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Short Communication

Eigenfrequencies of a combined system including two continua connected by discrete elements

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1. Introduction

Structural elements with attachments that can be considered as a combination of mechanical components with distributed and discrete physical parameters are encountered in many engineering applications from aero-spatial and naval systems to ground vehicles and buildings. Beams, bars, plates or shells to which bodies like devices, machinery etc., are attached, can be given as examples of such elements. In general, these attached bodies individually have small dimensions relative to the supporting structural elements, and hence, they can be considered as lumped masses and springs. Such an approach often results in "$\infty + n$" degree of freedom models. These attachments restrict free motion of main element besides the constraints due to its boundary conditions. For this reason, plates or beams with attachments are sometimes called "constrained systems". In some cases, supporting of continuous structural elements can be carried out by attachments. Distributed parameter elements with attachments have been increasingly the subject of interest, since deeper understanding of their dynamic, to be more exact, their vibrational characteristics is very significant to guarantee the entire system's performance. There has been a vast literature on combined systems, in other words, on systems with attachment or constrained systems in the last three decades, of which some presentative papers recently published, will be briefly mentioned here.

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Kukla and Posiadala presented the exact solution of the problem of free transverse vibrations of Bernoulli–Euler beams with elastically mounted masses by using Green function method [1]. The authors claim the solution they presented contains all possible end conditions, and can be applied to beams with rigidly attached masses or intermediate pinned or elastic supports. Kukla et al. studied the natural vibrations of two rods coupled by several translational springs [2]. They used the Green's functions method. Chan et al. studied the vibration of a simply supported beam partially loaded with distributed mass [3]. In addition to natural frequencies, they also obtained the mode shapes. Gürgöze [4], using Lagrange multipliers method, derived the frequency equation of a special combined dynamic system consisting of a clamped–free Bernoulli–Euler beam with a tip mass where a spring–mass system is attached to it. The author obtained the frequency equations of some simpler systems by using limiting process. Mermertaş and Gürgöze treated a similar system in Ref. [2] using the conventional method of separated variables [5]. Gürgöze [6] dealt with the determination of the frequency equation of a fixed–free longitudinally vibrating rod carrying a tip mass to which a spring–mass system is applied in-span. The author also gave an approximate formula for the fundamental frequency based on Dunkerley's procedure, and obtained frequency equation for some simpler cases by using limiting process. Gürgöze and Erol studied the system considered in Ref. [7] including the dampers in the attachment. They obtained an approximate formula for characteristic values as well as an exact equation. They also investigated how sensitive the frequencies of the system are to changing of parameters. İnceoğlu and Gürgöze extended the work presented in Ref. [2]. They studied the longitudinal vibrations of a mechanical system consisting of fixed–free rods carrying tip masses to which many double spring–mass systems are attached across the span. Using Green's function method they derived a general formulation for the exact solution of the system considered [8]. Gürgöze and İnceoğlu examined the problem of determining the stiffness coefficient of the spring to be placed at a specified position so that the fundamental frequency of the bending beam subject to various supporting conditions does not change despite the addition of a mass at a predefined position [9]. Cha used springs and masses as passive means of inducing multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. According to the author when the parameters of the elastically mounted masses are properly chosen, their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing nodes to be imposed at multiple locations anywhere along the combined assembly [10].

As is easily understood from the work cited above, most papers are concerned with determining the relations between the vibrational properties of systems and their physical properties. The present paper aims to present a generative model similar to those mentioned above. However, it differs from the previous ones in that it includes two continua connected by a discrete spring–mass system, performing longitudinal and transversal vibrations. The model that will be given here can be reduced into well-known simple and combined systems. To generalize obtained results, a non-dimensional analysis has been carried out. Furthermore, the limit values of physical parameters for which the system reduce into subsystems have been obtained.

2. Equations of motion and frequency equation

Consider the system shown in Fig. 1, which consists of a rod, a lumped mass, two linear springs and a beam. Assume that the rod and the beam have the lengths of L_1 and L_2, and they are made

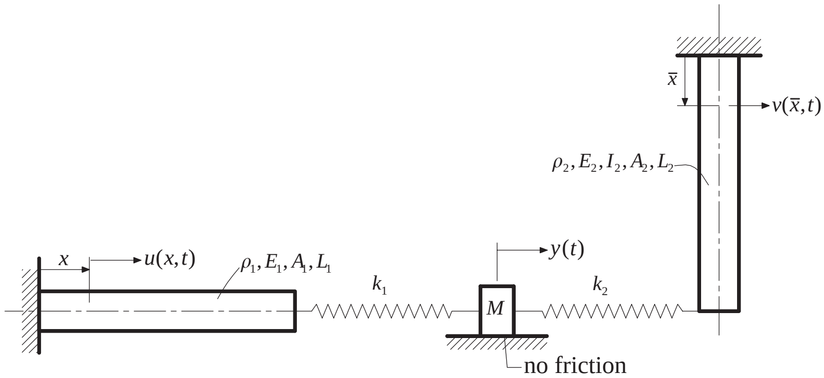

Fig. 1. The combined system.

of uniform and isotropic materials, having constant cross-sectional areas denoted by A_1 and A_2, respectively. The volumetric density and Young's modulus of the rod material are ρ_1 and E_1, while those of the beam are ρ_2 and E_2. The mass of lumped body is M, and the stiffness rates of springs by which the mass M is connected with the rod and the beam are k_1 and k_2, respectively. This system is conservative because there are no external and damping forces. Governing equations of motion of the system can be given directly as

