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Short Communication

The damped dynamic vibration absorbers: revisited and new result

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1. Introduction

The dynamic vibration absorber (DVA) or tuned-mass damper (TMD) is a widely used passive vibration control device. A simple DVA consists of a mass and a spring. When a mass–spring system or a primary system is excited by a harmonic force, its vibration can be suppressed by attaching a DVA as shown in Fig. 1(a). However, adding a DVA to a one-degree-of-freedom (dof) system results in a new 2-dof system. If the exciting frequency coincides one of the two natural frequencies of the new system, the system will be at resonance. To overcome this problem, a damper is added to DVA. Fig. 1(b) shows a primary system attached by a damped DVA. Equations of motion of the system are given as

$$m\ddot{x} + c_a\dot{x} - c_a\dot{x}_a + (k + k_a)x - k_a x_a = F_0 \sin(\omega t), \quad m_a\ddot{x}_a - c_a\dot{x} + c_a\dot{x}_a - k_a x + k_a x_a = 0, \tag{1}$$

where m and m_a are the primary mass and the absorber mass, respectively, k and k_a are the primary stiffness and the absorber stiffness, respectively, c_a is the damping value of the damper, F_0 is the force amplitude and ω is the exciting frequency. The normalized amplitude of the steady-state response of the primary mass is given as

$$G = \left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(2\zeta r)^2 + (\beta^2 - r^2)^2}{[1 - (1+\mu)r^2]^2(2\zeta r)^2 + [(1 - r^2)(\beta^2 - r^2) - \mu\beta^2 r^2]^2}}, \tag{2}$$

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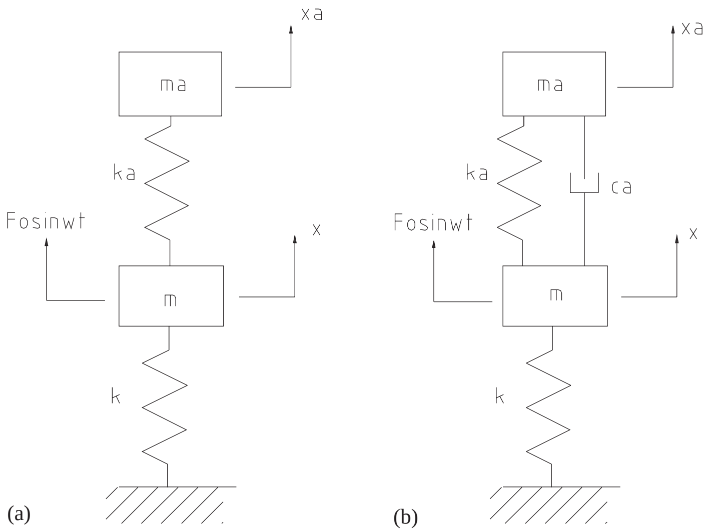

Fig. 1. Dynamic vibration absorber model A: (a) DVA; (b) damped DVA.

where following the notations used in Inman's textbook [1], the variables in the above equation are defined as

$$\omega_p = \sqrt{k/m}, \quad \omega_a = \sqrt{k_a/m_a}, \quad \beta = \omega_a/\omega_p, \quad \mu = m_a/m, \quad \zeta = c_a/(2m_a\omega_p), \quad r = \omega/\omega_p.$$

Using the following values: $m = 1.0\,\mathrm{kg}$, $k = 8900\,\mathrm{N/m}$, $m_a = .15\,\mathrm{kg}$, $\beta = 1$, Fig. 2 shows G vs. r for three different values of the damping ratio ζ. In a classical textbook on mechanical vibrations [2], Den Hartog pointed out a remarkable peculiarity in the figure that all three curves intersect at the two points P and Q. He proved that this is no accident and there exist the two fixed points independent of damping. Den Hartog further found that the optimum tuning parameter for β should be $\beta = 1/(1+\mu)$ such that the ordinates G of P and Q are equal. He also indicated that the optimum damping ratio ζ must be a value of ζ for which the curve passes horizontally through either P or Q. Den Hartog claimed that this optimum value can be found by differentiating Eq. (2) with respect to r, thus finding the slope, and equating the slope to zero for the point P or Q. Recognizing an undue amount of labor of this method, he did not present an analytical result. Subsequently, Brock [3] took a different approach which is quite clever, yet straightforward. No differentiation was needed. Based on the results, he suggested that the optimum damping ratio can be given by

$$\zeta_{\mathrm{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)^3}}. \tag{3}$$

It should be noted that frequently [5–7], the optimum damping ratio is also given as

$$\zeta_{\mathrm{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)}}.$$

In this case, the damping ratio is defined as $\zeta = c_a/(2m_a\omega_a)$.

