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Flexural vibrations of composite orthotropic annular plates with edge-damage

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Abstract

The free vibrational response of clamped homogeneous or symmetrically laminated orthotropic annular plates with edge damage on the outer circumference has been studied for the first time, using the finite difference method. The edge damage is modeled via the application of a non-uniform boundary condition. The sensitivity of vibrational frequencies to the extent of the edge damage is investigated and shown to be strongly influenced by the material properties and the relative thicknesses of the materials comprising the laminated plates. It is shown that the orthotropy ratio for the plate (defined here by $D_{rr}/D_{\theta\theta}$) is a convenient compact parameter for indicating the sensitivity to edge damage. Mode shapes are also computed and found to be profoundly affected by the orthotropy ratio. The behavior of higher modes of vibration is also discussed and the phenomenon of cross over, whereby two modes exist having the same frequency, is observed.

1. Introduction

Nowacki and Olesiak [1] are the first reported researchers who attempted to investigate the vibrations of a circular plate subject to a combination of support conditions. Subsequently, Bartlett [2] looked into the vibrational and buckling behavior of circular plates having

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non-uniform boundary conditions. He considered an isotropic plate clamped on part of its boundary and simply supported or free on the remainder and was able to obtain approximate analytic solutions to the dual series integral equation he derived. Noble [3] revisited Bartlett's work and suggested another simpler technique for solving the dual series integral equation. Interestingly, neither Bartlett nor Noble reproduced the predictions of Nowacki and Oleisak [1]. Bartlett [2] states that the accuracy of the approximation used by the latter authors is "invariably doubtful".

Narita and Leissa [4] tackled the problem of free transverse vibrations of a circular plate with non-uniform boundary conditions using a special analytic technique in which translational and rotational springs with space varying stiffness expanded into their Fourier components were taken around the circumference of the plate. Clamped boundary conditions were simulated by assigning large numerical values to the stiffness of both springs.

Eastep and Hemmig [5] used the finite element method to obtain numerical predictions of the natural frequencies and mode shapes of a vibrating circular plate with a partially free, partially clamped edge. In addition, they employed a laser holography method to experimentally determine the natural frequencies and mode shapes. Excellent agreement was found between the predicted and measured data.

All the aforementioned investigations involved *isotropic homogeneous circular plates.* But nowadays the utilization of composite multi-tiered orthotropic materials is rather widespread due to their low weight to strength ratio. In addition, it is noted that the existence of non-uniform boundary conditions can be supposed to model a situation in which there is edge-damage to the plate, (see also, [6–9]). In the current work the vibrational behavior of axisymetrically layered composite orthotropic annular plates is examined. Use is made of an accurate finite difference approach to obtain the frequencies of vibration and the mode shapes. The influence of the material properties of the plates is investigated and the sensitivity of the plate's vibrational response and mode shapes to the situation in which edge damage is present on the outer circumference is examined.

2. Problem definition and governing equation

A circular annular plate of external radius b, internal radius a and thickness h, composed of different polar orthotropic materials with elastic properties that are dependent on the thickness (z) is considered. If deformation of the plate is permitted in the radial, circumferential and transverse directions then the governing coupled displacement equations are given by Elishakoff and Stavsky [10] in the following general form

$$\sum_{j=1}^{3}(L_{ij}+\alpha_j)U_j=0, \quad i=1,2,3, \tag{1}$$

where

$$U_1=U,\ U_2=V,\ U_3=W \tag{2}$$

are the radial, circumferential and transverse displacements, respectively, and L_{ij} and α_j are differential operators. It can be further shown that if a reference plane is chosen from the bottom of the plate at a distance

$$h_0 = \int_0^h \rho z\, \mathrm{d}z \Big/ \int_0^h \rho\, \mathrm{d}z \tag{3}$$

and if the composite annular plate lamination is symmetric, then the equation of motion for the flexural displacements becomes uncoupled from the other two equations due to the disappearance of the coupling terms. After setting $W = w(r,\theta)\,\mathrm{e}^{\mathrm{i}\omega t}$, where ω is the vibrational frequency, the equation of motion for the flexural vibrations becomes

$$L_{33}w = R_0\omega^2 w, \tag{4}$$

where the operator L_{33} is given by

$$\begin{aligned} L_{33} = & -D_{rr}\left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3}\right) - D_{r\theta}\left(\frac{2}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{2}{r^3}\frac{\partial^3}{\partial r\partial\theta^2} + \frac{2}{r^4}\frac{\partial^2}{\partial\theta^2}\right) \\ & - D_{\theta\theta}\left(\frac{1}{r^4}\frac{\partial^4}{\partial\theta^4} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{2}{r^4}\frac{\partial^2}{\partial\theta^2}\right) \\ & - 4D_{ss}\left(\frac{1}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{1}{r^3}\frac{\partial^3}{\partial r\partial\theta^2} + \frac{1}{r^4}\frac{\partial^2}{\partial\theta^2}\right). \end{aligned} \tag{5}$$

The elastic constants and inertia terms are defined by

$$(D_{rr}, D_{\theta\theta}, D_{r\theta}, D_{ss}) = \int_{-h/2}^{h/2} z^2(E_{rr}, E_{\theta\theta}, E_{r\theta}, E_{ss})\,\mathrm{d}z, \tag{6}$$

$$R_0 = \int_{-h/2}^{h/2} \rho\,\mathrm{d}z. \tag{7}$$

3. Boundary conditions

Non-uniform boundary conditions are specified in order to model edge-damage (see [11]). Edge damage on the outer circumference only is considered. Here the outer edge is taken to be partially clamped and partially simply supported, whereas the inner edge is completely clamped. These conditions read:

$$\begin{aligned} \mathrm{BC1}: \quad & w(a) = w'(a) = 0, \\ & w(b) = M_r(b) = 0, \quad 0 \leqslant \theta \leqslant \theta_1, \\ \mathrm{BC2}: \quad & w(b) = w'(b) = 0, \quad \theta_1 < \theta \leqslant 2\pi \end{aligned}$$

(where w' denotes the derivative of w with respect to the radial coordinate).

