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A plane wave approximation to sound transmission in parallel sheared mean flow

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Abstract

In a number of fluid machinery applications, a simplified plane wave analysis of the problem of sound transmission in a sheared mean flow is often sufficient for engineering purposes. Although there exists an extensive literature on 3-D solution of this problem, a simplified non-isentropic plane wave formulation is not seen. A quasi 1-D theory is presented in this paper in a general form encompassing non-uniform ducts, compressible and axially non-uniform flows. An analytical solution of these equations is presented for uniform ducts carrying an incompressible mean flow and the wave field is shown to consist of superposed forward and backward acoustic waves and a hydrodynamic wave, which occur in general as coupled waves. The propagation constants and the corresponding modal matrix that determines the degree of the coupling are analyzed with reference to applications to some standard mean flow profile shapes and compared with previous results.

1. Introduction

The problem of sound transmission in a hard-walled duct carrying a parallel sheared flow is a classical one. The number of papers dealing with this problem in 3-D has been quite extensive, mainly due to its importance in aircraft turbofan applications. A thorough account of the previous work can be found in the article by Eversman [1]. On the other hand, in a number of fluid

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machinery applications, a simplified plane wave analysis of the problem is often found to be sufficient for engineering purposes. To simplify the solution of the problem further, the mean flow is assumed to have a uniform velocity profile over the duct cross section, even though the actual profile may vary from a parabolic shape characterizing a laminar flow, to a flat shape characterizing a fully developed turbulent flow. For an inviscid fluid, this results in the classical convected isentropic wave equation in 1D, which yields $1/(1+\overline{M}_0)$ and $-1/(1-\overline{M}_0)$ as propagation constants for the forward and backward sound waves, where $\overline{M}_0$ denotes the Mach number of the average mean flow velocity over the duct cross-section. A correction to these propagation constants for the effect of the mean flow velocity profile was derived in Ref. [2] by maintaining the isentropic condition for the wave propagation. This shows that the correction due to the mean velocity profile is of the order $O[\overline{M}_0^2]$ and that the predictions of the 1-D theory are in close agreement with the results of 3-D solutions for the fundamental mode propagation.

Strictly speaking, the condition of isentropic wave propagation is valid for the ideal case of a uniform mean flow profile, where it is tantamount to the energy equation. This is not true for a parallel sheared mean flow and the conservation law for energy should be used for the precise formulation of the problem. It is the purpose of the present paper to present this formulation which, to the author's knowledge, has not appeared elsewhere. The paper will derive a quasi-1-D energy equation that includes the mean flow velocity profile in the cross-sectional average sense. The continuity and momentum equations are also formulated similarly and are essentially similar to those given in Ref. [2]. The governing equations are given, for future reference, in a general form encompassing ducts having non-uniform cross section and axially non-uniform mean flow. Analytical solution of these equations is presented for the case of uniform ducts carrying an incompressible mean flow of an arbitrary velocity profile. The propagation constants for this case are derived in closed form and compared with the previous results.

2. Quasi-1-D conservation equations

Consider a straight hard-walled uniform duct carrying a steady axial mean flow. The propagation of plane sound waves in this fluid flow is governed by the linearized continuity, momentum and energy equations. First, consider the conservation equations for mass and axial momentum. Neglecting the viscosity effects on the wave motion, but allowing for an arbitrary mean flow velocity profile, these can be expressed, respectively, in quasi-1-D form as [2]

$$S\,\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho \int_S v\,\mathrm{d}S\right] = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\left[\rho \int_S v\,\mathrm{d}S\right] + \frac{\partial}{\partial x}\left[\rho \int_S v^2\,\mathrm{d}S\right] + S\,\frac{\partial p}{\partial x} = 0. \tag{2}$$

Here, S denotes the cross-sectional area of the duct, x denotes the duct axis and t denotes the time. The fluid density, $\rho = \rho_0 + \rho'$, the fluid pressure, $p = p_0 + p'$, and the particle velocity in the axial direction, $v = v_0 + v'$, are assumed to consist of acoustic fluctuations ρ', p' and v', superimposed on the time-averaged mean values ρ_0, p_0 and v_0, respectively, the prime ($'$) denoting a fluctuating part and the subscript '0' denoting a time-averaged mean part throughout the paper. The acoustic fluctuations are assumed to be small to first order and functions of t and x only, and ρ_0 and p_0 are assumed to be uniform over the duct cross section.

