation. These terms are present in Eq. (21) both explicitly and implicitly, i.e., they are contained in the pressure p_{11}. This component p_{11} and the component φ_{11} are actually defined as the 'residual' from formulation (18)–(20)—with parts (23) and (25) removed:

$$p_{11} = -\frac{\rho_{\mathrm{fl}}}{\rho}\left[\frac{\partial \varphi_{11}}{\partial T_0} + \frac{\partial \varphi_0}{\partial T_1} + \frac{1}{2}\left\{\left(\frac{\partial \varphi_0}{\partial x_0}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z_0}\right)^2\right\}\right], \tag{26}$$

$$\frac{\partial^2 \varphi_{11}}{\partial x_0^2} + \frac{\partial^2 \varphi_{11}}{\partial z_0^2} - \frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi_{11}}{\partial T_0^2} = -2\frac{\partial^2 \varphi_0}{\partial x_0 \partial x_1} - 2\frac{\partial^2 \varphi_0}{\partial z_0 \partial z_1} + 2\frac{c^2}{c_{\mathrm{fl}}^2}\frac{\partial^2 \varphi_0}{\partial T_0 \partial T_1}, \tag{27}$$

$$\frac{\partial \varphi_{11}}{\partial z_0} = -\frac{\partial \varphi_0}{\partial z_1} + \frac{\partial w_0}{\partial T_1} + \frac{\partial w_0}{\partial x_0}\frac{\partial \varphi_0}{\partial x_0}. \tag{28}$$

Furthermore, it is possible (because of the linearity of this formulation with respect to the functions p_{11} and φ_{11}) to write

$$p_{11} = p_{11}^{(1)} + p_{11}^{(2)}. \tag{29}$$

Here the first component is defined directly via $\varphi_0(x_0, z_0, T_0, x_1, z_1, T_1)$ as

$$p_{11}^{(1)} = -\frac{\rho_{\rm fl}}{\rho}\left[\frac{\partial \varphi_0}{\partial T_1} - \frac{1}{2}\left\{\left(\frac{\partial \varphi_0}{\partial x_0}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z_0}\right)^2\right\}\right]. \tag{30}$$

The second component is

$$p_{11}^{(2)} = -\frac{\rho_{\rm fl}}{\rho}\frac{\partial \varphi_{11}}{\partial T_0}. \tag{31}$$

In this formula, the velocity potential φ_{11} is a solution of the problem

$$\frac{\partial^2 \varphi_{11}}{\partial x_0^2} + \frac{\partial^2 \varphi_{11}}{\partial z_0^2} - \frac{c^2}{c_{\rm fl}^2}\frac{\partial^2 \varphi_{11}}{\partial T_0^2} = -2\frac{\partial^2 \varphi_0}{\partial x_0 \partial x_1} - 2\frac{\partial^2 \varphi_0}{\partial z_0 \partial z_1} + 2\frac{c^2}{c_{\rm fl}^2}\frac{\partial^2 \varphi_0}{\partial T_0 \partial T_1}, \tag{32a}$$

$$\frac{\partial \varphi_{11}}{\partial z_0} = \frac{\partial w_0}{\partial x_0}\frac{\partial \varphi_0}{\partial x_0} \quad (z_0 = 0). \tag{32b}$$

Finally, the continuity condition for the functions $\varphi_0(x_0, z_0, T_0, x_1, z_1, T_1)$ and $w_0(x_0, T_0, x_1, T_1)$ in 'slow' coordinates also follows from the continuity condition (28) when condition (32b) is subtracted:

$$\frac{\partial \varphi_0}{\partial z_1} = \frac{\partial w_0}{\partial T_1} \quad (z_1 = 0). \tag{33}$$