In these conditions it can be shown that

$$M_r = -D_{rr}\frac{d^2w}{dr^2} - \frac{D_{r\theta}}{r}\frac{dw}{dr}, \tag{8}$$

where the appropriate value of the outer radius replaces r. Note that the angle θ_1 will be referred to as the angle of imperfection.

4. Method of solution

A numerical finite difference approach is adopted in order to solve the current problem. The annular plate is covered with a mesh of nodes in the radial and circumferential directions (see Fig. 1). The increments in the r and θ directions are denoted by Δr and $\Delta\theta$, respectively. At each point of the mesh the governing differential equation is replaced by a finite difference equation based on the use of central differences to fourth order. Such an approach involves imaginary points that are beyond the outer radius. In order to link the value of w within the domain to imaginary external values use is made of the boundary conditions that involve

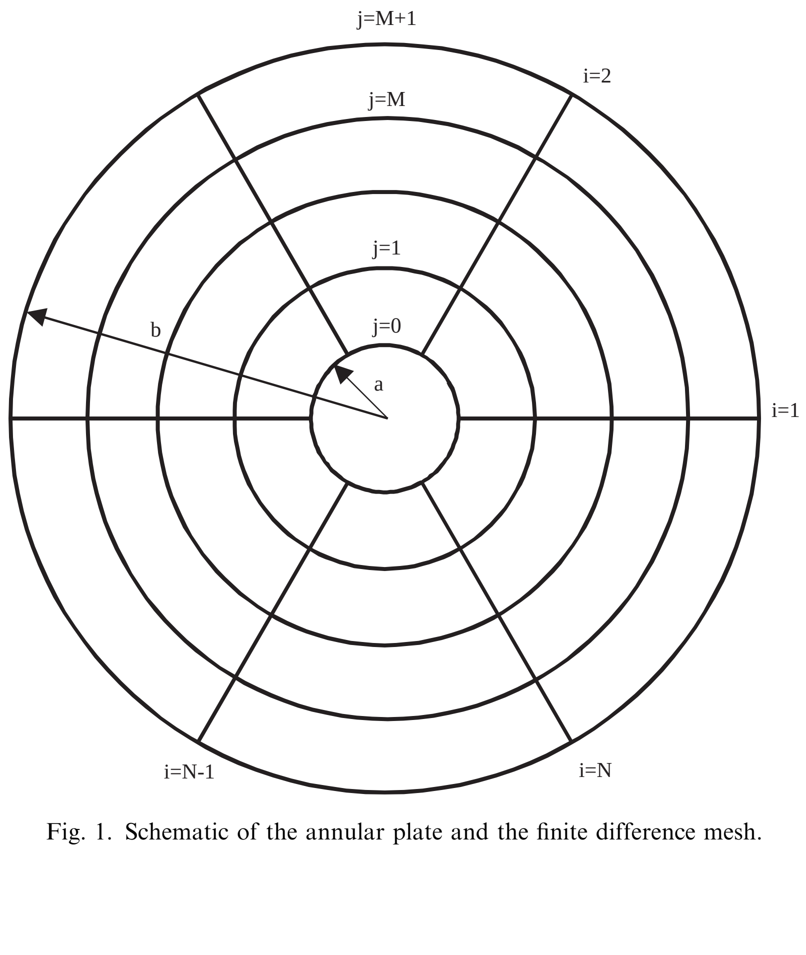

Fig. 1. Schematic of the annular plate and the finite difference mesh.

derivatives. However, these derivatives are now approximated using shifted differences. For example,

$$\frac{\mathrm{d}w(b)}{\mathrm{d}r} \approx \frac{1}{\Delta r}\left(\nabla - \frac{\nabla^2}{2} - \frac{\nabla^3}{6} - \frac{\nabla^4}{12}\right) w(b + \Delta r), \tag{9}$$

where ∇ is the backwards difference operator. As can be seen differences up to the fourth order are employed, for consistency with the level of accuracy at points within the solution domain (see, e.g. [12]).

If there are M points in the radial direction and N points in the circumferential direction at which the displacement w is unknown the following set of NM linear algebraic equations that replaces the original differential equation and boundary conditions is obtained

$$(\mathbf{D} - \omega^2 R_0 \mathbf{I})\mathbf{W} = 0, \tag{10}$$

where **D** is the matrix of the finite difference coefficients, **I** is the unit matrix and **W** is the vector of flexural displacements at the mesh points. Equation (10) has a non-trivial solution if

$$\det(\mathbf{D} - \omega^2 R_0 \mathbf{I}) = 0. \tag{11}$$

The problem is thus reduced to computing the eigenvalues of the matrix **D**. This was achieved using a routine of the Matlab software. The eigenvectors (i.e. mode shapes) are produced as output with the eigenvalues.