Upon carrying out the integrations over the duct cross section, Eqs. (1) and (2) can be expressed as

$$S\left(\frac{\partial\rho}{\partial t}+\bar{v}\,\frac{\partial\rho}{\partial x}\right)+\rho\,\frac{\partial S\bar{v}}{\partial x}=0, \tag{3}$$

$$\rho S\left(\frac{\partial\bar{v}}{\partial t}+\bar{v}\,\frac{\partial\bar{v}}{\partial x}\right)+\frac{\partial}{\partial x}\,[S\rho(\overline{v^2}-\bar{v}^2)]+S\,\frac{\partial p}{\partial x}=0, \tag{4}$$

respectively. Here, an overbar denotes averaging over the duct cross-sectional area.

Eqs. (3) and (4) are next expanded into mean and fluctuating parts and the products of acoustic perturbations are neglected as second-order small quantities. This usual linearization procedure yields the acoustic continuity and momentum equations as

$$S\left(\frac{\partial\rho'}{\partial t}+\bar{v}_0\,\frac{\partial\rho'}{\partial x}+\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\,v'+\rho_0\,\frac{\partial v'}{\partial x}+\frac{\mathrm{d}\bar{v}_0}{\mathrm{d}x}\,\rho'\right)+\frac{\mathrm{d}S}{\mathrm{d}x}\,(\rho_0 v'+\bar{v}_0\rho')=0, \tag{5}$$

$$\rho_0 S\left(\frac{\partial v'}{\partial t}+\bar{v}_0\,\frac{\partial v'}{\partial x}+\frac{\mathrm{d}\bar{v}_0}{\mathrm{d}x}\,v'\right)+S\bar{v}_0\,\frac{\mathrm{d}\bar{v}_0}{\mathrm{d}x}\,\rho'+\frac{\partial}{\partial x}\,(S\beta\bar{v}_0^2\rho')+S\,\frac{\partial p'}{\partial x}=0, \tag{6}$$

respectively, where the parameter β and $\bar{v}_0$, the cross-section averaged mean flow velocity, are defined by

$$(1+\beta)\bar{v}_0^2=\frac{1}{S}\int_S v_0^2\,\mathrm{d}S,\quad \bar{v}_0=\frac{1}{S}\int_S v_0\,\mathrm{d}S. \tag{7}$$

The derivation of Eqs. (5) and (6) assumes that Eqs. (3) and (4) are satisfied identically by the mean flow. Accordingly, from the continuity equation, it is deduced that the product $S\bar{v}_0\rho_0$ is constant and that $(\mathrm{d}/\mathrm{d}x)[(1+\beta)S\rho_0\bar{v}_0^2]+S(\mathrm{d}/\mathrm{d}x)p_0=0$ from the momentum equation.

Now consider the energy equation. Again, assuming plane wave motion and neglecting the visco-thermal effects, but allowing similarly for an arbitrary mean flow velocity profile, this can be expressed in quasi-1-D form as

$$\frac{\partial}{\partial t}\left[\rho\int_S e\,\mathrm{d}S\right]+\frac{\partial}{\partial x}\left[\rho\int_S h^0 v\,\mathrm{d}S\right]=0, \tag{8}$$

where

$$e=u+\tfrac{1}{2}v^2,\quad h^0=e+\frac{p}{\rho}. \tag{9}$$

Here, e and h^0 denote, respectively, the specific total energy and the specific stagnation enthalpy of the fluid and u denotes the specific internal energy. The objective of the subsequent analysis is to transform Eq. (8) into a form that gives, upon linearization, an equation that closes Eqs. (5) and (6) for the determination of ρ', p' and v'.