It does not present serious difficulties to formulate the solvability conditions for problem (21)–(33) and therefore to find the dependence of the functions $w_0(x_0, T_0, x_1, T_1)$ and $\varphi_0(x_0, z_0, T_0, x_1, z_1, T_1)$ on the 'slow' variables x_1, z_1, T_1. However, from a practical viewpoint it is important to obtain only a stationary response, if it exists. This response implies that all derivatives in 'slow' coordinates vanish and the amplitude of vibrations of a plate is actually constant in 'slow' variables. Then the governing equation (21) is simplified to

$$\begin{aligned} &\frac{\partial^4 w_1}{\partial x_0^4} - \frac{12\sigma(1-v^2)}{E}\frac{\partial^2 w_1}{\partial x_0^2} + 12(1-v^2)\frac{\partial^2 w_1}{\partial T_0^2} - 12(1-v^2)p_{10} \\ &= \frac{12(1-v^2)\tilde{q}}{E} + \frac{3}{2}\frac{\partial^4 w_0}{\partial x_0^4}\left(\frac{\partial w_0}{\partial x_0}\right)^2 + 3\left(\frac{\partial^2 w_0}{\partial x_0^2}\right)^3 \\ &\quad + 9\frac{\partial w_0}{\partial x_0}\frac{\partial^2 w_0}{\partial x_0^2}\frac{\partial^3 w_0}{\partial x_0^3} + 12(1-v^2)p_{11}. \end{aligned} \tag{34}$$

In this equation,

$$p_{11} = -\frac{\rho_{\rm fl}}{\rho}\left[\frac{\partial \varphi_{11}}{\partial T_0} + \frac{1}{2}\left\{\left(\frac{\partial \varphi_0}{\partial x_0}\right)^2 + \left(\frac{\partial \varphi_0}{\partial z_0}\right)^2\right\}\right], \tag{35a}$$

$$\frac{\partial^2 \varphi_{11}}{\partial x_0^2} + \frac{\partial^2 \varphi_{11}}{\partial z_0^2} - \frac{c^2}{c_{\rm fl}^2} \frac{\partial^2 \varphi_{11}}{\partial T_0^2} = 0, \tag{35b}$$

$$\frac{\partial \varphi_{11}}{\partial z_0} = \frac{\partial w_0}{\partial x_0} \frac{\partial \varphi_0}{\partial x_0} \quad (z_0 = 0). \tag{35c}$$

The leading order terms in expansions (11) are formulated as

$$\begin{aligned} w_0(x,t) = {} & W_{01} \exp({\rm i}kx_0 - {\rm i}\omega T_0) + W_{02} \exp(-{\rm i}kx_0 - {\rm i}\omega T_0) \\ & + W_{03} \exp({\rm i}kx_0 + {\rm i}\omega T_0) + W_{04} \exp(-{\rm i}kx_0 + {\rm i}\omega T_0), \end{aligned} \tag{36a}$$

$$\begin{aligned} \varphi_0(x,z,t) = {} & \frac{{\rm i}\omega h/c}{\sqrt{k^2 - (\omega h/c_{\rm fl})^2}} [-W_{01} \exp({\rm i}kx_0 - {\rm i}\omega T_0) \\ & - W_{02} \exp(-{\rm i}kx_0 - {\rm i}\omega T_0) + W_{03} \exp({\rm i}kx_0 + {\rm i}\omega T_0) \\ & + W_{04} \exp(-{\rm i}kx_0 + {\rm i}\omega T_0)] \exp(z_0 \sqrt{k^2 - (\omega h/c_{\rm fl})^2}). \end{aligned} \tag{36b}$$

A resonant standing wave is generated when the driving load is

$$\begin{aligned} \tilde{q} = {} & \frac{Q_0}{4} [\exp({\rm i}k_q x - {\rm i}\omega_q t) + \exp(-{\rm i}k_q x - {\rm i}\omega_q t) \\ & + \exp({\rm i}k_q x + {\rm i}\omega_q t) + \exp(-{\rm i}k_q x + {\rm i}\omega_q t)]. \end{aligned} \tag{37}$$