5. Results and discussion

For all the calculations performed using the aforedescribed approach the number of points in the radial and circumferential directions was $M = 20$ and $N = 36$, respectively. Results computed using more points showed negligible variation. The inner radius was assigned the value $a = 0.08$ m whereas the outer radius was taken to be $b = 0.16$ m. The thickness of the annuli was held fixed at a value of $h = 0.001$ m. A number of different material lay-ups was considered. Material properties relevant to the calculations are listed in Table 1, based on [13]. The computer program was thoroughly checked by comparing with data from the literature for uniform boundary conditions for homogeneous isotropic and orthotropic annuli [14,15], with good agreement .

Table 1
Material properties

Material (abbreviation)	Density ρ kg/m$^3 \times 10^3$	E_{rr} N/m$^2 \times 10^{10}$	$E_{r\theta}$ N/m$^2 \times 10^{10}$	$E_{\theta\theta}$ N/m$^2 \times 10^{10}$
Steel (ST)	7.492	20.6	6.8	20.6
S-glass epoxy (SGE)	2.002	5.21	0.3	1.18
Ultra high modulus graphite epoxy (UHMG)	1.613	31.00	0.16	0.62
Ultra high modulus graphite epoxy inverted (UHMGI)	1.613	0.62	0.16	31.00

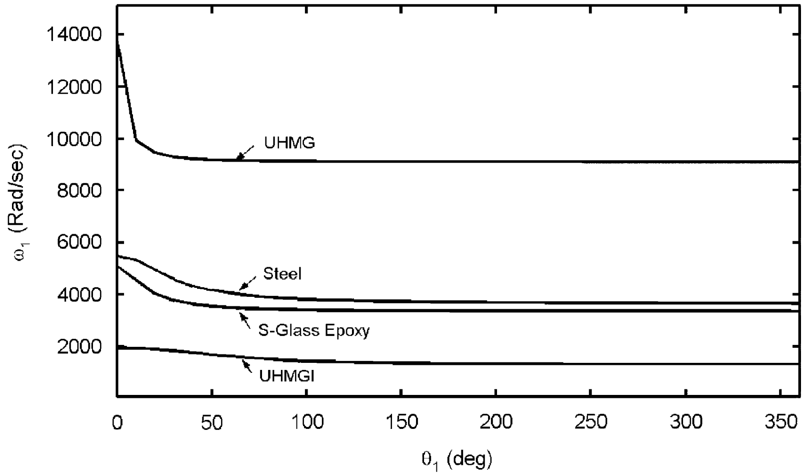

Fig. 2. Sensitivity of various annular plates to edge damage angle.

5.1. Homogeneous annular plates

In Fig. 2 the sensitivity of the lowest frequency of vibration to the angle of edge damage θ_1 is illustrated for steel, s-glass epoxy, ultra high modulus graphite epoxy and inverted ultra high modulus graphite epoxy annular plates. These materials have orthotropy ratios $(D_{rr}/D_{\theta\theta})$ of 1, 4.42, 50 and 0.02, respectively. Recall that the edge-damage is modeled by the simple support condition. Thus, when $\theta_1 = 0$ the entire annulus is clamped along the outer circumference and there is no edge damage. As θ_1 increases the edge damage increases until eventually when $\theta_1 = 2\pi$ the entire outer circumference is simply supported. In all cases the ratio of these two limiting frequencies is about 1.5. For the steel plate the frequency drops drastically even with slight damage at the edge. For example, the frequency drops by almost 74% when less than 20% of the circumference (~18.8%) is damaged. For the s-glass epoxy annulus the sensitivity to the edge damage is greater so that the fully clamped annulus's frequency is reduced by 74% when only approximately 7% of the edge is damaged.

The sensitivity to edge damage of the homogeneous UHMG annulus is even more pronounced—edge damage of a mere 3% is sufficient to produce a reduction in the frequency by 74%. In contrast, the effect of edge damage on a UHMGI annulus is rather more gradual. A reduction in the frequency by 74% is now achieved when 27% of the circumference is simply supported.

Summarizing the findings so far described would seem to lead to the conclusion that, for the problem at hand, the greater the orthotropy ratio of the material comprising the annulus the greater its sensitivity to edge damage.

5.2. Triple layered annular plates

Attention is now turned to composite orthotropic annular plates having a symmetric lay-up. In view of the previous findings annuli comprised of UHMG and UHMGI are considered. The

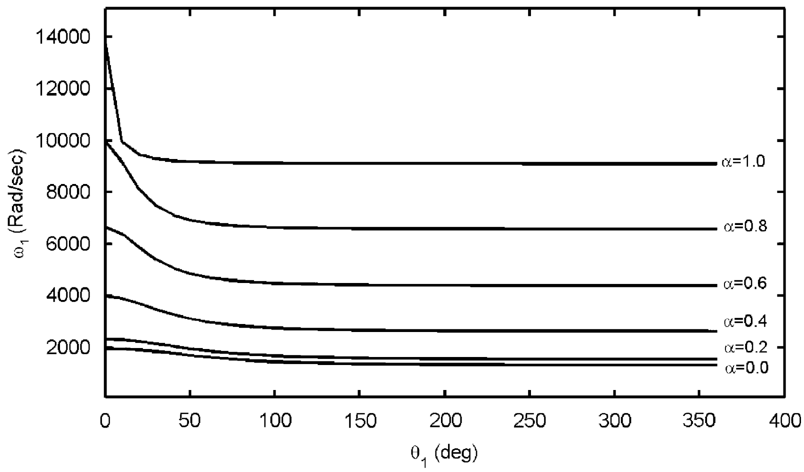

Fig. 3. Sensitivity of triple layered UHMGI/UHMG/UHMGI annular plates to edge damage angle, for different symmetric lay-ups.

