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Short Communication

Normalizations in acoustic optimization with Rayleigh integral

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Abstract

The numerical difficulties of the acoustic optimization subject to an acoustic constraint, when structural boundary conditions are imposed on the velocity profiles with the finite element solution, have been discussed in this paper. A structural constraint is proposed for the acoustic optimization. This constraint leads to a radiation efficiency index, which is the ratio of the acoustic energy radiated by the structure to its potential energy. The structural constraint allows for the boundary conditions to be satisfied prior to the optimization, leading to velocity profiles physically feasible to elastic structures.
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1. Introduction

Present strategies for optimizing acoustic power radiated from a structure can be separated into two groups. The first group entails a conventional approach: the response of a structure for a given set of design parameters is obtained and the acoustic power radiated from the structure is calculated. Optimization methods are then used to search the parameter space in order to achieve minimum radiated acoustic power. The second group decouples the acoustic and structural domains: the surface velocity profiles of the structure that result in minimum acoustic radiation, called "weak radiators", are found first via an acoustic optimization problem. Then, structural design parameters are optimized to meet the desired velocity profiles. This approach is more suited to the exterior acoustic field than the first approach [1].

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Cunefare, Naghshineh and Koopmann have completed extensive studies with this decoupling technique [2–4]. In order to determine the weak radiators for baffled, undamped or lightly damped structures subject to harmonic excitation at a certain frequency, radiation modes must first be extracted. This can be accomplished by means of an acoustic optimization subject to certain constraints. Subsequently, those radiation modes are grouped to satisfy the prescribed structural boundary conditions, which can be a difficult task. In the acoustic optimization, the constraint is needed to avoid the trivial solution. Cunefare considered structural constraints [5]. However, these constraints did not impose structural boundary conditions. When an acoustic constraint is used, such as the radiated power of a piston with the same mean square velocity as that of the flexible structure, the acoustic optimization problem causes numerical difficulties with the finite element method, as will be demonstrated herein.

This paper proposes a structural constraint to the acoustic optimization problem, leading to a definition of radiation efficiency based on the structural energy. With the structural constraint, boundary conditions of the structure can be satisfied prior to the acoustic optimization, and the numerical difficulties associated with the finite element solution are eliminated. Furthermore, grouping the weak radiation modes to satisfy structural boundary conditions becomes a trivial task.

The outline of the paper is as follows. In Section 2, we review the classical formulation of radiation efficiency in terms of the Rayleigh integral and the acoustic optimization problem for finding weak adiators. Section 3 discusses the normalization schemes as constraints to the acoustic optimization problem. We point out the numerical difficulties with finite element solutions when structural boundary conditions are imposed to the acoustic optimization problem subject to an acoustic constraint. A structural constraint is then proposed. Section 4 presents numerical examples with different boundary conditions. Section 5 concludes the paper.

2. Formulation

Consider a baffled beam of width b and length L vibrating in air with frequency ω. The acoustic pressure $p(\mathbf{r}_{S'})$ at any observation point $\mathbf{r}_{S'}$ on the beam surface S due to the beam normal surface velocity $v(\mathbf{r}_S)$ can be written using the Rayleigh integral as

$$p(\mathbf{r}_{S'}) = \frac{\mathrm{i}\omega\rho_f}{2\pi}\int_S v(\mathbf{r}_S)\frac{\mathrm{e}^{-\mathrm{i}kR}}{R}\,\mathrm{d}S, \tag{1}$$

where ρ_f is the air density, k is the acoustic wavenumber, $\mathbf{r}_S$ is the position of the surface element $\mathrm{d}S$ with normal velocity $v(\mathbf{r}_S)$, $\mathbf{r}_{S'}$ is the position of the surface receiver point, and $R = |\mathbf{r}_S - \mathbf{r}_{S'}|$. The acoustic power W radiated from the baffled beam is given by

$$\begin{aligned} W &= \frac{1}{2}\int_{S'}\int_S v(\mathbf{r}_S)\left(\frac{\omega\rho_f}{2\pi}\frac{\sin(kR)}{R}\right)v^*(\mathbf{r}_{S'})\,\mathrm{d}S\,\mathrm{d}S' \\ &= \frac{\omega\rho_f b^2}{4\pi}\int_0^L\int_0^L \frac{\sin kR(x,r)}{R(x,r)}\,v(x)v^*(r)\,\mathrm{d}x\,\mathrm{d}r, \end{aligned} \tag{2}$$

where the asterisk indicates complex conjugate. Note that, for the beam, we have $\mathrm{d}S = b \cdot \mathrm{d}x$ and $\mathrm{d}S' = b \cdot \mathrm{d}r$.