As shown in Ref. [4], the fluid-introduced nonlinearity does not contribute to the modulation equation in this case of 'weak' resonant excitation. The origin of the secular terms lies only in the terms that describe the structural nonlinearity (namely, the nonlinear curvature) and a standard algebra gives a set of the 'amplitude modulation' equations for a stationary response in this loading case. With $\tilde{Q}_0 \equiv (12(1-v^2)Q_0)/E$, these equations are

$$k^6[15 W_{01} W_{02} W_{03} + \tfrac{15}{2} W_{01}^2 W_{04}] = \tfrac{1}{4} \tilde{Q}_0, \tag{38a}$$

$$k^6[15 W_{02} W_{03} W_{04} + \tfrac{15}{2} W_{04}^2 W_{01}] = \tfrac{1}{4} \tilde{Q}_0, \tag{38b}$$

$$k^6[15 W_{01} W_{02} W_{04} + \tfrac{15}{2} W_{02}^2 W_{03}] = \tfrac{1}{4} \tilde{Q}_0, \tag{38c}$$

$$k^6[15 W_{01} W_{03} W_{04} + \tfrac{15}{2} W_{03}^2 W_{02}] = \tfrac{1}{4} \tilde{Q}_0. \tag{38d}$$

The solution is

$$W_{01} = W_{02} = W_{03} = W_{04} = W = k^{-2} \left(\frac{\tilde{Q}_0}{90} \right)^{1/3}. \tag{39}$$

As seen from Eq. (39), the resonant growth in the amplitude is bounded by the presence of the 'structural' nonlinearity.

3.2. 'Strong' sub-resonant excitation

Now the 'strong' excitation ($\delta_\mu = 0$) at frequency $\omega_q = \omega/2$ and wavenumber $k_q = k/2$ is addressed. In this case, the 'fluid-produced' quadratic nonlinearity generates interaction between a wave of the above parameters and a resonant wave. A stationary solution in 'slow' variables of the problem to the order $O(1)$ is sought as

$$\begin{aligned} w_0(x,t) = {} & W_{01}\exp(\mathrm{i}kx_0 - \mathrm{i}\omega T_0) + W_{02}\exp(-\mathrm{i}kx_0 - \mathrm{i}\omega T_0) + W_{03}\exp(\mathrm{i}kx_0 + \mathrm{i}\omega T_0) \\ & + W_{04}\exp(-\mathrm{i}kx_0 + \mathrm{i}\omega T_0) + W_{q1}\exp\left(\frac{\mathrm{i}kx_0 - \mathrm{i}\omega T_0}{2}\right) + W_{q2}\exp\left(\frac{-\mathrm{i}kx_0 - \mathrm{i}\omega T_0}{2}\right) \\ & + W_{q3}\exp\left(\frac{\mathrm{i}kx_0 + \mathrm{i}\omega T_0}{2}\right) + W_{q4}\exp\left(\frac{-\mathrm{i}kx_0 + \mathrm{i}\omega T_0}{2}\right), \end{aligned} \tag{40a}$$

$$\begin{aligned} \varphi_0(x,z,t) = {} & \mathrm{i}\,\frac{\omega h}{c}\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{-1/2} [-W_{01}\exp(\mathrm{i}kx_0 - \mathrm{i}\omega T_0) - W_{02}\exp(-\mathrm{i}kx_0 - \mathrm{i}\omega T_0) \\ & + W_{03}\exp(\mathrm{i}kx_0 + \mathrm{i}\omega T_0) + W_{04}\exp(\mathrm{i}kx_0 + \mathrm{i}\omega T_0)]\exp\left(z_0\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{1/2}\right) \\ & + \mathrm{i}\,\frac{\omega h}{c}\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{-1/2}\left[-W_{q1}\exp\left(\frac{\mathrm{i}kx_0 - \mathrm{i}\omega T_0}{2}\right) - W_{q2}\exp\left(\frac{-\mathrm{i}kx_0 - \mathrm{i}\omega T_0}{2}\right) \right. \\ & \left. + W_{q3}\exp\left(\frac{\mathrm{i}kx_0 + \mathrm{i}\omega T_0}{2}\right) + W_{q4}\exp\left(\frac{-\mathrm{i}kx_0 + \mathrm{i}\omega T_0}{2}\right)\right] \\ & \times\exp\left(\frac{z_0}{2}\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{1/2}\right). \end{aligned} \tag{40b}$$