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{E_1}{\rho_1}, \tag{1}$$

$$v_{tt} = -\beta^2 v_{\bar{x}\bar{x}\bar{x}\bar{x}}, \quad \beta^2 = \frac{E_2 I_2}{\rho_2 A_2}, \tag{2}$$

$$M\ddot{y} + k_1(y - u(L_1, t)) + k_2(y - v(L_2, t)) = 0 \tag{3}$$

along with the associated boundary conditions for the rod and the beam

$$u(0, t) = 0, \tag{4a}$$

$$E_1 A_1 u_x(L_1, t) = k_1(y - u(L_1, t)), \tag{4b}$$

$$v(0, t) = 0, \tag{4c}$$

$$v_{\bar{x}}(0, t) = 0, \tag{4d}$$

$$E_2 I_2 v_{\bar{x}\bar{x}}(L_2, t) = 0, \tag{4e}$$

$$E_2 I_2 v_{\bar{x}\bar{x}\bar{x}}(L_2, t) + k_2(y - v(L_2, t)) = 0, \tag{4f}$$

where $y = y(t)$ describes the displacement of lumped mass, and $u = u(x, t)$ represents the longitudinal displacement of any cross-section at x, while $v = v(\bar{x}, t)$ denotes the transversal displacement of any cross-section at $\bar{x}$ (see Fig. 1). The subscripts x, $\bar{x}$ indicate partial derivatives of relevant dependent variables, whereas dots denote derivatives with respect to time.

Since synchronous motions of the system are to be investigated, one can use the method of separation of variables, which proposes the solutions in the form

$$u(x,t) = U(x)T(t), \tag{5}$$

$$v(\bar{x},t) = V(\bar{x})T(t) \tag{6}$$

and

$$y(t) = Y_0 T(t) \tag{7}$$

for rod, beam and lumped mass vibrations, respectively. Considering the boundary conditions (4a), (4c) and (4d) the conventional procedure of this method leads to the eigenfunctions for rod and beam, respectively, as

$$U(x) = \bar{B} \sin \frac{\omega}{c} x \tag{8}$$

and

$$V(\bar{x}) = \bar{C}\{\cos(d\bar{x}) - \cosh(d\bar{x})\} + \bar{D}\{\sin(d\bar{x}) - \sinh(d\bar{x})\}, \tag{9}$$

where ω is the natural frequency of free vibrations of the system, while $\bar{B}$, $\bar{C}$, $\bar{D}$ are unknown amplitude coefficients, and $d = \sqrt{\omega/\beta}$. By substituting Eqs. (7)–(9) into (3), (4b), (4e) and (4f), the following relationships which contain four unknowns $Y_0, \bar{B}, \bar{C}, \bar{D}$ are obtained:

$$Y_0\left\{1 + \frac{k_1}{k_2} - \frac{M}{k_2}\omega^2\right\} - \bar{B}\frac{k_1}{k_2}\sin\lambda - \bar{C}\{\cos\gamma - \cosh\gamma\} - \bar{D}\{\sin\gamma - \sinh\gamma\} = 0, \tag{10}$$

$$Y_0 - \bar{B}\left\{\sin\lambda + \frac{\lambda}{\xi_1}\cos\lambda\right\} = 0, \tag{11}$$

$$\bar{C}\{\cos\gamma + \cosh\gamma\} + \bar{D}\{\sin\gamma + \sinh\gamma\} = 0, \tag{12}$$

$$\begin{aligned} Y_0 &+ \bar{C}\{\alpha\sin\gamma - \alpha\sinh\gamma - \cos\gamma + \cosh\gamma\} \\ &+ \bar{D}\{-\alpha\cos\gamma - \alpha\cosh\gamma - \sin\gamma + \sinh\gamma\} = 0, \end{aligned} \tag{13}$$

where

$$\lambda = \omega\left(\frac{\rho_1}{E_1}\right)^{1/2} L_1, \quad \gamma = \omega^{1/2}\left(\frac{\rho_2 A_2}{E_2 I_2}\right)^{1/4} L_2, \quad \alpha = \frac{E_2 I_2}{k_2 L_2^3}\gamma^3. \tag{14}$$

Eqs. (10)–(13) can be written in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 1 & a_{42} & 0 & 0 \end{bmatrix} \begin{Bmatrix} Y_0 \\ \bar{B} \\ \bar{C} \\ \bar{D} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{15}$$

where

$$a_{11} = \left\{1 + \frac{k_1}{k_2} - \frac{M}{k_2}\,\omega^2\right\}, \quad a_{12} = -\frac{k_1}{k_2}\sin\lambda, \quad a_{13} = -\{\cos\gamma - \cosh\gamma\},$$
$$a_{14} = -\{\sin\gamma - \sinh\gamma\}, \quad a_{23} = \{\alpha\sin\gamma - \alpha\sinh\gamma - \cos\gamma + \cosh\gamma\},$$
$$a_{24} = \{-\alpha\cos\gamma - \alpha\cosh\gamma - \sin\gamma + \sinh\gamma\}, \quad a_{33} = \{\cos\gamma + \cosh\gamma\},$$
$$a_{34} = \{\sin\gamma + \sinh\gamma\}, \quad a_{42} = -\left\{\sin\lambda + \frac{E_1A_1}{k_1L_1}\,\lambda\cos\lambda\right\}. \tag{16}$$