The radiation efficiency σ of the beam is defined as

$$\sigma = \frac{W}{\frac{1}{2}\rho_f c S \langle \bar{v}^2 \rangle}, \tag{3}$$

where c is the speed of sound in the air, S is the beam surface area and $\langle \bar{v}^2 \rangle$ is the spatial average of the normal surface velocity of the beam. σ is the ratio of the radiated power to that from a baffled piston with the same area and the normal velocity $\sqrt{\langle \bar{v}^2 \rangle}$.

2.1. Optimization of radiated acoustic power

Let the spatial distribution of the velocity be expressed as

$$v(x) = \mathbf{a}^{\mathrm{T}} \xi(x), \tag{4}$$

where $\mathbf{a}$ represents an $N \times 1$ vector of real modal coefficients and ξ represents an $N \times 1$ vector of expansion functions. Then, the radiated acoustic power can be written in a quadratic form as $W = \frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a}$, where the matrix $\mathbf{B}$ computed from the integration can be shown to be Hermitian.

Consider the problem of minimizing W with respect to $\mathbf{a}$ subject to a constraint $\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a} = 1$. The Hermitian matrix $\mathbf{E}$ is determined from a physical constraint. By using the method of Lagrange multipliers, we have

$$L = \tfrac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a} - \lambda\left(\tfrac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a} - 1\right) \tag{5}$$

and the necessary conditions for optimality

$$\frac{\partial L}{\partial \mathbf{a}} = 0, \quad \frac{\partial L}{\partial \lambda} = 0. \tag{6}$$

This leads to an eigenvalue problem,

$$\mathbf{B}\mathbf{a} = \lambda \mathbf{E}\mathbf{a}. \tag{7}$$

Since both matrices in Eq. (7) are Hermitian, the eigenvalues must be all real so that we can order them as $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. The corresponding eigenvectors $\mathbf{a}_i$ are also real. The radiation efficiency of the velocity profile $v_i(x) = \mathbf{a}_i^{\mathrm{T}} \xi(x)$ is determined by λ_i.

3. Normalization schemes

3.1. Acoustic constraint

Pre-multiplying Eq. (7) by $\frac{1}{2}\mathbf{a}^{\mathrm{T}}$, we have

$$\lambda = \frac{\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a}}{\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a}} = \frac{W}{\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a}}. \tag{8}$$

Clearly, the constraint $\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a}$ behaves like a normalization term in the above equation. When we choose the constraint to be the acoustic power radiated by the fore-mentioned piston such that $\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a} = \frac{1}{2}\rho_f c S\langle\bar{v}^2\rangle$, we have $\sigma = \lambda$. In this case, the eigenvalues λ_i are the classical radiation efficiency defined in Eq. (3) for the corresponding velocity profile.

3.2. A remark

Note that the velocity profiles determined in this eigenvalue problem do not necessarily satisfy any structural boundary conditions. Hence, it can be difficult to apply the results obtained here to structural designs for minimal sound radiation. Furthermore, we point out that when the boundary conditions are imposed in the acoustic optimization problem, poor numerical behavior associated with the finite element solution occurs.

For the purpose of illustration, we consider the beam and the acoustic radiation problem studied in Ref. [3]. Two methods are applied to this problem: the finite element method and the modal expansion approach with comparison functions of the beam. The modal expansion approach provides an accurate solution as a basis to check the finite element solutions. Both methods considered in this paper can produce the same solutions as reported in Ref. [3] under the same conditions.

In the acoustic optimization, cantilever boundary conditions of the beam are imposed. Fig. 1 compares the third radiation mode, as an example, obtained with the finite element method using five, 20, and 80 C^1- and C^2-elements with that obtained by the modal expansion approach. The shape functions of the C^1- and C^2-elements are listed in Appendix A.