The amplitude of the directly excited sub-resonant standing wave is found from the elementary linear equation for $W_{q1} = W_{q2} = W_{q3} = W_{q4} = W_q$:

$$W_q\left\{\left(\frac{k}{2}\right)^4 + \left(\frac{c_M}{c_L}\right)^2\left(\frac{k}{2}\right)^2 - \left(\frac{\omega h}{2c_L}\right)^2 - \frac{\rho_{\mathrm{fl}}}{\rho}\left(\frac{\omega h}{2c_L}\right)^2\left[\left(\frac{k}{2}\right)^2 - \left(\frac{\omega h}{2c_{\mathrm{fl}}}\right)^2\right]^{-1/2}\right\} = \tfrac{1}{4}\tilde{Q}_0. \tag{41}$$

Although the frequency ω and the wavenumber k are found from the condition that the expression in curly brackets—the dispersion function—vanishes for the pair $(\omega, k(\omega))$, the pair $(\omega/2, k(\omega)/2)$ does not satisfy this dispersion equation. The 'amplitude modulation' equations are

$$\begin{aligned} & k^6\left[15W_{01}W_{02}W_{03} + \tfrac{15}{2}W_{01}^2W_{04} + 3W_{01}W_q^2\right] \\ & + \frac{1}{8}\frac{\rho_{\mathrm{fl}}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left\{1 - 3k^2\left[k^2 - \left(\frac{\omega h}{c_{\mathrm{fl}}}\right)^2\right]^{-1}\right\}W_q^2 = 0, \end{aligned}$$

$$k^6\left[15W_{01}W_{02}W_{04}+\tfrac{15}{2}W_{02}^2W_{03}+3W_{02}W_q^2\right]$$
$$+\frac{1}{8}\frac{\rho_{\rm fl}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left\{1-3k^2\left[k^2-\left(\frac{\omega h}{c_{\rm fl}}\right)^2\right]^{-1}\right\}W_q^2=0,$$
$$k^6\left[15W_{01}W_{03}W_{04}+\tfrac{15}{2}W_{03}^2W_{02}+3W_{03}W_q^2\right]$$
$$+\frac{1}{8}\frac{\rho_{\rm fl}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left\{1-3k^2\left[k^2-\left(\frac{\omega h}{c_{\rm fl}}\right)^2\right]^{-1}\right\}W_q^2=0,$$
$$k^6\left[15W_{02}W_{03}W_{04}+\tfrac{15}{2}W_{04}^2W_{01}+3W_{04}W_q^2\right]$$
$$+\frac{1}{8}\frac{\rho_{\rm fl}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left\{1-3k^2\left[k^2-\left(\frac{\omega h}{c_{\rm fl}}\right)^2\right]^{-1}\right\}W_q^2=0. \quad (42)$$

In this case, the stationary solution is $W_{01}=W_{02}=W_{03}=W_{04}=W_0$, and the amplitude of the resonant standing wave is found from the cubic equation

$$W_0^3+fW_q^2W_0+gW_q^2=0. \quad (43)$$

The coefficients in this equation are

$$f=1/15,\ g=\frac{1}{180}k^{-6}\frac{\rho_{\rm fl}}{\rho}\left(\frac{\omega h}{c_L}\right)^2\left\{1-3k^2\left[k^2-\left(\frac{\omega h}{c_{\rm fl}}\right)^2\right]^{-1}\right\}.$$