To obtain non-trivial solutions, the determinant of coefficient matrix of Eq. (15) must be zero. Hence, the frequency equation of system is obtained as

$$\begin{aligned}\mathrm{DET} = {} & (\sin\gamma\cosh\gamma - \cos\gamma\sinh\gamma)\left[(M\omega^2)\sin\lambda + (M\omega^2 - k_1)\frac{E_1A_1}{k_1L_1}\,\lambda\cos\lambda\right] \\ & + \frac{E_2I_2}{k_2L_2^3}\,\gamma^3(1+\cos\gamma\cosh\gamma)\left[(M\omega^2 - k_2)\sin\lambda + (M\omega^2 - k_1 - k_2)\frac{E_1A_1}{k_1L_1}\,\lambda\cos\lambda\right] = 0,\end{aligned} \tag{17}$$

where the parentheses within the brackets represent the effects of discrete elements on the system frequencies. When both of the continuous elements, i.e. the rod and the beam, participate in vibratory motion, it is immaterial whether the terms explicitly including ω in the parentheses are expressed in terms of γ or λ. Accordingly, Eq. (17) can have one of the following forms:

$$\begin{aligned}\mathrm{DET1} = {} & (\sin\gamma\cosh\gamma - \cos\gamma\sinh\gamma)\left[\left(\frac{\mu_1}{\xi_1}\,\lambda^2\right)\sin\lambda + \left(\frac{\mu_1\lambda^2}{\xi_1\xi_2} - \frac{1}{\xi_2}\right)\lambda\cos\lambda\right] \\ & + \alpha(1+\cos\gamma\cosh\gamma)\left[\left(\frac{\mu_1}{\xi_2}\,\lambda^2 - 1\right)\sin\lambda + \left(\frac{\mu_1\lambda^2}{\xi_1\xi_2} - \frac{1}{\xi_1} - \frac{1}{\xi_2}\right)\lambda\cos\lambda\right] = 0\end{aligned} \tag{18}$$

or

$$\begin{aligned}\mathrm{DET2} = {} & (\sin\gamma\cosh\gamma - \sinh\gamma\cos\gamma)\left[\left(\frac{\bar{\mu}_1}{\bar{\xi}_2}\,\gamma^4\right)\sin\lambda + \left(\frac{\bar{\mu}_1}{\bar{\xi}_2\bar{\xi}_1}\,\gamma^4 - \frac{1}{\bar{\xi}_2}\right)\bar{\xi}\lambda\cos\lambda\right] \\ & + \alpha(1+\cos\gamma\cosh\gamma)\left[\left(\frac{\bar{\mu}_1}{\bar{\xi}_2}\,\gamma^4 - 1\right)\sin\lambda + \left(\frac{\bar{\mu}_1}{\bar{\xi}_2\bar{\xi}_1}\,\gamma^4 - \frac{1}{\bar{\xi}_1} - \frac{1}{\bar{\xi}_2}\right)\bar{\xi}\lambda\cos\lambda\right] = 0,\end{aligned} \tag{19}$$

where

$$\xi = \frac{E_2I_2L_1}{L_2^3E_1A_1}, \quad \bar{\xi} = \xi^{-1}, \quad \mu_1 = \frac{M}{\rho_1A_1L_1}, \quad \bar{\mu}_1 = \frac{M}{\rho_2A_2L_2}, \quad \xi_1 = \frac{k_1}{E_1A_1/L_1}, \quad \bar{\xi}_1 = \frac{k_1}{E_2I_2/L_2^3},$$
$$\xi_2 = \frac{k_2}{E_1A_1/L_1}, \quad \bar{\xi}_2 = \frac{k_2}{E_2I_2/L_2^3}, \quad \alpha = \frac{\xi}{\xi_2}\,\gamma^3 = \frac{1}{\bar{\xi}_2}\,\gamma^3. \tag{20}$$

Let

$$\mu = \frac{\rho_2A_2L_2}{\rho_1A_1L_1}, \quad \bar{\mu} = \mu^{-1}. \tag{21}$$

If one considers the definitions given by Eqs. (20) and (21) together the following relationships among some non-dimensional parameters can be concluded

$$\bar{\xi}_1 = \frac{\xi_1}{\xi}, \quad \bar{\xi}_2 = \frac{\xi_2}{\xi}, \quad \bar{\mu}_1 = \frac{\mu_1}{\mu}. \tag{22}$$

Furthermore, for finite λ and γ, these two parameters can always be related to each other as follows:

$$\lambda = \gamma^2 \frac{\bar{\mu}}{\bar{\xi}}, \quad \gamma = \lambda^{1/2} \left(\frac{\mu}{\xi}\right)^{1/2}. \tag{23}$$

2.1. *Case study*

In this section some limit cases will be discussed by utilizing two different forms of Eq. (17) which can be viewed as generating equations that cover all the possible situations encountered by changing of the non-dimensional parameters. For instance, for the first five cases to be studied in what follows Eq. (18) is the appropriate form of frequency equation while Eq. (19) is used in the last case since Eq. (18) leads to some mathematical handicaps for that case.

Case 1 (Uncoupling of Systems ($\mu_1 = 0, \xi_1 = 0$, while ξ_2, μ, ξ are not zero)): In this case taking ξ_1 and μ_1 equal to zero means that no physical connection between rod and beam exists as shown in Fig. 2. The frequency Eq. (18) takes the following form when $\mu_1, \xi_1 \to 0$ at the limit:

$$\lim_{\mu_1, \xi_1 \to 0} \mathrm{DET1} = \{1 + \cos\gamma \cosh\gamma\} \cos\lambda = 0. \tag{24}$$

As is immediately seen from Eq. (24), the value of γ rendering the parenthesis zero is the non-dimensional frequency of beam, while the roots of $\cos\lambda$ are the eigenvalues of rod.