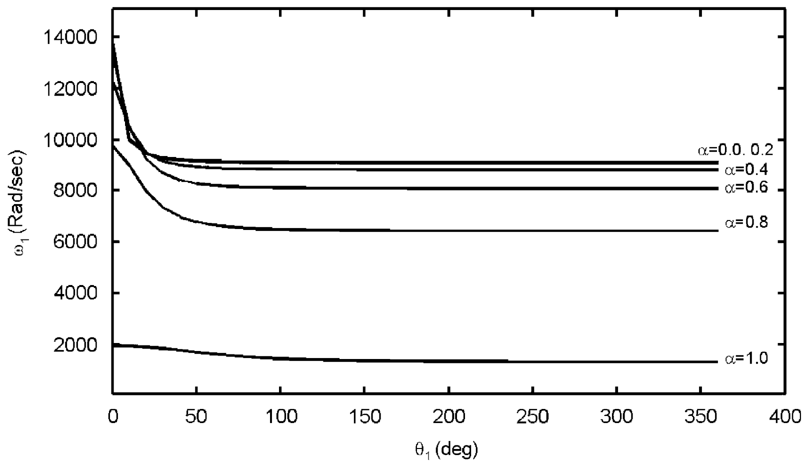

Fig. 4. Sensitivity of triple layered UHMG/UHMGI/UHMG annular plates to edge damage angle, for different symmetric lay-ups.

central layer is taken as UHMG of thickness αh whereas the outer layers are UHMGI, each of thickness $0.5(1-\alpha)h$, where $0 \leqslant \alpha \leqslant 1$ will be referred to as the relative thickness parameter. The influence of both the plates lay-up and the edge damage are illustrated in Fig. 3. It is clear that as the influence of the outer layers becomes greater the sensitivity to edge damage is reduced. If the roles of the outer and inner layers are reversed so that UHMGI is the central core whereas UHMG comprises the outer layers the vibrational response with edge damage is as presented in Fig. 4. It is of interest to note that the dominant effect of the outer layers is clearly visible as can be seen by the clustering of the curves for values of α between 0.8 and 0. In general, comparing Figs.

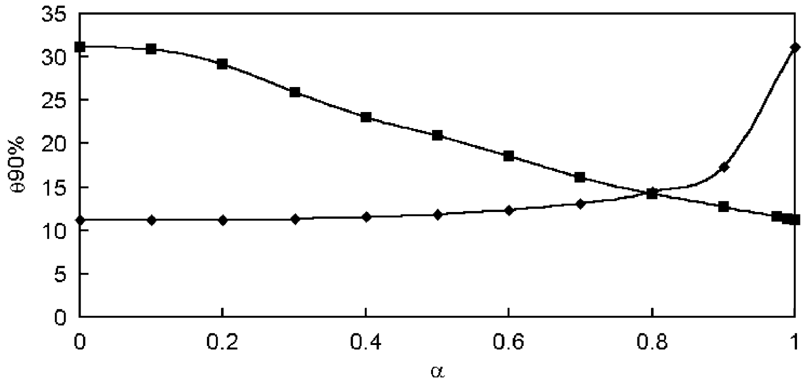

Fig. 5. Effect of thickness parameter on the edge damage angle at which the frequency is reduced by 90%—triple layered UHMG/UHMGI/UHMG (diamonds) and UHMGI/UHMG/UHMGI (squares) annular plates.

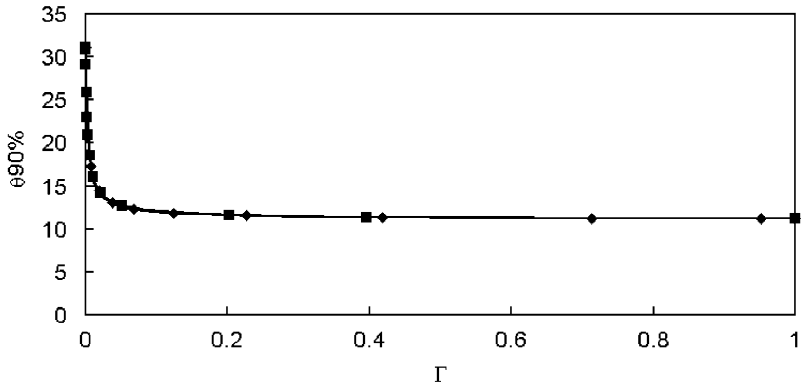

Fig. 6. Dependence of edge damage angle at which a 90% drop in frequency occurs on the normalized orthotropy ratio for UHMG/UHMGI/UHMG (diamonds) and UHMGI/UHMG/UHMGI (squares) annular plates.