Upon carrying out the integrations over the duct cross section, Eq. (8) can be written as

$$S\frac{\partial}{\partial t}\left[\rho\left\{\bar{e}+\tfrac{1}{2}(\overline{v^2}-\bar{v}^2)\right\}\right]+\frac{\partial}{\partial x}\left[S\rho\bar{v}\left\{\bar{h}^0+\frac{1}{2}\left(\frac{\overline{v^3}}{\bar{v}}-\bar{v}^2\right)\right\}\right]=0, \tag{10}$$

where

$$\bar{e}=u+\tfrac{1}{2}\bar{v}^2, \quad \bar{h}^0=\bar{e}+\frac{p}{\rho}. \tag{11}$$

When Eqs. (3) and (4) are substituted, Eq. (10) simplifies to

$$S\rho\left(\frac{\partial u}{\partial t}+\bar{v}\frac{\partial u}{\partial x}\right)-\bar{v}\frac{\partial}{\partial x}[S\rho(\overline{v^2}-\bar{v}^2)]+p\frac{\partial S\bar{v}}{\partial x}=0, \tag{12}$$

which, upon using the perfect gas state equation $(\gamma-1)\rho\,\mathrm{d}u=\mathrm{d}p-(p/\rho)\,\mathrm{d}\rho$, can be expressed as

$$\begin{aligned}S\left(\frac{\partial p}{\partial t}+\bar{v}\frac{\partial p}{\partial x}\right)+\gamma p\frac{\partial S\bar{v}}{\partial x}+\frac{\gamma-1}{2}\left[S\frac{\partial}{\partial t}[\rho(\overline{v^2}-\bar{v}^2)]+\frac{\partial}{\partial x}[S\rho(\overline{v^3}-\bar{v}^3)]\right.\\ \left.-2\bar{v}\frac{\partial}{\partial x}[S\rho(\overline{v^2}-\bar{v}^2)]\right]=0,\end{aligned} \tag{13}$$

where γ denotes the ratio of the specific heat coefficients. It can be shown that Eq. (13) will hold for any real gas if the factor $\gamma-1$ is replaced by $(\gamma-1)/\kappa T$, and the factor γp is replaced by $(\gamma-1)\rho c_p/\kappa^2 T$, where κ denotes the isobaric compressibility, T denotes the temperature and c_p is the specific heat coefficient at constant pressure.

Eq. (13) can be linearized as usual by partitioning into mean and fluctuating parts, assuming the mean part is satisfied by the mean flow and neglecting the products of acoustic perturbations as second-order small quantities. This process yields for the acoustic fluctuations

$$\begin{aligned}S\left(\frac{\partial p'}{\partial t}+\bar{v}_0\frac{\partial p'}{\partial x}+\gamma p_0\frac{\partial v'}{\partial x}\right)+\gamma p_0\frac{\mathrm{d}S}{\mathrm{d}x}v'+\frac{\gamma-1}{2}\left[S\bar{v}_0^2\beta\frac{\partial\rho'}{\partial t}-2\bar{v}_0\frac{\partial}{\partial x}(S\bar{v}_0^2\rho\beta)\right.\\ \left.-2v'\rho_0\frac{\partial}{\partial x}(S\bar{v}_0^2\beta)+\frac{\partial}{\partial x}(3S\rho_0\bar{v}_0^2v'\beta+S\chi\bar{v}_0^3\rho')\right]=0,\end{aligned} \tag{14}$$

where the parameter χ is defined by

$$(1+\chi)\bar{v}_0^3=\frac{1}{S}\int_S v_0^3\,\mathrm{d}S. \tag{15}$$

This completes the derivation of the basic equations, Eqs. (5), (6) and (14), that govern the propagation of plane sound waves in a duct carrying a steady mean flow having an arbitrary velocity profile. For a non-uniform duct, or a uniform duct carrying an axially non-uniform mean flow (v_0 is a function of x also), or a duct carrying a compressible mean flow, these equations do not admit an obvious analytical solution and have to be solved numerically.