Case 2 (No connection between beam and lumped mass ($\xi_2 = 0$, while other parameters are non-zero)): This case also shows another type of uncoupled system as shown in Fig. 3. In the limit case there exist two separate systems, one of which is rod–mass attachment system, while the other is vibrating beam, alone. The frequency equation related to this case is found as follows:

$$\lim_{\xi_2 \to 0} \mathrm{DET1} = \{1 + \cos\gamma \cosh\gamma\} \left[\mu_1 \lambda \sin\lambda + \left(\frac{\mu_1 \lambda^2}{\xi_1} - 1\right) \cos\lambda\right] = 0. \tag{25}$$

Here, the formula given in brackets enables us to find the eigenvalues of rod–lumped mass attachment system, while the rest delivers those of transversally vibrating beam.

Case 3 (No intermediate spring ($\xi_1 = 0$ only)): In contrast to the previous case, lumped mass remains attached to beam and is separated from rod by decreasing stiffness of intermediate spring infinitely so that it affects no longer the system (see Fig. 4). Consequently, the frequency equation giving the eigenvalues of both rod and beam–lumped mass combined system is

$$\lim_{\xi_1 \to 0} \mathrm{DET1} = \cos\lambda \left[\alpha(1 + \cos\gamma \cosh\gamma)\left(\frac{\mu_1 \lambda^2}{\xi_2} - 1\right) + (\sin\gamma \cosh\gamma - \cos\gamma \sinh\gamma)\left(\frac{\mu_1 \lambda^2}{\xi_2}\right)\right] = 0. \tag{26}$$

The first factor on the right-hand side of Eq. (26) is the term giving the eigenvalues of a longitudinally vibrating rod, while the bracket yields the eigenvalues of a beam with spring–mass attachment.

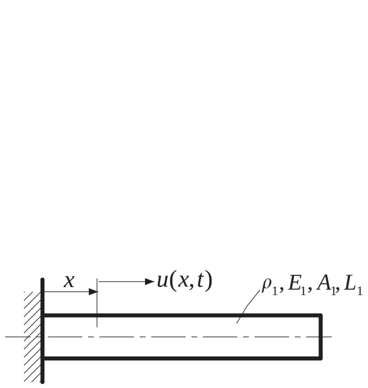

Fig. 2. Uncoupled system corresponding to Case 1 ($\mu_1 = 0, \xi_1 = 0$ with ξ_2, μ, ξ being not zero).

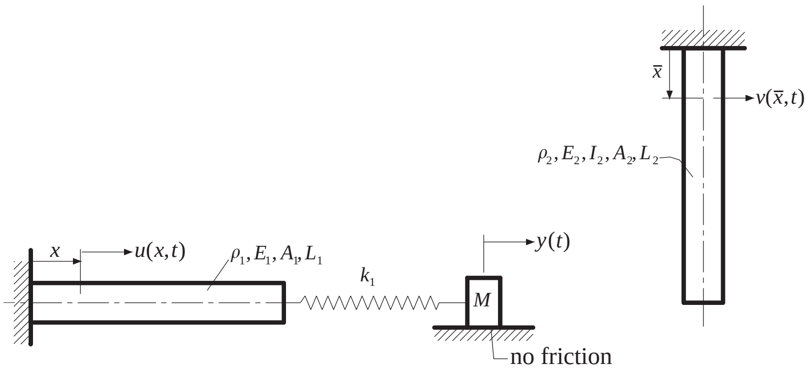

Fig. 3. Uncoupled system corresponding to Case 2 ($\xi_2 = 0$).

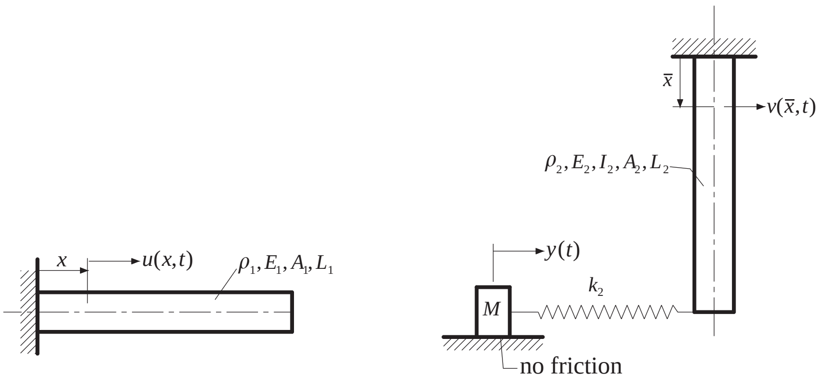

Fig. 4. Uncoupled system corresponding to Case 3 ($\xi_1 = 0$).

Case 4 (No lumped mass ($\mu_1 = 0$ only)):

$$\lim_{\mu_1 \to 0} \text{DET1} = \alpha(1 + \cos\gamma\cosh\gamma)\left(\sin\lambda + \left(\frac{\xi_2 + \xi_1}{\xi_2\xi_1}\right)\lambda\cos\lambda\right) + (\sin\gamma\cosh\gamma - \cos\gamma\sinh\gamma)\left(\frac{\lambda}{\xi_2}\cos\lambda\right) = 0. \tag{27}$$

Similar to above considerations, the limit case figure and frequency equation of the system are represented by Fig. 5 and Eq. (27), respectively.