3 and 4, it can be deduced that the sensitivity of the plates having UHMG in the outer layers is greater than the sensitivity of the plates having UHMGI in the outer layers. Another way of viewing this greater sensitivity is shown in Fig. 5 in which graphs of the value of θ_1 at which the frequency of the clamped plate is reduced by 90%, $\theta_{90\%}$, as a function of the thickness parameter α, for the UHMGI/UHMG/UHMGI and UHMG/UHMGI/UHMG composite annuli are drawn. For the former lay-up the curve drops rather gradually as the amount of UHMG in the core is reduced and the amount of UHMGI increases accordingly. For the latter lay-up the dominance of the properties of the UHMG layers is strikingly evident until at least about 80–90% of the plate is comprised of UHMGI.

All the previously described results provide a clear indication of the major role played by the annuli's orthotropy in determining their sensitivity to the edge damage angle. In Fig. 6 a graph is drawn of the dependence of the edge angle at which the frequency was reduced by 90% on the normalized plate orthotropy ratio which we define according to $\Gamma = (D_{rr}/D_{\theta\theta})/(D_{rr}/D_{\theta\theta})_{\max}$. where the subscript max refers to the maximum value of the orthotropy ratio in the range of values covered (i.e. for $0 \leqslant \alpha \leqslant 1$). The points on the graph are taken from the symmetrically layered UHMG/UHMGI/UHMG (the diamond points) and UHMGI/UHMG/UHMGI (the square points) material combinations. Remarkably the two curves are identical. This clearly highlights the prime role of the orthotropy ratio of the annulus in determining vibrational response to the non-uniform boundary conditions.

5.3. Mode shapes

Consideration is now given to the mode shapes associated with the frequencies of the edge damaged annuli discussed above. Fig. 7 illustrates the mode shape for the lowest frequency of vibration of a UHMG annulus subjected to an edge damage angle of 10°. Both a three-dimensional representation and a contour map are presented for clarity. The localized effect of the simply supported damaged section is readily observable. In Fig. 8 the mode shape is drawn for a steel annulus having a 10° edge damage angle. It is clear that the reduction in the orthotropy ratio from 50 to unity leads to a less pronounced localization of the displacement. A greater area in the circumferential direction is apparently influenced by the edge damage. Finally, in Fig. 9, the mode shape for the 10° edge damaged UHMGI annulus is drawn. The increased extent to which the edge damage is felt in the mode shape is strikingly evident. The last three figures suggest that the mode shape is more extensively circumferentially influenced beyond the locale of the actual edge damage as the value of the orthotropy ratio decreases. This stands in contrast to the situation with the regard to the vibrational frequency for which it was shown that as the orthotropy ratio decreased the less sensitive the frequency response was to the edge damage angle.

5.4. Higher frequencies and associated mode shapes

Until now the discussion has been restricted to the effect of non-uniform boundary conditions on the lowest frequency of vibration. The influence of the edge damage angle on the higher vibrational frequencies and mode shapes is now briefly examined.

In Fig. 10 the sensitivity of the first seven frequencies of a UHMGI annular plate are drawn. A number of observations are in place. First, the higher frequencies are influenced by the edge damage angle but they decrease less rapidly than the lowest frequency as the angle increases. For the undamaged plate ($\theta_1 = 0$) the frequencies of the second and third frequencies are identical (1994.2 rad/sec) as are those of the fourth and fifth (2264.2 rad/sec), the sixth and seventh (3168.5 rad/sec)and eighth and ninth pairs (4672.9 rad/sec)) (the latter pair is not shown in the figure). This pattern then repeats itself from the 10th frequency onwards. Similar behavior occurs for the case of complete edge damage ($\theta_1 = 2\pi$). For other edge damage angles it can be seen that there are numerous points of cross over of the aforementioned pairs of frequency curves. Eastep and Hemmig [5] report similar cross over behavior in their numerical and experimental work on isotropic plates with non-uniform boundary conditions. Such behavior must be carefully scrutinized by the experimentalist since it indicates the permissibility of two modes having identical frequencies.

The behavior of the higher frequencies of a UHMG annular plate (not shown here) is rather different. The extreme sensitivity to the edge damage angle essentially follows that shown in Fig. 2, and no points of cross over exist.

In Fig. 11 the contours of the first six mode shapes are drawn for a UHMG plate with $\theta_1 = 80^\circ$. Note that for all these mode shapes the displacements are primarily confined to the damaged