For a uniform duct carrying an axially uniform incompressible mean flow, Eqs. (5), (6) and (14) simplify to

$$\frac{\partial \rho'}{\partial t} + \bar{v}_0 \frac{\partial \rho'}{\partial x} + \rho_0 \frac{\partial v'}{\partial x} = 0, \tag{16}$$

$$\rho_0 \left(\frac{\partial v'}{\partial t} + \bar{v}_0 \frac{\partial v'}{\partial x} \right) + \beta \bar{v}_0^2 \frac{\partial \rho'}{\partial x} + \frac{\partial p'}{\partial x} = 0, \tag{17}$$

$$\frac{\partial p'}{\partial t} + \bar{v}_0 \frac{\partial p'}{\partial x} + [\gamma p_0 + (\gamma - 1)\beta \rho_0 \bar{v}_0^2] \frac{\partial v'}{\partial x} + \frac{\gamma - 1}{2} (\chi - 3\beta) \bar{v}_0^3 \frac{\partial \rho'}{\partial x} = 0, \tag{18}$$

respectively. Analytical solution of these equations is presented in the next section.

To the author's knowledge, the foregoing quasi-1-D theory has not appeared elsewhere. As can be expected, the continuity equation of this theory, Eq. (5) or (16), is same as the continuity equation for the uniform mean flow velocity profile case, and, for $\beta = 0$, the momentum equation, Eq. (6) or (17), reduces to the momentum equation for that ideal case. On the other hand, for $\beta = 0$ and $\chi = 0$, the energy equation (14) becomes a statement of isentropic propagation. This is discussed in some detail in Ref. [3]. It is much simpler to show this in Eq. (18). As can be readily verified, for $\beta = 0$ and $\chi = 0$, Eqs. (18) and (16) yield the well-known isentropic relationship $p' = c_0^2 \rho'$, where c_0 denotes the speed of sound, $c_0^2 = \gamma p_0 / \rho_0$.

3. Plane sound waves in a uniform duct carrying incompressible sheared mean flow

From the perfect gas state equation $(p/c_v)\,\mathrm{d}s = \mathrm{d}p - c_0^2\,\mathrm{d}\rho$, where c_v is the specific heat coefficient at constant volume and s denotes the specific entropy, it follows that, when β or χ is different from zero, the entropy fluctuation does not vanish and, therefore, the acoustic pressure and density are strictly related by [5]

$$p' = c_0^2 \rho' + \varepsilon, \tag{19}$$

where ε denotes an additional pressure fluctuation. For relatively flat mean flow velocity profiles or low subsonic Mach numbers, ε will be a small fluctuating quantity, as β or χ give rise to terms that are proportional to square or cube of the average mean flow velocity.

In dealing with Eqs. (16)–(18), it is also convenient to define the following decompositions:

$$p' = p^+ + p^-, \quad \rho_0 c_0 v' = p^+ - p^-. \tag{20}$$

In the uniform mean flow profile case, these decompositions uncouple Eqs. (16)–(18) into waves traveling in the forward ($+x$) and backward directions, which are represented in Eq. (20) by the pressure components distinguished by the superscripts '+' and '−', respectively. For an arbitrary mean flow velocity profile, adhering to this decomposition is expected to incur a simplicity similar to the solution of the governing equations.