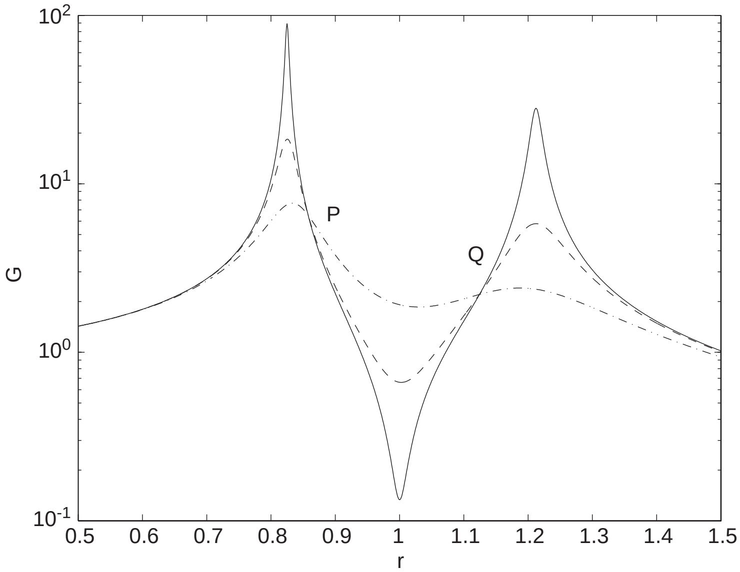

Fig. 2. Normalized amplitude of the steady-state response of the primary mass of model A: $\zeta = .01$ (solid line); $\zeta = .05$ (dashed line); $\zeta = .15$ (dash–dot line).

Den Hartog included this result in the late edition of his book [4]. However, he still stated that the result can be obtained by the approach given in Ref. [2] (see the description given in Ref. [4, p. 103]). We attempted Den Hartog's approach and encountered a prohibitive complexity. We also note that in a recent textbook by Kelly [5], a different way is suggested. Kelly states that the optimum value ζ can be obtained by setting $\mathrm{d}G/\mathrm{d}\zeta = 0$ using $\beta = 1/(1+\mu)$ (see the paragraph above Eq. (8.63) of Ref. [5]). We tried this way and failed to get any result.

Another less common way to attach a damper to a DVA is shown in Fig. 3. In active suspension systems [8–10], depending on how the inertia reference frame is interpreted, the damper in Fig. 3(a) is referred to as skyhook damper while the damper in Fig. 3(b) is referred to as groundhook damper. The force a skyhook damper would deliver has been shown to be the optimal force to minimize the sprung mass (m_a for the present system) motion. Although the skyhook damper has been perceived as an ideal active suspension device, it is not practical in most situations. Apparently, the objective of using a damper for a vibration absorber is different from that of using a skyhook damper for active suspension systems. Nevertheless, connecting a damper between the inertial reference frame and the absorber mass provides an alternative, which sometimes becomes a sole choice, for example, when a damper is too massive.

In this study, we refer to the damped DVA in Fig. 1 as model A and to the damped DVA in Fig. 3 as model B. To the best of our knowledge, no study has been reported on the optimum parameters of model B. The contribution of this technical note is twofold: clarification of the derivation process of Eq. (3) for model A and development of the optimum parameters for model B.

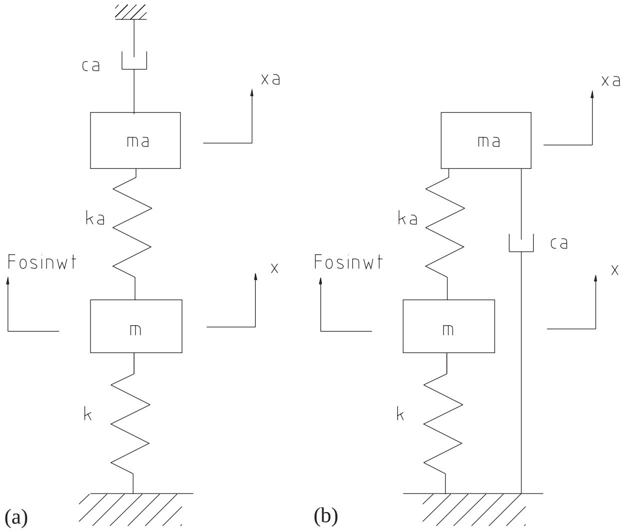

Fig. 3. Dynamic vibration absorber model B: (a) skyhook damper; (b) groundhook damper.