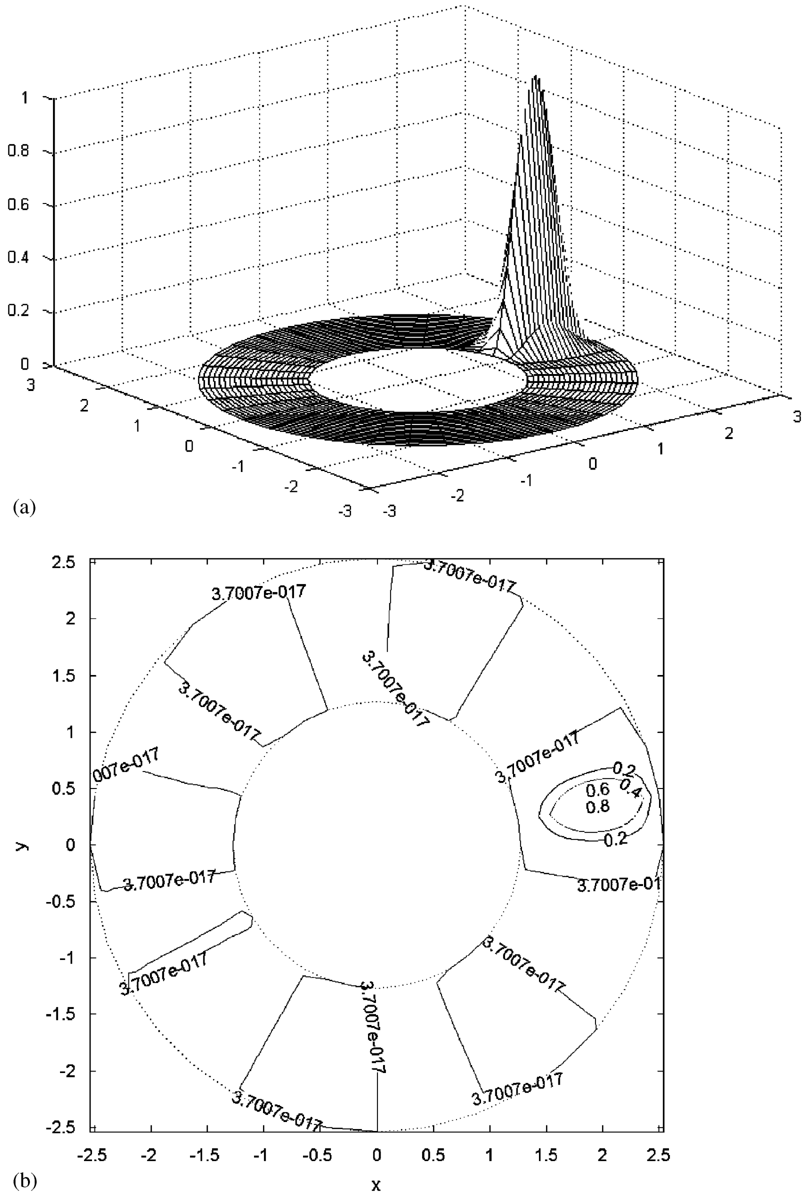

Fig. 7. (a) Mode shape and (b) mode shape contours for a UHMG annular plate subjected to 10° edge damage.

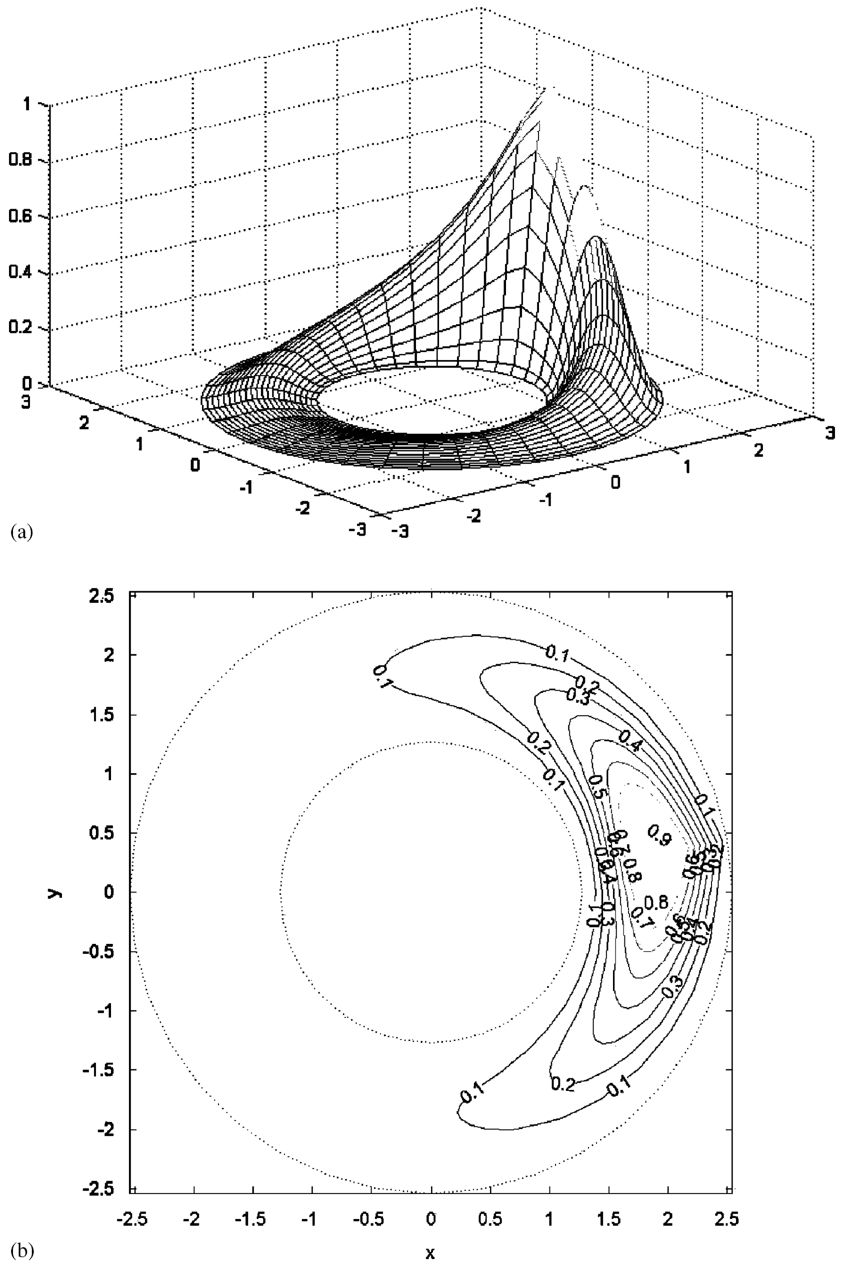

Fig. 8. (a) Mode shape and (b) mode shape contours for a steel annular plate subjected to 10° edge damage.

section of the plate with a symmetric/antisymmetric circumferential wave shape for odd/even mode numbers.