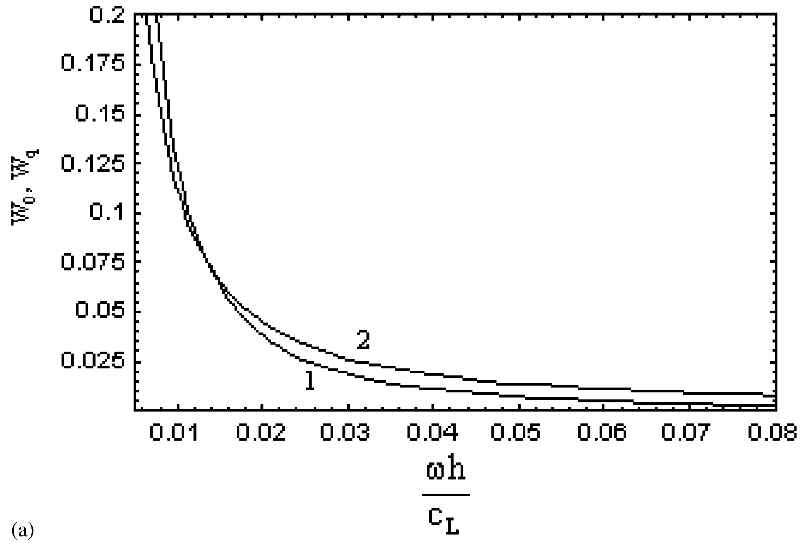
Although there is no difficulty in finding the roots of this equation by, e.g., the symbolic manipulator Mathematica [11], its real root is elementarily defined in the form of an asymptotic expansion in the small parameter W_q (it is entirely realistic to assume that non-resonant vibrations of a plate at frequency $\omega_q=\omega/2$ and wavenumber $k_q=k/2$ have a fairly small amplitude $W_q\ll 1$). The expansion is

$$W_0\cong -g^{1/3}W_q^{2/3}+\tfrac{1}{3}fg^{-1/3}W_q^{4/3}. \quad (44)$$

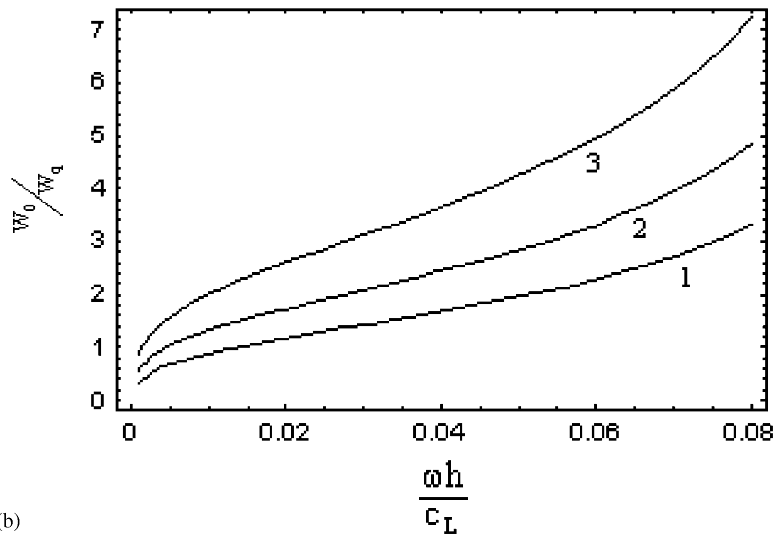
The ratio of the resonant amplitude to the amplitude of the directly excited wave is

$$W_0/W_q\cong -g^{1/3}W_q^{-1/3}+\tfrac{1}{3}fg^{-1/3}W_q^{1/3}. \quad (45)$$

As the amplitude of a driving force tends to zero, both the directly excited non-resonant wave and the resonant wave indirectly excited by nonlinear modal coupling have vanishing amplitudes (see formulae (41) and (44)). However, this formula suggests that the smaller the amplitude of a driving load, the larger the ratio between the amplitudes of these waves. The dependence of the amplitudes W_0 (curve 1) and W_q (curve 2) on the frequency parameter $\omega h/c_L$ is shown in Fig. 2a for the non-dimensional excitation amplitude $\tilde{Q}_0=1.0\times 10^{-5}$. Vibrations of a steel plate in water are considered, for which $\rho=7.8\times 10^3\,\mathrm{kg/m^3}$, $E=2.1\times 10^5\,\mathrm{MPa}$, $\nu=0.3$, $\rho_{\rm fl}=1.0\times 10^3\,\mathrm{kg/m^3}$, $c_{\rm fl}=1.5\times 10^3\,\mathrm{m/s}$. The dimensional amplitude of a distributed load is $Q_0=$