Case 5 (Beam with comparatively high stiffness ($\xi \to \infty$ only)): For this case, the beam acts as a fixed rigid wall and does not participate in general motion of the system. Hence, the complete system of Fig. 1 reduces to one with longitudinally vibrating rod-linear spring–lumped mass attachment (see Fig. 6). Frequency equation of the relevant system is

$$\lim_{\xi \to \infty} \text{DET1} = \left(\frac{\mu_1\lambda^2}{\xi_2} - 1\right)\sin\lambda + \left(\frac{\mu_1\lambda^2}{\xi_2\xi_1} - \frac{\xi_2 + \xi_1}{\xi_2\xi_1}\right)\lambda\cos\lambda = 0. \tag{28}$$

Case 6 (Rod with comparatively high stiffness): While Eq. (18) is viable for numerical calculations of eigenvalues provided the rod has finite stiffness, it leads to mathematical complications in the extreme case corresponding to an ideally rigid rod. Therefore Eq. (19) is preferable for this case. Dividing all terms of Eq. (19) by the product $\bar{\xi}\lambda$, then limiting for $\lambda \to 0$,

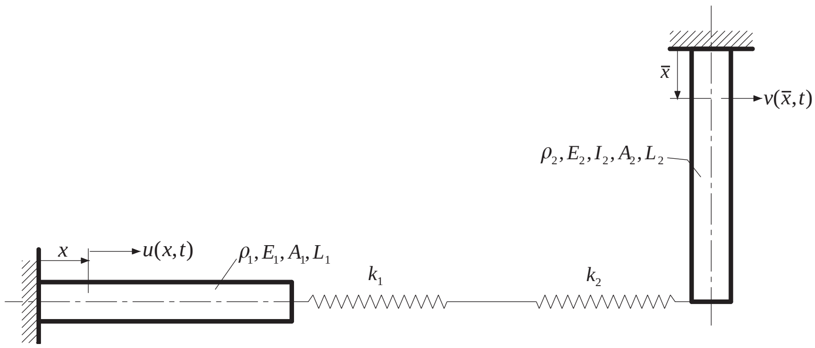

Fig. 5. Rod–beam system connected by spring attachments ($\mu_1 = 0$, corresponding to Case 4).

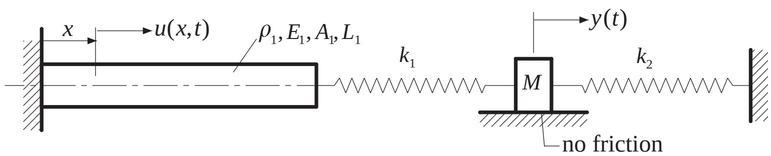

Fig. 6. Rod with spring–mass attachment for Case 5 ($\xi \to \infty$).

$\bar{\bar{\xi}} \to \infty$ and rearranging the remaining terms yields

$$\lim_{\lambda \to 0, \bar{\bar{\xi}} \to \infty} \mathrm{DET2} = \bar{\alpha}(1 + \cos\gamma \cosh\gamma)\{\bar{\bar{\xi}}_2 + \bar{\bar{\xi}}_1 - \bar{\mu}_1 \gamma^4\} + (\cosh\gamma \sin\gamma - \cos\gamma \sinh\gamma)\{\bar{\bar{\xi}}_1 - \gamma^4 \bar{\mu}_1\} = 0. \tag{29}$$

The results associated with this case are given in Table 5.

3. Numerical results

In this section, one will show when some of those limit cases mentioned above practically occur. To this end, comparisons will be made between the eigenvalues obtained by Eqs. (18) or (19) and those obtained through reduced equations in the limit cases. For Case 1, for example, for what numerical values of μ_1 and ξ_1 uncoupling take place will be introduced, while other non-dimensional parameters are fixed. Several MATLAB codes were written to carry out numerical computations.

Table 1 shows how the first three eigenvalues of the complete system in Fig. 1 vary with regard to (μ_1, ξ_1). Observation of decrease in the eigenvalues versus increase in μ_1 for a fixed ξ_1 or decrease in ξ_1 for a fixed μ_1 is consistent with the dynamic behaviour of the system. The roots obtained from Eq. (18) converge to the ones from Eq. (24) with simultaneous decrease in ξ_1 and μ_1. In the table, bold-typed numbers denote the eigenvalues of the uncoupled system given by Eq. (24). Thus, **1.5708** and **4.7124** are the first and second eigenvalues of longitudinally vibrating rod, respectively. Similarly, **1.1112** gives the first eigenvalue of the transversely vibrating beam in terms of λ (see Eq. (23)). The values for ξ and μ given in Table 1 are chosen considering practical applications. However, to examine their effects on the change of the system's eigenvalues, in addition to Table 1(a) two more tables were introduced, one of which is Table 1(b) where $\mu/\xi = 100$, the other one is Table 1(c) with $\mu/\xi = 1000$. Nevertheless, their effects to the decoupling of the system are negligible. Since ξ_2 does not appear in Eq. (24), it has no effect on the conversion of the system in Fig. 1 into the one shown in Fig. 2. Consequently, since the absolute errors are small enough to ignore, it can be concluded that separation of the system, i.e. the rod vibrates on its own and so does the beam, is encountered practically when the numbers (μ_1, ξ_1) are both less than or about 0.001, simultaneously, regardless of the μ/ξ ratio and the ξ_2 value.