In contrast, the mode shapes computed in Fig. 12 are for a UHMGI annular plate having the same edge damage angle. The picture that emerges is conspicuously different from that of Fig. 12

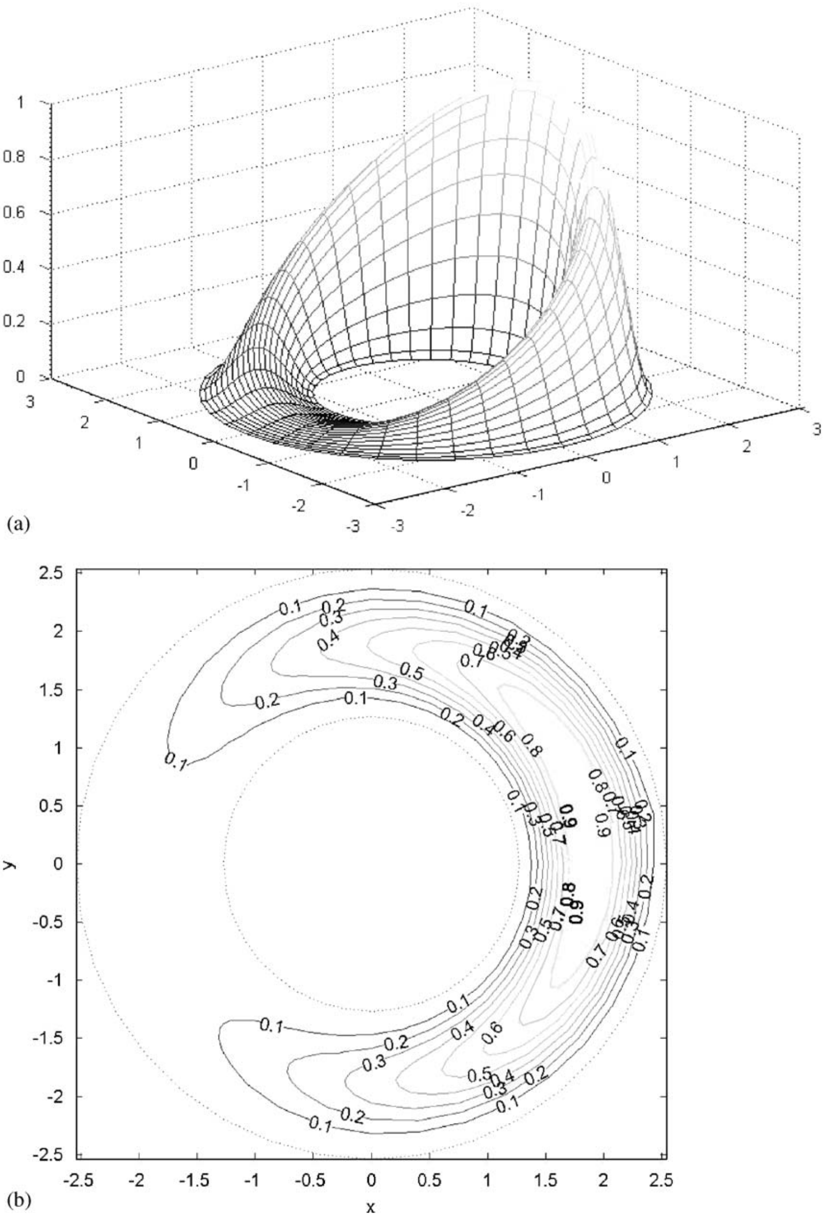

Fig. 9. (a) Mode shape and (b) mode shape contours for a UHMGI annular plate subjected to 10° edge damage.

and clearly illustrates the way in which a greater area of the plate is influenced by the non-uniformity of the boundary conditions, in consonance with our previous discussed observation for the first mode shape. In addition, it can be seen that both mode shapes four and five have four circumferential waves. However, mode shape five is symmetric whereas mode shape four is antisymmetric. These shapes interchange at an edge damage angle of $\theta_1 = 70^{\circ}$ (not shown here) due to the cross over of the fourth and fifth frequencies in this vicinity (see Fig. 10).