2. Optimum parameters of model B

Equations of motion for model B are given as

$$m\ddot{x} + (k + k_a)x - k_a x_a = F_0 \sin(\omega t), \;\; m_a \ddot{x}_a + c_a \dot{x}_a - k_a x + k_a x_a = 0. \tag{4}$$

The normalized amplitude of the steady-state response of the primary mass is given as

$$G = \left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(2\zeta r)^2 + (\beta^2 - r^2)^2}{(1 + \mu\beta^2 - r^2)^2(2\zeta r)^2 + [(1 - r^2)(\beta^2 - r^2) - \mu\beta^2 r^2]^2}}. \tag{5}$$

Using the following values: $m = 1.0\,\text{kg}$, $k = 8900\,\text{N/m}$, $m_a = .15\,\text{kg}$, $\beta = 1$, the normalized amplitudes are evaluated for three different damping ratios and the results are shown in Fig. 4.

Comparison of Figs. 2 and 4 reveals that both the models behave similarly. To find the abscissas of points P and Q, Den Hartog expressed Eq. (2) in the form (note an error in Ref. [4, Eq. (3.24)])

$$G = \sqrt{\frac{A\zeta^2 + B}{C\zeta^2 + D}} \tag{6}$$

which is independent of damping if $A/C = B/D$. Following the same procedure, we solve the equation

$$\frac{1}{(1 + \mu\beta^2 - r^2)^2} = \frac{(\beta^2 - r^2)^2}{[(1 - r^2)(\beta^2 - r^2) - \mu\beta^2 r^2]^2}. \tag{7}$$

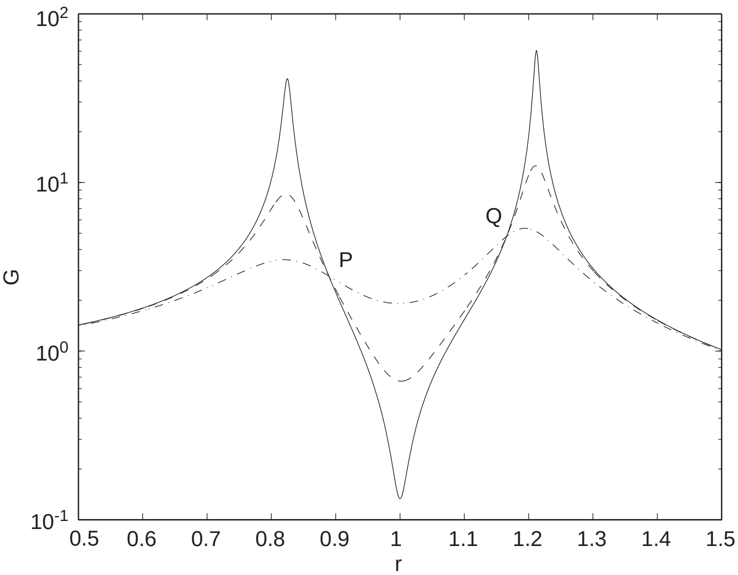

Fig. 4. Normalized amplitude of the steady-state response of the primary mass of model B: $\zeta = .01$ (solid line); $\zeta = .05$ (dashed line); $\zeta = .15$ (dash–dot line).