In Table 2 are presented the variations of the first three eigenvalues of the combined system with respect to ξ_1 and ξ_2 for three different values of μ_1. The μ and ξ ratios are fixed in order to investigate how the attachment properties affect decoupling of the two systems. With ξ_2 relatively small, the complete system of Fig. 1 separates into two independent vibrating systems, one of which is the rod and spring–mass attachment system and the other is the vibrating beam. Here, to find out whether changing the mass ratio (μ_1) of the lumped mass will influence the separation of the systems, an interval for μ_1 ranging in $1, \ldots, 0.01$ is introduced. Except for the value **1.1119** which represents the first eigenvalue of the beam, all other bold-typed numbers given in the rightmost column of Table 1 belong to the rod–spring mass attachment system. It can be clearly

Table 1
Variation of the first three eigenvalues (λ) of the combined system in Fig. 1 with respect to μ_1 and ξ_1 with the other parameters fixed (Case 1)

μ_1	ξ_1														
	1			0.1			0.01			0.001			0.0001		
	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.
(a)															
1	0.6867	2.1044	2.2311	0.3348	1.6342	2.2177	0.1803	1.5772	2.2174	0.1541	1.5714	2.2174	0.1512	1.5709	2.2174
0.1	1.3796	2.1740	3.3062	0.9686	1.6563	2.3891	0.5358	1.5774	2.3578	0.4584	1.5714	2.3552	0.4498	1.5709	2.3550
0.01	1.5271	2.1775	4.6842	1.4872	1.7812	4.4551	1.0784	1.5781	3.6688	0.9434	1.5714	3.5861	0.9278	1.5709	3.5780
0.001	1.5410	2.1778	4.7280	1.5249	1.8240	4.7218	1.2402	1.5787	4.7143	1.1077	1.5714	4.7126	1.0918	1.5709	4.7124
0.0001	1.5424	2.1778	4.7316	1.5279	1.8284	4.7230	1.2584	1.5788	4.7143	1.1273	1.5714	4.7126	1.1114	1.5709	4.7124
										First eig. by Eq. (24)			**1.1112**		
				$\mu = 0.1,\ \xi = 0.01,\ \xi_2 = 0.1$							Second eig. by Eq. (24)			**1.5708**	
												Third eig. by Eq. (24)			**4.7124**
(b)															
1	0.5814	0.8269	2.1182	0.3112	0.7537	1.6344	0.1695	0.7456	1.5772	0.1450	0.7448	1.5714	0.1422	0.7447	1.5709
0.1	1.6490	1.4818	2.2918	0.5217	1.3563	1.6927	0.3413	1.1596	1.5776	0.2983	1.1340	1.5714	0.2934	1.1315	1.5709
0.01	1.6523	1.6119	2.2932	0.5488	1.5970	2.2366	0.3927	1.5764	2.1532	0.3503	1.5714	2.1336	0.3453	1.5709	2.1314
0.001	1.6526	1.6236	2.2933	0.5513	1.6012	2.2500	0.3985	1.5765	2.2079	0.3565	1.5714	2.1998	0.3515	1.5709	2.1989
0.0001	1.6527	1.6247	2.2933	0.5515	1.6016	2.2511	0.3991	1.5766	2.2114	0.3571	1.5714	2.2039	0.3521	1.5709	2.2031
										First eig. by Eq. (24)			**0.3516**		
				$\mu = 1,\ \xi = 0.01,\ \xi_2 = 0.1$							Second eig. by Eq. (24)			**1.5708**	
												Third eig. by Eq. (24)			**2.2034**
(c)															
1	0.2062	0.6896	0.7774	0.1652	0.4438	0.7356	0.1079	0.3668	0.7340	0.0943	0.3586	0.7338	0.0928	0.3578	0.7338
0.1	0.2072	0.7205	1.4806	0.1738	0.7050	1.3377	0.1242	0.6808	1.0844	0.1108	0.6747	1.0485	0.1092	0.6740	1.0449
0.01	0.2073	0.7211	1.6132	0.1746	0.7102	1.5975	0.1260	0.6982	1.5764	0.1127	0.6956	1.5714	0.1111	0.6954	1.5709
0.001	0.2073	0.7211	1.6250	0.1747	0.7106	1.6016	0.1262	0.6993	1.5765	0.1129	0.6969	1.5714	0.1113	0.6967	1.5709
0.0001	0.2073	0.7211	1.6262	0.1747	0.7106	1.6019	0.1262	1.6994	1.5766	0.1129	0.6971	1.5714	0.1114	0.6968	1.5709
										First eig. by Eq. (24)			**0.1112**		
				$\mu = 10,\ \xi = 0.01,\ \xi_2 = 0.1$							Second eig. by Eq. (24)			**0.6968**	
												Third eig. by Eq. (24)			**1.5708**

Table 2
Variation of the first three eigenvalues (λ) of the combined system in Fig. 1 with respect to ξ_1 and ξ_2 with the other parameters fixed (Case 2)