Upon substituting Eqs. (19) and (20) and assuming $\exp(-\mathrm{i}\omega t)$ time dependence for all fluctuating quantities, Eqs. (16)–(18) can be expressed in the following state space form:

$$\begin{bmatrix} 1+\overline{M}_0 & -1+\overline{M}_0 & \overline{M}_0 \\ 1+\overline{M}_0+\beta\overline{M}_0^2 & 1-\overline{M}_0+\beta\overline{M}_0^2 & \beta\overline{M}_0^2 \\ 1+\overline{M}_0+(\gamma-1)\overline{M}_0^2(\beta+\alpha\overline{M}_0) & -1+\overline{M}_0-(\gamma-1)\overline{M}_0^2(\beta-\alpha\overline{M}_0) & \alpha(\gamma-1)\overline{M}_0^3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} p^+ \\ p^- \\ \varepsilon \end{bmatrix}$$
$$= \mathrm{i}k\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} p^+ \\ p^- \\ \varepsilon \end{bmatrix}. \tag{21}$$

Here $\overline{M}_0 = \bar{v}_0/c_0$ is the Mach number of the average mean flow velocity over the duct cross section, $k = \omega/c_0$ is the wavenumber and $\alpha = (\chi - 3\beta)/2$. Although straightforward, after some cumbersome manipulations, Eq. (21) can be re-cast as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} p^+ \\ p^- \\ \varepsilon \end{bmatrix} = \mathrm{i}k\mathbf{M}\begin{bmatrix} p^+ \\ p^- \\ \varepsilon \end{bmatrix}, \tag{22}$$

where

$$\mathbf{M} = \begin{bmatrix} \begin{matrix} -[\beta^2(\gamma-1)\overline{M}_0^3 - \beta\overline{M}_0^2\gamma \\ -(\gamma-1)\alpha\overline{M}_0^2(2-\overline{M}_0) \\ +2(\overline{M}_0-1)]/2D \end{matrix} & \begin{matrix} -\overline{M}_0^2[\beta^2(\gamma-1)\overline{M}_0 \\ -\beta(2-\gamma) \\ -\alpha(\gamma-1)\overline{M}_0]/2D \end{matrix} & \begin{matrix} -\overline{M}[\beta^2\overline{M}_0^2(\gamma-1) \\ +\beta(1-\overline{M}_0) \\ -\alpha\overline{M}_0(\gamma-1)(\overline{M}_0-1)]/2D \end{matrix} \\ \\ \begin{matrix} -\overline{M}_0^2[\beta^2(\gamma-1)\overline{M}_0 \\ +\beta(2-\gamma) \\ -\alpha(\gamma-1)\overline{M}_0]/2D \end{matrix} & \begin{matrix} -[\beta^2(\gamma-1)\overline{M}_0^3 + \beta\overline{M}_0^2\gamma \\ -(\gamma-1)\alpha\overline{M}_0^2(2+\overline{M}_0) \\ +2(\overline{M}_0+1)]/2D \end{matrix} & \begin{matrix} -\overline{M}[\beta^2\overline{M}_0^2(\gamma-1) \\ +\beta(1+\overline{M}_0) \\ -\alpha\overline{M}_0(\gamma-1)(\overline{M}_0+1)]/2D \end{matrix} \\ \\ \begin{matrix} (\gamma-1)\overline{M}[\beta^2\overline{M}_0^2 \\ +\beta(1-\overline{M}_0) \\ -\alpha\overline{M}_0(\overline{M}_0-1)]/D \end{matrix} & \begin{matrix} (\gamma-1)\overline{M}[\beta^2\overline{M}_0^2 \\ +\beta(1+\overline{M}_0) \\ -\alpha\overline{M}_0(\overline{M}_0+1)]/D \end{matrix} & \begin{matrix} [\beta^2(\gamma-1)\overline{M}_0^4 + \beta\overline{M}_0^2\gamma \\ -(\gamma-1)\alpha\overline{M}_0^4 \\ +(1-\overline{M})(1+\overline{M}_0)]/\overline{M}_0 D \end{matrix} \end{bmatrix}, \tag{23}$$

$$D = \beta\overline{M}_0^2\gamma - \alpha(\gamma-1)\overline{M}_0^2 + (1-\overline{M}_0)(1+\overline{M}_0). \tag{24}$$