The fluctuations in the radiation mode by the finite element method are concentrated in the first two elements close to the clamped boundary. As the number of elements increases, and the order of the element shape functions increases from C^1 to C^2, the fluctuations become sharper and closer to the edge of the beam. Such a phenomenon is clearly due to the poor numerical behavior of the formulation of the acoustic optimization problem since the solution obtained by the modal expansion approach is rather smooth.

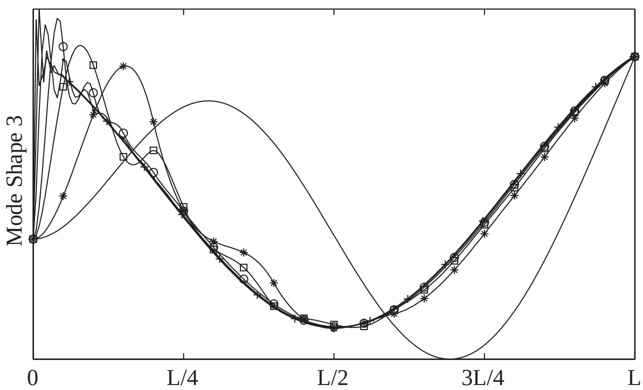

Fig. 1. Radiation mode 3 with clamped–free boundary conditions: -∗-, five C^1 elements; -o-, 20 C^1 elements; - + -, 80 C^1 elements; -□-, five C^2 elements; -◇-, 20 C^2 elements; ---, 80 C^2 elements. Solid line: solution by modal expansion.

Because the ultimate goal of the acoustic optimization problem is to design a structure with prescribed boundary conditions that will yield minimum sound radiation, the oscillatory behavior in the radiation modes obtained by the finite element method is not acceptable.

3.3. Structural constraint

Motivated by the poor numerical behavior of the optimization problem with acoustic constraint in the finite element solution, we propose a structural constraint. Consider the strain energy of the beam given by

$$W_P = \frac{1}{2} EI \int_S \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \frac{\mathrm{d}^2 v^*}{\mathrm{d}x^2} \,\mathrm{d}S. \tag{9}$$

EI will be chosen to be unity so that the acoustic optimization problem is independent of structural parameters. Whether we use the finite element method or the modal expansion approach, W_P can be expressed in a quadratic form,

$$W_P = \tfrac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}. \tag{10}$$

The acoustic optimization problem now becomes minimizing W with respect to $\mathbf{a}$ subject to a constraint $\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}$ =constant. This leads to an eigenvalue problem,

$$\mathbf{B}\mathbf{a} = \lambda \mathbf{K}\mathbf{a}. \tag{11}$$

Since

$$\lambda = \frac{\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a}}{\frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}} = \frac{W}{W_P}, \tag{12}$$

we can define a structure-based radiation efficiency normalized by the structural strain energy as

$$\sigma_P = \lambda = \frac{W}{W_P}. \tag{13}$$

σ_P measures the percentage of the structural strain energy radiated into the air.

4. Numerical examples

Having introduced the structural constraint, we study several examples of the baffled beam with different boundary conditions. With the structural constraint, results obtained by the finite element method and the modal expansion approach are identical, and will not be further compared.

Fig. 2 compares the radiation efficiency σ_P for different boundary conditions including pinned–pinned, clamped–free and clamped–clamped. The first three radiation modes for the clamped–free boundary condition are shown in Fig. 3. Fig. 4 shows those for the clamped–clamped boundary condition. It is seen from the figures that the number of half-waves in the velocity profiles increases as the radiation efficiency decreases.

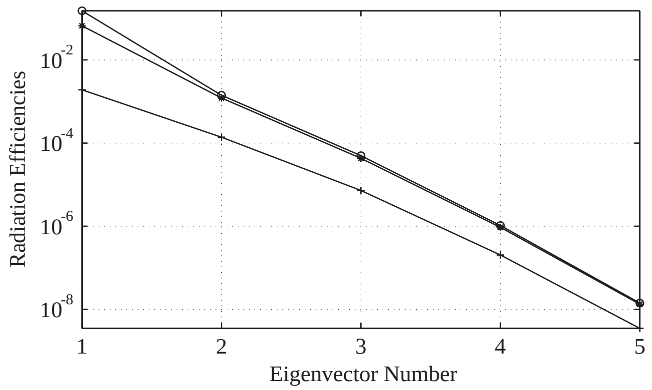

Fig. 2. Comparison of radiation efficiencies with different boundary conditions: -o-, pinned–pinned; -∗-, clamped–free; -+-, clamped–clamped.