The solutions are

$$r_{1,2} = \sqrt{\frac{1 + (1+\mu)\beta^2 \mp \sqrt{1 + 2(\mu - 1)\beta^2 + (\mu^2 + 1)\beta^4}}{2}}. \tag{8}$$

The ordinates of points P and Q can be found by letting $\zeta \to \infty$ in Eq. (5):

$$G = \sqrt{\frac{1}{(1 + \mu\beta^2 - r^2)^2}}. \tag{9}$$

The optimum value for β is obtained by setting $G(r_1) = G(r_2)$:

$$\frac{1}{1 + \mu\beta^2 - r_1^2} = \frac{-1}{1 + \mu\beta^2 - r_2^2} \tag{10}$$

which yields

$$\beta^* = \frac{1}{\sqrt{1 - \mu}}. \tag{11}$$

Substituting r_1 and β^* into Eq. (9) gives the common ordinate

$$G(r_1) = G(r_2) = \frac{2(1 - \mu)}{\sqrt{2\mu}}. \tag{12}$$

Now we are ready to apply Brock's approach to find the optimum damping ratios. In order to cause the curve of G vs. r to pass horizontally through point P, we first require that it pass through a point P' of abscissa $r^2 = r_1^2 + \delta$ and ordinate $G = 2(1-\mu)/\sqrt{2\mu}$, and then let δ approach zero as a limit. Solving Eq. (5) for ζ^2, we obtain (note an error in Ref. [3, Eq. (3)])

$$\zeta^2 = \frac{(\beta^2 - r^2)^2 - G^2[(1-r^2)(\beta^2 - r^2) - \mu\beta^2 r^2]^2}{4r^2[G^2(1+\mu\beta^2 - r^2)^2 - 1]}. \tag{13}$$

Following Brock's idea, after substituting $r^2 = r_1^2 + \delta$ and $G = 2(1-\mu)/\sqrt{2\mu}$ into the above equation, we should have a result of the form

$$\zeta^2 = \frac{A_0 + A_1\delta + A_2\delta^2 + A_3\delta^3 + \cdots}{B_0 + B_1\delta + B_2\delta^2 + B_3\delta^3 + \cdots}. \tag{14}$$

Since Eq. (13) assumes the indeterminate form 0/0 if $\delta = 0$, we know that $A_0 = B_0 = 0$. As δ is a very small number, we can neglect the higher order terms (this is how we interpret Brock's approach as he did not explicitly say it) and the desired result is given by

$$\zeta^2 = \frac{A_1}{B_1}. \tag{15}$$

Finding A_1 and B_1 proved to be quite tedious. As Brock did not show the procedure, we give a brief description of how we did it. Because we only need to find all the terms associated with δ, we use the approximations

$$r^4 \approx r_1^4 + 2r_1^2\delta, \quad r^6 \approx r_1^6 + 3r_1^4\delta, \quad r^8 \approx r_1^8 + 4r_1^6\delta. \tag{16}$$

This way, we obtain

$$\begin{aligned} A_1 &= 2\beta^2\{G^2[1+(1+\mu)\beta^2] - 1\} + 2\{1 - 2G^2\beta^2 - G^2[1+(1+\mu)\beta^2]^2\}r_1^2 \\ &\quad + 6G^2[1+(1+\mu)\beta^2]r_1^4 - 4G^2r_1^6, \\ B_1 &= 4G^2(1+\mu\beta^2)^2 - 4 - 16G^2(1+\mu\beta^2)r_1^2 + 12G^2r_1^4. \end{aligned}$$

Now substituting r_1^2 of Eq. (8), β^* of Eq. (11), and $G(r_1, \beta^*)$ of Eq. (12) into the above equations yields

$$\zeta_1^2 = \frac{3\mu\sqrt{2\mu}}{8(\sqrt{2\mu} - \mu)(1-\mu)}. \tag{17}$$

By a similar procedure with r_2^2 of Eq. (8), we obtain

$$\zeta_2^2 = \frac{3\mu\sqrt{2\mu}}{8(\sqrt{2\mu} + \mu)(1-\mu)}. \tag{18}$$

As suggested by Brock, a convenient average value

$$\zeta_{\text{opt}} = \sqrt{\frac{\zeta_1^2 + \zeta_2^2}{2}} = \frac{1}{2}\sqrt{\frac{3\mu}{(1-\mu)(2-\mu)}} \tag{19}$$

can be used as the optimum damping ratio.