The general solution of Eq. (22) can be expressed in the wave transfer form as (the time dependence of the state variables being suppressed)

$$\begin{bmatrix} p^+(x) \\ p^-(x) \\ \varepsilon(x) \end{bmatrix} = \mathbf{T}(x)\begin{bmatrix} p^+(0) \\ p^-(0) \\ \varepsilon(0) \end{bmatrix}, \tag{25}$$

where the transfer matrix is given by

$$\mathbf{T}(x) = \mathbf{\Phi}\begin{bmatrix} e^{ikK^+x} & 0 & 0 \\ 0 & e^{ikK^-x} & 0 \\ 0 & 0 & e^{ik\mu x} \end{bmatrix}\mathbf{\Phi}^{-1}. \tag{26}$$

Here, K^+, K^- and μ denote the propagation constants, which are equal to the eigenvalues of matrix $\mathbf{M}$, and $\mathbf{\Phi}$ is the modal matrix whose columns are the corresponding eigenvectors. Again, after some cumbersome algebra, the propagation constants can be shown to be given by the roots of the polynomial

$$Q(\lambda) = 2(M\lambda - 1)[[1 + (\gamma\beta - 1)\overline{M}_0^2]\lambda^2 + 2\overline{M}_0\lambda - 1] - 2(\gamma - 1)\alpha\overline{M}_0^3\lambda^3. \tag{27}$$

Since this is a cubic polynomial, in theory, the propagation constants can be expressed analytically. However, except for the case $\alpha = 0$, the roots of Eq. (27) do not come out in useful analytic forms. The case $\alpha = 0$, however, allows factorization and simple expressions are found for the roots. Fortunately, this case still covers the standard mean flow profile types and will be considered separately in what follows. The analysis is limited to subsonic mean flow velocities.

3.1. Uniform mean flow

It is convenient to begin with this ideal case, for which $\beta = 0$ and $\chi = 0$, and the matrix $\mathbf{M}$ reduces to

$$\mathbf{M} = \begin{bmatrix} 1/(1+\overline{M}_0) & 0 & 0 \\ 0 & -1/(1-\overline{M}_0) & 0 \\ 0 & 0 & 1/\overline{M}_0 \end{bmatrix}. \tag{28}$$

Therefore, in this case, the propagation constants are simply,

$$K^+ = \frac{1}{1+\overline{M}_0}, \quad K^- = \frac{-1}{1-\overline{M}_0}, \quad \mu = \frac{1}{\overline{M}_0}, \tag{29}$$

and the modal matrix $\mathbf{\Phi}$ (the first, second and third columns of which always correspond to K^+, K^- and μ, respectively, in this paper) constitutes a unit matrix. Therefore, $K^+$and K^- are the classical propagation constants quoted in the Introduction and correspond to acoustic waves traveling in forward ($+x$) and backward directions, respectively, with the velocity of speed of sound relative to the mean flow. The third propagation constant, μ, represents a hydrodynamic wave that travels with the mean flow. In this case it is trivial, as the propagation is isentropic, that is, $p' = c_0^2\rho'$, and, therefore, $\varepsilon = 0$.

Eq. (25) shows that the plane sound wave field in a duct carrying a sheared flow will, in general, consist of superimposed waves having the propagation constants K^+, K^- and μ. In the case of a uniform mean flow, the forward and backward sound waves are uncoupled and, therefore, can be associated physically with sources and boundary reflections, respectively. In the case of a sheared mean flow, however, there will always be some degree of coupling, which implies that the plane sound wave field generated by a source will, strictly speaking, propagate with continuous reflections due to cross-sectional inhomogeneity of a sheared mean flow. The cases considered next show that under certain conditions such reflections are weak and their effect on the plane sound wave field may not be discernible.