ξ_1	ξ_2																	
	1			0.1			0.01			0.001			0.0001			Eigenvalues by Eq. (25)		
	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st eig.	2nd eig.	3rd eig.	1st	2nd	3rd
(a)																		
1	0.6884	2.1150	4.2276	0.6867	2.1044	2.2311	0.6806	1.2787	2.1173	0.6769	1.1297	2.1171	0.6763	1.1137	2.1171	**0.6763**	**1.1119**	**2.1171**
0.5	0.5838	1.8720	4.2270	0.5809	1.8711	2.2199	0.5717	1.2787	1.8728	0.5667	1.1297	1.8727	0.5660	1.1137	1.8727	**0.5659**	**1.1119**	**1.8727**
0.1	0.3422	1.6342	4.2264	0.3348	1.6342	2.2177	0.3130	1.2786	1.6343	0.3027	1.1297	1.6343	0.3012	1.1137	1.6343	**0.3011**	**1.1119**	**1.6343**
0.05	0.2739	1.6026	4.2263	0.2639	1.6026	2.2175	0.2345	1.2786	1.6026	0.2203	1.1297	1.6026	0.2184	1.1137	1.6026	**0.2181**	**1.1119**	**1.6026**
0.01	0.1954	1.5772	4.2262	0.1803	1.5772	2.2174	0.1318	1.2785	1.5772	0.1043	1.1297	1.5772	0.1000	1.1137	1.5772	**0.0995**	**1.1119**	**1.5772**
$\mu = 0.1,\ \xi = 0.01,\ \mu_1 = 1$																		
(b)																		
1	1.3824	2.8723	4.5790	1.3796	2.1740	3.3062	1.2661	1.4167	3.1664	1.1296	1.3996	3.1565	1.1137	1.3987	3.1555	**1.1119**	**1.3986**	**3.1554**
0.5	1.3390	2.3029	4.5620	1.3364	2.0676	2.7418	1.2573	1.3881	2.4656	1.1296	1.3601	2.4531	1.1137	1.3588	2.4519	**1.1119**	**1.3587**	**2.4518**
0.1	0.9704	1.6587	4.5282	0.9686	1.6563	2.3891	0.9567	1.2967	1.6669	0.9391	1.1301	1.6647	0.9351	1.1137	1.6645	**0.9346**	**1.1119**	**1.6645**
0.05	0.7872	1.6082	4.5218	0.7777	1.6078	2.3702	0.7310	1.2911	1.6098	0.6937	1.1299	1.6093	0.6876	1.1137	1.6093	**0.6869**	**1.1119**	**1.6093**
0.01	0.5642	1.5774	4.5163	0.5358	1.5774	2.3578	0.4137	1.2880	1.5774	0.3296	1.1298	1.5774	0.3162	1.1137	1.5774	**0.3146**	**1.1119**	**1.5774**
$\mu = 0.1,\ \xi = 0.01,\ \mu_1 = 0.1$																		
(c)																		
1	1.5342	3.5307	4.8090	1.5271	2.1775	4.6842	1.2720	1.5645	4.6565	1.1296	1.5556	4.6534	1.1137	1.5549	4.6531	**1.1119**	**1.5549**	**4.6531**
0.5	0.5322	3.1306	4.7802	1.5244	2.1153	4.6801	1.2704	1.5644	4.6376	1.1296	1.5552	4.6321	1.1137	1.5545	4.6315	**1.1119**	**1.5545**	**4.6315**
0.1	0.5050	1.9628	4.7316	1.4872	1.7812	4.4551	1.2568	1.5639	3.3133	1.1294	1.5515	3.1764	1.1137	1.5505	3.1628	**1.1119**	**1.5504**	**3.1613**
0.05	0.4208	1.6733	4.7225	1.3955	1.6414	4.0432	1.2380	1.5631	2.4810	1.1292	1.5437	2.2869	1.1137	1.5417	2.2683	**1.1119**	**1.5414**	**2.2662**
0.01	1.0787	1.5784	4.7145	1.0784	1.5781	3.6688	1.0741	1.5373	1.6094	1.0320	1.1379	1.5816	0.9973	1.1137	1.5814	**0.9924**	**1.1119**	**1.5813**
$\mu = 0.1,\ \xi = 0.01,\ \mu_1 = 0.01$																		

seen from those tables that $\xi_2 = 0.01$ can be regarded as the limit value indicating separation point.

Numerical results for the Case 3 will not be given here because this case is dynamically analogous to Case 2.

Table 3 which corresponds to Case 4 shows the change of the eigenvalues of the combined system with regard to μ_1, while other parameters are fixed. The rightmost column denotes the eigenvalues obtained from Eq. (27). Table 3 indicates that lumped mass no longer affects the complete system's frequencies for the values lower than $\mu_1 = 0.001$.

Table 4, on the other hand, expresses after which value of ξ the beam acts as a rigid wall and does not participate in the system's total motion, and so in the frequency. Clearly, $\xi = 100$ can be regarded as the limit value of Case 5.

Table 5 indicates which values of the parameters ξ, ξ_1, ξ_2 convert the system in Fig. 1 into the one shown in Fig. 7. According to the definitions in Eq. (20), as the longitudinal stiffness of the rod increases three stiffness parameters approach to zero simultaneously. However, different from Eq. (18), Eq. (19) provides us with the same limit conditions by changing only one parameter, namely $\bar{\bar{\xi}}$.

Table 3
Variation of the first five eigenvalues (λ) of the combined system in Fig. 1 with respect to μ_1 (Case 4)

μ_1	1	0.1	0.01	0.001	0.0001	Eigenvalues by Eq. (27)
1st eig.	0.3112	0.5217	0.5488	0.5513	0.5515	**0.55156**
2nd eig.	0.7537	1.3563	1.5970	1.6012	1.6016	**1.6016**
3rd eig.	1.6344	1.6927	2.2366	2.2500	2.2511	**2.2512**
4th eig.	2.3000	2.3273	4.4181	4.7217	4.7229	**4.7230**
5th eig.	4.7336	4.7345	4.8026	6.1822	6.1857	**6.1861**
$\mu = 1,\ \xi = 0.01,\ \xi_1 = 0.1,\ \xi_2 = 0.1$						

Table 4
Variation of the first five eigenvalues (λ) of the combined system in Fig. 1 with respect to ξ with the other parameters fixed (Case 5)