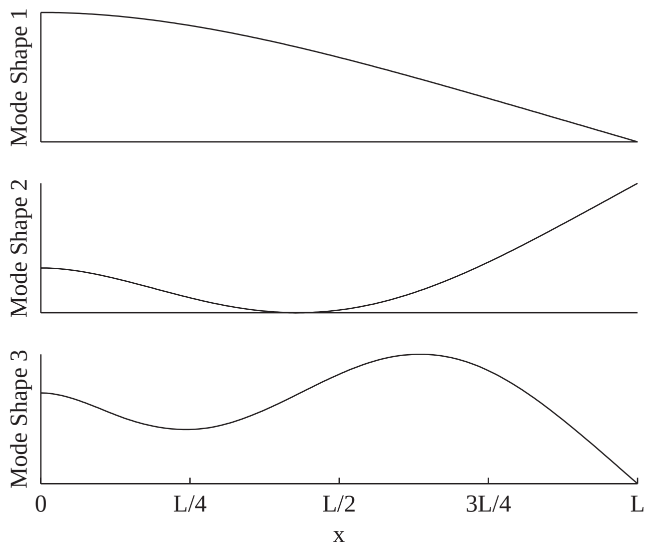

Fig. 3. The first three radiation modes for the clamped–free boundary condition obtained with the structural constraint.

It should be pointed out that by using the radiation efficiency alone, it is not easy to classify the radiation modes as weak or strong radiators. In order to do so, we compute the wavenumber transform of the velocity profiles and present the results in Fig. 5 for the clamped–free boundary condition and in Fig. 6 for the clamped–clamped boundary condition. From the wavenumber analysis, we conclude that the first two radiation modes for both boundary conditions can be identified as strong radiators, and the remaining ones as weak radiators at the given frequency.

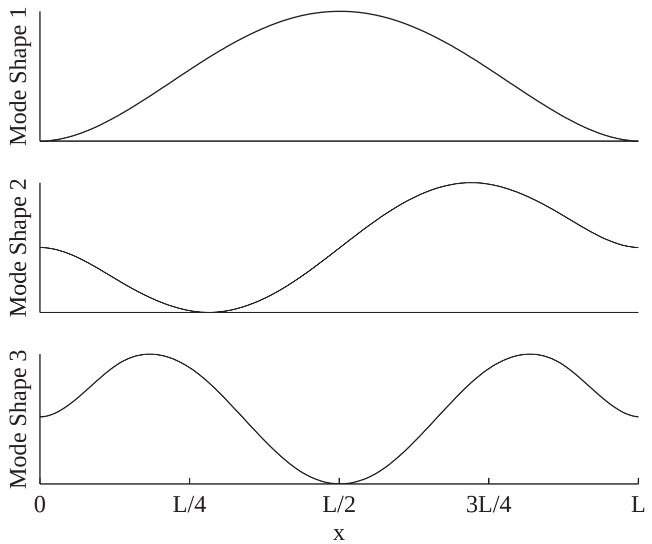

Fig. 4. The first three radiation modes for the clamped–clamped boundary condition obtained with the structural constraint.

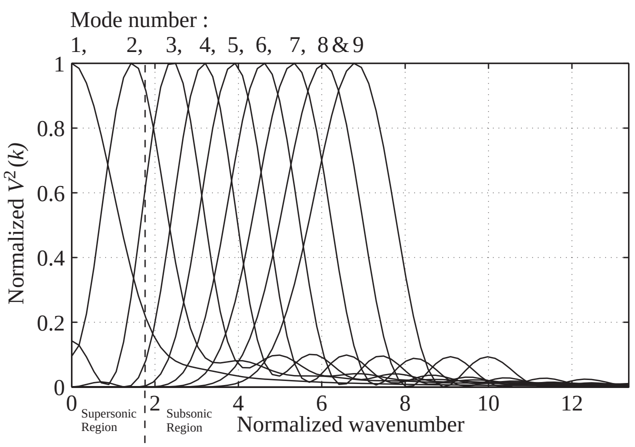

Fig. 5. Wavenumber transforms of radiation modes for the clamped–free boundary condition. The wavenumber k is normalized by a factor π/L. All the amplitudes of the transformation are normalized by their own peak values. The vertical dashed line separating the supersonic and subsonic regions marks the normalized acoustic wavenumber $\omega L/(c\pi)$.