3.2. '1/n'th power-law turbulent flow

The '$1/n$'th power law is commonly used in modelling the velocity profiles in turbulent flows. The profile parameters β, χ and α for this type of a flow velocity profile are shown in Fig. 1 as functions of n. It is seen that the parameter α tends to zero as n is increases. For $n > 5$, which is the usual application range of the '$1/n$' power law, the approximation $\alpha = 0$ holds true with less than 0.2% absolute error. Thus, for this type of a mean flow velocity profile, the present theory can be implemented, without any significant loss of accuracy, by taking $\alpha = 0$. This is convenient, as, for $\alpha = 0$, the matrix $\mathbf{M}$ simplifies to

$$\mathbf{M} = \begin{bmatrix} \begin{matrix} -[\beta^2(\gamma-1)\overline{M}_0^3 - \beta\overline{M}_0^2\gamma \\ +2(\overline{M}_0 - 1)]/2D \end{matrix} & \begin{matrix} -\overline{M}_0^2[\beta^2(\gamma-1)\overline{M}_0 \\ -\beta(2-\gamma)]/2D \end{matrix} & \begin{matrix} -\overline{M}[\beta^2\overline{M}_0^2(\gamma-1) \\ +\beta(1-\overline{M}_0)]/2D \end{matrix} \\ \begin{matrix} -\overline{M}_0^2[\beta^2(\gamma-1)\overline{M}_0 \\ +\beta(2-\gamma)]/2D \end{matrix} & \begin{matrix} -[\beta^2(\gamma-1)\overline{M}_0^3 + \beta\overline{M}_0^2\gamma \\ +2(\overline{M}_0 + 1)]/2D \end{matrix} & \begin{matrix} -\overline{M}[\beta^2\overline{M}_0^2(\gamma-1) \\ +\beta(1+\overline{M}_0)]/2D \end{matrix} \\ \begin{matrix} (\gamma-1)\overline{M}[\beta^2\overline{M}_0^2 \\ +\beta(1-\overline{M}_0)]/D \end{matrix} & \begin{matrix} (\gamma-1)\overline{M}[\beta^2\overline{M}_0^2 \\ +\beta(1+\overline{M}_0)]/D \end{matrix} & \begin{matrix} [\beta^2(\gamma-1)\overline{M}_0^4 + \beta\overline{M}_0^2\gamma \\ +(1-\overline{M})(1+\overline{M}_0)]/\overline{M}_0 D \end{matrix} \end{bmatrix}, \quad (30)$$

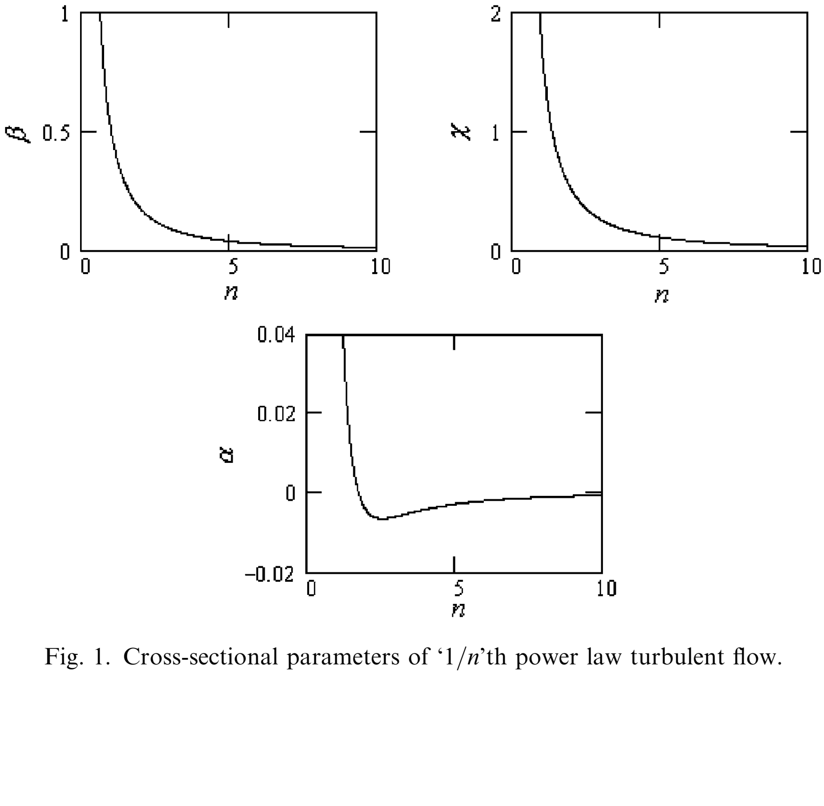


Fig. 1. Cross-sectional parameters of '$1/n$'th power law turbulent flow.