(a)



(b)

Fig. 2. Forced nonlinear response of a plate with fluid loading: (a) Amplitudes of the sub-harmonic and the resonant waves. Curve 1—the amplitude of the sub-harmonic wave; curve 2—the amplitude of the resonant wave. (b) The amplitude ratio W_0/W_q at $\tilde{Q}_0 = 1.0 \times 10^{-5}$ (curve 1), $\tilde{Q}_0 = 3.3 \times 10^{-6}$ (curve 2), $\tilde{Q}_0 = 1.0 \times 10^{-6}$ (curve 3).

0.183 MPa and the static axial pre-stress is $\sigma = 21$ MPa. As seen from the graph, starting from $\omega h/c_L \approx 0.135$ (for a steel plate of thickness $h = 10$ mm, corresponding to a frequency of $f \approx 3.3$ kHz), the indirectly excited resonant response (curve 2) becomes larger than the directly excited sub-harmonic response of a fluid-loaded plate (curve 1). Both are fairly small—the dimensional amplitudes at $f \approx 3.3$ kHz are $W_0^{\dim} = W_q^{\dim} = 0.075h = 0.75$ mm. The amplitude ratio W_0/W_q versus the frequency parameter $\omega h/c_L$ is shown inFig. 2b. As seen, the weaker the excitation of the sub-harmonic wave, the larger the amplitude ratio W_0/W_q.

4. Conclusions

The stationary nonlinear dynamics of an infinitely long uniform elastic plate under heavy fluid loading has been analysed by the method of multiple scales. The excitation regimes, when nonlinear modal interaction effects are pronouncedly developed, are identified and studied. In the case of weak resonant excitation, the stationary response is controlled by the structure-originated nonlinearity, and the steady-state amplitude of the standing flexural wave is proportional to the cube root of the excitation force. In the case of strong sub-harmonic excitation, the stationary response is controlled by the fluid-originated nonlinearity, and the resonant wave excited by the nonlinear modal coupling may have larger amplitude than a directly excited non-resonant wave.

References

[1] J.H. Ginsberg, Multi-dimensional nonlinear acoustic wave propagation, part 2: the nonlinear interaction of an acoustic field, *Journal of Sound and Vibration* 40 (1975) 359–379.
[2] A.H. Nayfeh, S.G. Kelly, Non-linear interaction of acoustic fields and plate under harmonic excitation, *Journal of Sound and Vibration* 60 (1978) 371–378.
[3] I.D. Abrahams, Acoustic scattering by a finite nonlinear elastic plate. Primary, secondary and combinatory resonances, *Proceedings of the Royal Society of London A* 414 (1987) 237–253.
[4] S.V. Sorokin, S.G. Kadyrov, Modelling of nonlinear oscillations of elastic structures in heavy fluid loading conditions, *Journal of Sound and Vibration* 222 (1999) 425–451.
[5] S.V. Sorokin, Non-linear oscillations of a baffled elastic plate in heavy fluid loading conditions, *Journal of Sound and Vibration* 232 (2000) 619–643.
[6] N. Peake, Nonlinear stability of a fluid-loaded elastic plate with mean flow, *Journal of Fluid Mechanics* 434 (2001) 101–118.
[7] D.G. Crighton, A. Dowling, J.E. Fowcs Williams, M. Heckl, F. Leppington, *Modern Methods in Analytical Acoustics*, Springer, Berlin, 1992.
[8] M. Junger, D. Feit, *Sound, Structures and their Interaction*, second ed., MIT Press, Cambridge, MA, 1994.
[9] A.H. Nayfeh, *Perturbation Methods*, Wiley, New York, 1973.
[10] C.J. Chapman, S.V. Sorokin, The forced vibration of an elastic plate under significant fluid loading, *Journal of Sound and Vibration* 281 (2005) 719–741; doi:10.1016/j.jsv.2004.02.013.
[11] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer*, Addison-Wesley, Reading, MA, 1991.