5. Concluding remarks

We have studied the effect of constraints to the optimization of radiated acoustic power calculated with the Rayleigh integral. An infinitely baffled finite beam vibrating at a single frequency is used as an example to demonstrate the discussion. We have found that the acoustic

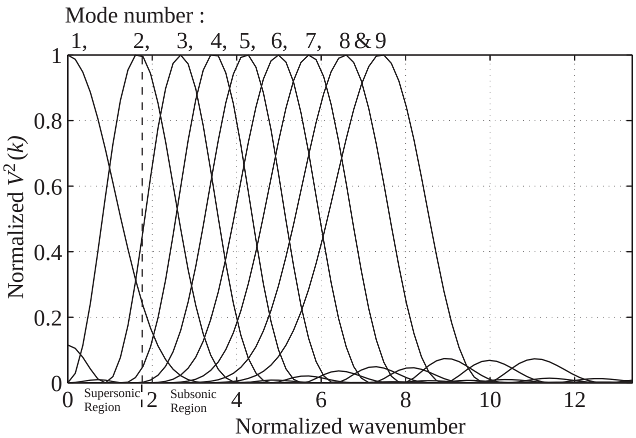

Fig. 6. Wavenumber transforms of radiation modes for the clamped–clamped boundary condition. The wavenumber k is normalized by a factor π/L. All the amplitudes of the transformation are normalized by their own peak values. The vertical dashed line separating the supersonic and subsonic regions marks the normalized acoustic wavenumber $\omega L/(c\pi)$.

optimization subject to an acoustic constraint specified by the radiated power of a piston can cause numerical difficulties with the finite element solution when boundary conditions are imposed to the velocity distribution of the beam. This numerical predicament has been verified with the help of the finite element solutions with C^1 and C^2 elements as well as the modal expansion approach with comparison functions. We then introduced the structural constraint to the acoustic optimization problem. The eigenvalue of the eigenvalue problem resulting from the optimization represents the ratio of the radiated acoustic power to the strain energy of the structure. The acoustic optimization problem for the baffled beam with clamped–free and clamped–clamped boundary conditions has been studied to demonstrate the theory. The optimized velocity profiles with the structural constraint offer a feasible target for the structural optimization study to create quiet structures.

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Appendix A. Mathematical details

Here, we list some mathematical details that are not presented in the main body of the paper.

A.1. Element functions

This comparison function solution is valid for problems that admit closed-form expressions for the natural vibration modes. Problems with complex geometries that do not allow closed-form solutions can be solved with the powerful finite element method. This paper seeks a physically-realizable surface velocity profile that will minimize the radiated acoustic power; as such, the continuity of the velocity and its slope should be maintained. For this reason, C^1- and C^2-continuous functions (third- and fifth-order Hermitian polynomials, respectively) are used to approximate the velocity distribution in each element.

The beam is divided lengthwise into N elements with length l_e. The natural coordinates of an element can be written as

$$\zeta = \frac{2(x - x_j)}{l_e} - 1, \tag{A.1}$$

where x_j is the location of the first node of the jth element. The velocity in an element of the beam is given by

$$\hat{v}^{\mathrm{T}}(\zeta) = \mathbf{H}^{\mathrm{T}}(\zeta)\mathbf{v}_i, \tag{A.2}$$

where for C^1 elements,

$$\mathbf{H}(\zeta) = \begin{bmatrix} \frac{1}{4}(1-\zeta)^2(2+\zeta) \\ \frac{1}{4}(1-\zeta)^2(\zeta+1) \\ \frac{1}{4}(1+\zeta)^2(2-\zeta) \\ \frac{1}{4}(1+\zeta)^2(\zeta-1) \end{bmatrix}, \tag{A.3}$$

$$\mathbf{v}_i^{\mathrm{T}} = \begin{bmatrix} v_i & \left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)_i & v_{i+1} & \left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)_{i+1} \end{bmatrix} = [\, v_i \quad \theta_i \quad v_{i+1} \quad \theta_{i+1} \,] \tag{A.4}$$