ξ	0.01	0.1	1	10	100	Eigenvalues by Eq. (28)
1st eig.	0.4005	0.5641	0.6106	0.6154	0.6159	**0.61594**
2nd eig.	0.8261	1.3004	1.6371	1.6371	1.6371	**1.6371**
3rd eig.	1.6371	1.6373	3.5733	4.7337	4.7337	**4.7337**
4th eig.	2.3020	4.7337	4.7337	7.8667	7.8667	**7.8667**
5th eig.	4.7337	6.9969	7.8667	11.0047	11.0047	**11.005**
$\mu = 1,\ \mu_1 = 0.5,\ \xi_1 = 0.1,\ \xi_2 = 0.1$						

Table 5
The variation of the first five eigenvalues (γ) of the combined system in Fig. 1 with respect to $\bar{\xi}$ (Case 6)

$\bar{\xi}$	0.01	1	10	100	1000	10000	Limit values of
ξ	100	1	0.1	0.01	0.001	0.0001	γ from
ξ_1	10	0.1	0.01	0.001	0.0001	0.00001	Eq. (29)
ξ_2	10	0.1	0.01	0.001	0.0001	0.00001	
1st eig.	0.5060	0.7814	0.7909	0.7918	0.7919	0.7919	**0.7919**
2nd eig.	0.6746	1.2795	1.8903	1.8903	1.8903	1.8903	**1.8903**
3rd eig.	0.7925	1.8903	2.2333	3.9641	4.6951	4.6951	**4.6951**
4th eig.	0.9429	2.1757	3.8612	4.6951	7.0480	7.855	**7.855**
5th eig.	1.0858	2.8048	4.6951	6.8648	7.8550	10.996	**10.996**

$\bar{\mu}=1,\ \bar{\mu}_1=0.5,\ \bar{\xi}_1=0.1,\ \bar{\xi}_2=0.1,\ \mu=1,\ \mu_1=0.5$

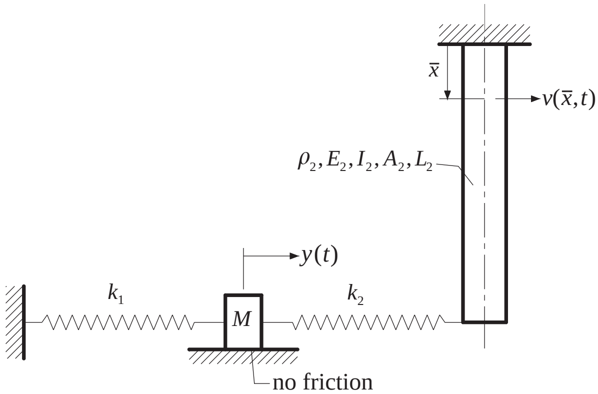

Fig. 7. Beam with spring–mass attachment system corresponding to Case 6 ($\xi \to \infty$).

4. Conclusions

The present study concerns a combined system consisting of a rod and a beam which vibrate longitudinally and transversely, respectively, and are connected via a double spring–mass system. Since such systems have many engineering applications, their vibrational properties must be examined. However, determining the vibration characteristics of all the individual components that make up a combined system is not always enough to get insight into the overall system behaviour. Therefore, the derivation of the frequency equation associated with a combined system in terms of meaningful non-dimensional parameters, such as mass and stiffness ratios will certainly be useful. While such a relationship is helpful in understanding the effects of physical properties of individual components on the combined system, it can also be used for determining parameter values for which the interaction of system components weakens or vanishes. The work performed so far on one-dimensional structural elements with attachments has developed in two main directions, i.e., rods and beams, in other words, longitudinally or transversely vibrating elements. In this regard the present study can be considered as an attempt to bridge the gap between two trends. To the authors' knowledge, the combined system studied here and the frequency equations derived in this paper are novel. The results presented here are thought to be useful to design engineers.

References

[1] S. Kukla, B. Posiadala, Free vibrations of beams with elastically mounted masses, *Journal of Sound and Vibration* 175 (4) (1994) 557–564.

[2] S. Kukla, J. Przyblski, L. Tomski, Longitudinal vibration of rods coupled by translational springs, *Journal of Sound and Vibration* 185 (4) (1995) 717–722.

[3] K.T. Chan, T.P. Leung, W.O. Wong, Free vibration of simply supported beam partially loaded with distributed mass, *Journal of Sound and Vibration* 191 (4) (1996) 590–597.

[4] M. Gürgöze, On the eigenfrequencies of a cantilever beam with attached tip mass and a spring–mass system, *Journal of Sound and Vibration* 190 (2) (1996) 149–162.

[5] V. Mermertaş, M. Gürgöze, Longitudinal vibrations of rods coupled by a double spring–mass system, *Journal of Sound and Vibration* 202 (5) (1997) 748–755.

[6] M. Gürgöze, On the eigenfrequencies of longitudinally vibrating rods carrying a tip mass and spring–mass in-span, *Journal of Sound and Vibration* 216 (2) (1998) 295–308.

[7] M. Gürgöze, H. Erol, On the eigencharacteristics of longitudinally vibrating rods carrying a tip mass and viscously damped spring–mass in-span, *Journal of Sound and Vibration* 225 (3) (1999) 573–580.

[8] S. İnceoğlu, M. Gürgöze, Longitudinal vibration of rods coupled by several spring–mass, *Journal of Sound and Vibration* 234 (5) (2000) 895–905.

[9] M. Gürgöze, S. İnceoğlu, Preserving the fundamental frequencies of beams despite mass attachments, *Journal of Sound and Vibration* 235 (2) (2000) 345–359.

[10] P.D. Cha, Specifying nodes at multiple locations for any normal mode of a linear elastic structure, *Journal of Sound and Vibration* 250 (5) (2002) 923–934.