With the optimum parameters for both the models, a comparison is conducted. Using the following values: $m = 1.0\,\text{kg}, k = 8900\,\text{N/m}, m_a = .15\,\text{kg}$, we list some values for each model in Table 1. In the table, $\bar{G}$ is the average normalized amplitude defined as

$$\bar{G} = \frac{1}{1.5 - .5} \int_{.5}^{1.5} G\,\mathrm{d}r. \tag{20}$$

Using the optimum parameters, we plot the curves of G vs. r for both the models in Fig. 5. Numerically we can find the peak values $G_{pi}, i = 1, 2$ of the normalized amplitude and their corresponding frequency ratios $r_{pi}, i = 1, 2$ and the value G_v of the dip between the two peaks and its corresponding frequency ratio r_v. The results are summarized in Table 2. We make the following observations. First we note that two peaks in each of the curves are almost equal in height. As expected, $G(r_1) = G(r_2) \approx G_{p1} \approx G_{p2}$. For model B to be optimum, a larger damping

Table 1
Comparison of the two models

Model	β^*	ζ_{opt}	$G(r_1) = G(r_2)$[a]	$G(r = \beta^*)$	$\bar{G}$
A	0.8696	0.1923	3.786	3.476	2.522
B	1.085	0.2675	3.104	2.795	2.376

[a] $r_1 = 0.7999$, $r_2 = 1.049$ for Model A; $r_1 = 0.9243$, $r_2 = 1.224$ for Model B.

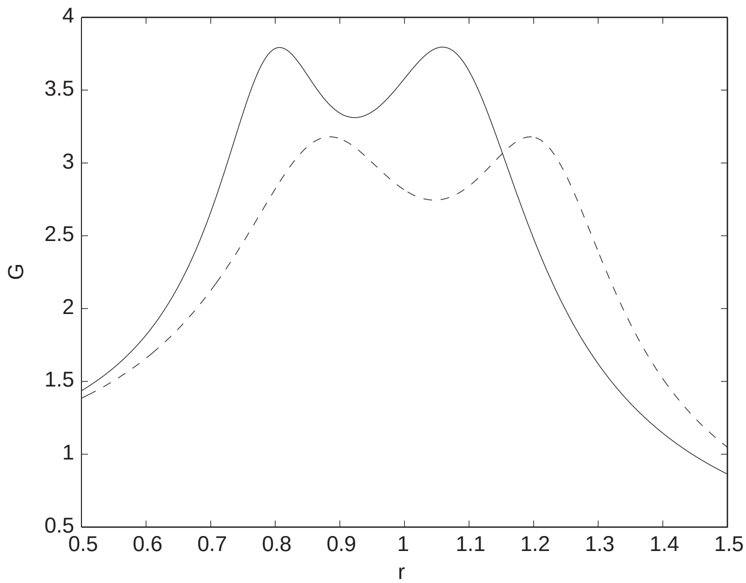

Fig. 5. Comparison of the optimum model A and model B: model A (solid line); model B (dashed line).

Table 2
Summary of some key values from Fig. 5

Model	G_{p1}	r_{p1}	G_{p2}	r_{p2}	G_v	r_v
A	3.793	0.807	3.795	1.059	3.311	0.923
B	3.180	0.884	3.179	1.195	2.745	1.046

ratio is required. Overall, model B gives better vibration suppression, evidenced by the smaller G_{pi}, the smaller G_v, and the smaller $\bar{G}$. We also note that the performance at the anti-resonance frequency of model B does not significantly degrade due to an increase of damping, i.e., around $r = \beta^*$, there is still a dip.

Brock also found the result for constant tuning. The constant tuning is defined as the case when $\beta = 1$. In this case, the ordinates of P and Q are not equal. For model A, the ordinate of point P is greater than that of point Q. The optimum damping ratio is considered to be the value for which the (r, G) curve passes horizontally through point P. Brock found that this value is

$$\zeta_{\text{opt}} = \sqrt{\frac{\mu(3+\mu)[1+\sqrt{\mu/(2+\mu)}]}{8(1+\mu)}}. \tag{21}$$

In the case of model B, the ordinate of point Q is greater than that of point P. We found that the optimum damping ratio is of the form

$$\zeta_{\text{opt}} = \frac{\sqrt{\mu[\mu+6-\sqrt{\mu(\mu+2)}]}}{4}. \tag{22}$$

3. Summary

We have revisited a classical problem: optimum damped dynamic vibration absorber named as model A in this study. We have failed to obtain the results using Den Harto's method and Kelly's method. After comparing Brock's approach with the previous two methods, we have realized that Brock employed a perturbation method instead of differentiating a high-order equation. We have applied Brock's approach to a different type of damped vibration absorber named model B in this study. We have found the optimum parameters for model B. We have verified all the results numerically and presented a comparison of the two models.

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