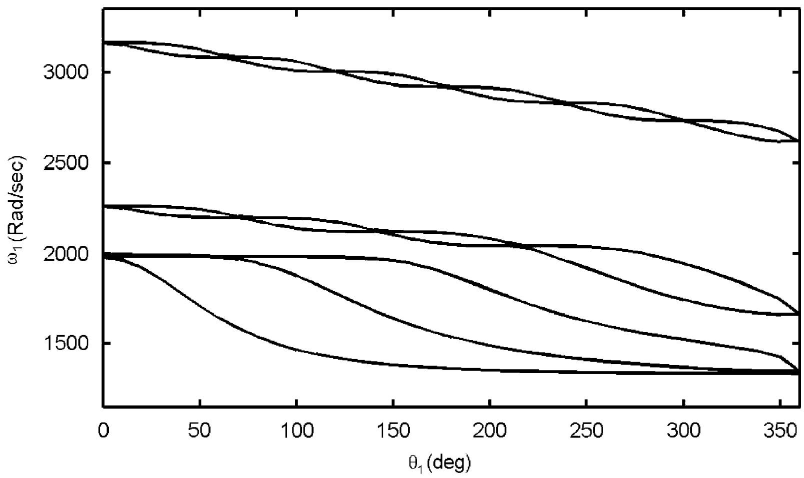

Fig. 10. Sensitivity of inverted ultra high modulus graphite epoxy annular plate to edge damage angle—first seven frequencies.

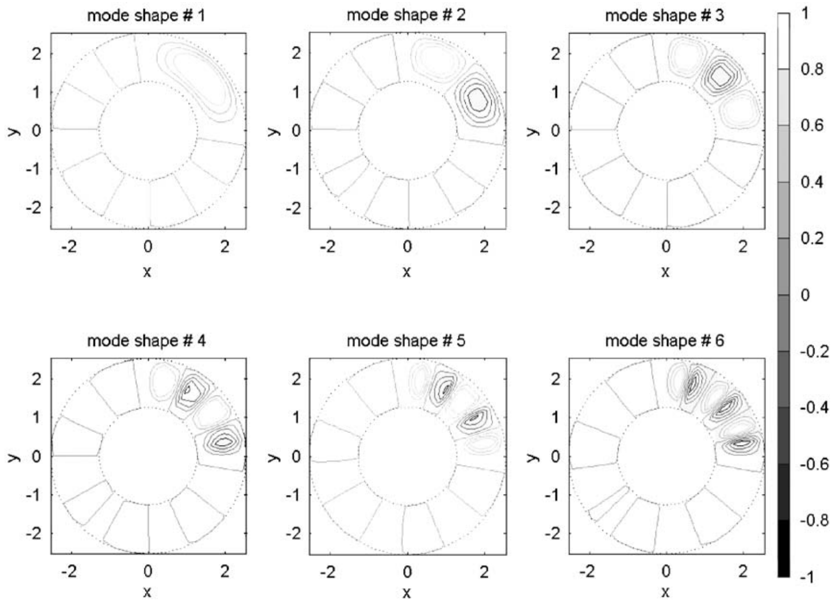

Fig. 11. Mode shapes for first six frequencies of UHMG annular plate with 80° edge damage angle.

6. Conclusions

The free vibrational response of clamped homogeneous or symmetrically laminated orthotropic annular plates with edge damage on the outer circumference has been studied for the first time, using the finite difference method. The edge damage was modeled via the application of a non-

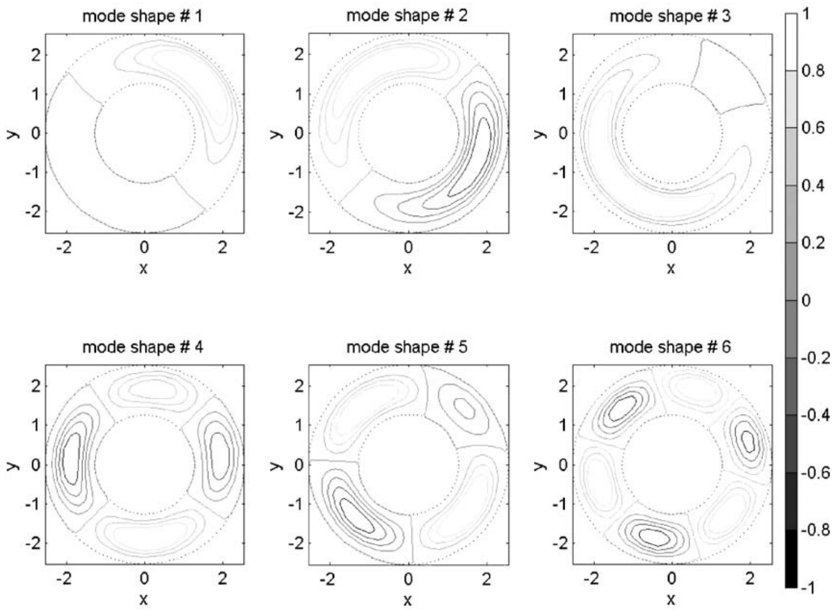

Fig. 12. Mode shapes for first six frequencies of UHMGI annular plate with 80° edge damage angle.

uniform boundary condition. The sensitivity of vibrational frequencies to the extent of the edge damage was investigated and shown to be strongly influenced by the material properties and the relative thicknesses of the materials comprising the laminated plates. It was revealed that the orthotropy ratio for the plate (defined here by $D_{rr}/D_{\theta\theta}$) is a convenient compact parameter for indicating the sensitivity to edge damage. Mode shapes were also computed and found to be profoundly effected by the orthotropy ratio. The behavior of higher modes of vibration was also discussed and the phenomenon of cross over, whereby two modes exist having the same frequency, is noted.

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