and for C^2 elements,

$$\mathbf{H}(\zeta) = \begin{bmatrix} \frac{1}{2} - \frac{15}{16}\zeta + \frac{5}{8}\zeta^3 - \frac{3}{16}\zeta^5 \\ \frac{5}{16} - \frac{7}{16}\zeta - \frac{3}{8}\zeta^2 + \frac{5}{8}\zeta^3 + \frac{1}{16}\zeta^4 - \frac{3}{16}\zeta^5 \\ \frac{1}{16} - \frac{1}{16}\zeta - \frac{1}{8}\zeta^2 + \frac{1}{8}\zeta^3 + \frac{1}{16}\zeta^4 - \frac{1}{16}\zeta^5 \\ \frac{1}{2} + \frac{15}{16}\zeta - \frac{5}{8}\zeta^3 + \frac{3}{16}\zeta^5 \\ -\frac{5}{16} - \frac{7}{16}\zeta + \frac{3}{8}\zeta^2 + \frac{5}{8}\zeta^3 - \frac{1}{16}\zeta^4 - \frac{3}{16}\zeta^5 \\ \frac{1}{16} + \frac{1}{16}\zeta - \frac{1}{8}\zeta^2 - \frac{1}{8}\zeta^3 + \frac{1}{16}\zeta^4 + \frac{1}{16}\zeta^5 \end{bmatrix}, \tag{A.5}$$

$$\begin{aligned} \mathbf{v}_i^T &= \begin{bmatrix} v_i & \left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)_i & \left(\frac{\mathrm{d}^2v}{\mathrm{d}\zeta^2}\right)_i & v_{i+1} & \left(\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right)_{i+1} & \left(\frac{\mathrm{d}^2v}{\mathrm{d}\zeta^2}\right)_{i+1} \end{bmatrix} \\ &= [\, v_i \quad \theta_i \quad \kappa_i \quad v_{i+1} \quad \theta_{i+1} \quad \kappa_{i+1} \,]. \end{aligned} \tag{A.6}$$

A.2. Expressions of the matrices **B** *and* **E**

Recall that $W = \frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a}$ and $\frac{1}{2}\rho_f c S\langle \bar{v}^2\rangle = \frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{E}\mathbf{a}$.
For the modal expansion approach, we have

$$\mathbf{B} = \frac{\omega\rho_f b^2}{2\pi}\int_0^L\int_0^L \frac{\sin kR(x,r)}{R(x,r)}\xi(x)\xi^{\mathrm{T}}(r)\,\mathrm{d}x\,\mathrm{d}r, \tag{A.7}$$

$$\mathbf{E} = \rho_f cb\int_0^L \xi(x)\xi^{\mathrm{T}}(x)\,\mathrm{d}x. \tag{A.8}$$

For the finite element method, we have

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{v}_i^{\mathrm{T}}\left(\int_{-1}^{1}\int_{-1}^{1}\frac{\omega\rho b^2 l_e^2}{8\pi}\frac{\sin(kR_{ij}(\zeta,\eta))}{R_{ij}(\zeta,\eta)}\mathbf{H}^{\mathrm{T}}(\zeta)\mathbf{H}(\eta)\,\mathrm{d}\zeta\,\mathrm{d}\eta\right)\mathbf{v}_j. \tag{A.9}$$

Let

$$\mathbf{a}_{ij} = \int_{-1}^{1}\int_{-1}^{1}\frac{\omega\rho b^2 l_e^2}{8\pi}\frac{\sin(kR_{ij}(\zeta,\eta))}{R_{ij}(\zeta,\eta)}\mathbf{H}^{\mathrm{T}}(\zeta)\mathbf{H}(\eta)\,\mathrm{d}\zeta\,\mathrm{d}\eta.$$

We have

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{v}_j^{\mathrm{T}}\mathbf{a}_{ij}\mathbf{v}_i = \frac{1}{2}\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a}, \tag{A.10}$$

where $\mathbf{a}$ is the global nodal vector consisting of the assembly of all $\mathbf{v}_i$ and $\mathbf{B}$ is the assembly of all $\mathbf{a}_{ij}$. Likewise, we have $\mathbf{E}$ is the assembly of $\rho_f cb\int_{-1}^{1}\mathbf{H}^{\mathrm{T}}(\zeta)\mathbf{H}(\zeta)\,\mathrm{d}\zeta$ over all elements.

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