where $D = 1 - (1 - \gamma\beta)\overline{M}_0^2$, and the propagation constants are given by

$$K^+ = \frac{1}{\sqrt{1 + \gamma\beta\overline{M}_0^2} + \overline{M}_0}, \tag{31}$$

$$K^- = \frac{-1}{\sqrt{1 + \gamma\beta\overline{M}_0^2} - \overline{M}_0}, \tag{32}$$

$$\mu = \frac{1}{\overline{M}_0}. \tag{33}$$

Here, $\overline{M}_0 = 2n^2 M_0/(1+n)(1+2n)$, where M_0 denotes the Mach number of the mean flow velocity at the duct centre. The corresponding modal matrix $\mathbf{\Phi}$, however, does not lend itself to an analytical form that is simple enough for presentation here. However, some general results can be deduced by numerical analysis of the eigenvectors of matrix $\mathbf{M}$. For this purpose, it suffices to consider the parameter β in the range $\beta < 1/3$, as β remains in this range for $n > 5$ and the larger the β, the greater the deviation of the eigenvectors from unit vectors. For $n > 5$, therefore, the largest deviation occurs for $\beta = 1/3$. The modal matrix corresponding to this most critical case is *for* $M_0 = 0.1$:

$$\mathbf{\Phi} = \begin{bmatrix} 1 & -4.977 \times 10^{-4} & -1.661 \times 10^{-3} \\ -4.977 \times 10^{-4} & 1 & -1.661 \times 10^{-3} \\ -1.331 \times 10^{-3} & -1.331 \times 10^{-3} & 1 \end{bmatrix}, \tag{34}$$

and for $\overline{M}_0 = 0.7$:

$$\mathbf{\Phi} = \begin{bmatrix} 0.998 & -0.02 & -0.07 \\ -0.02 & 0.998 & -0.07 \\ -0.06 & -0.06 & 0.995 \end{bmatrix}, \tag{35}$$

where the usual value of $\gamma = 7/5$ is used for the ratio of the specific heat coefficients. For relatively small $\overline{M}_0$, the modal matrix $\mathbf{\Phi}$ retains the unit matrix form, as expected. For $\overline{M}_0 = 0.7$, the deviation of the modal matrix from the unit matrix is less than 6% for the acoustic modes, which are characterized by the propagation constants K^+ and K^-, and less than 7% for the hydrodynamic mode. This shows that, for a turbulent mean flow having a '$1/n$' power-law velocity profile, Eq. (22) can be uncoupled without significant loss of accuracy and its solution can be expressed as

$$p^+(x) = \mathrm{e}^{\mathrm{i}kK^+x} p^+(0), \quad p^-(x) = \mathrm{e}^{\mathrm{i}kK^-x} p^-(0), \quad \varepsilon(x) = \mathrm{e}^{\mathrm{i}kx/\overline{M}_0}\varepsilon(0), \tag{36}$$

up to subsonic Mach numbers as high as $\overline{M}_0 = 0.7$ or 0.8. Thus, the K^+ and K^- waves can be characterized as forward and backward traveling acoustic waves, respectively, as in the case of a uniform mean flow velocity profile. The hydrodynamic wave can propagate at speeds comparable with the speed of sound for relatively high subsonic $\overline{M}_0$. Note that, to this approximation, these waves exist if they exist at the origin, $x = 0$, say. Therefore, if the propagation is isentropic at the origin, then there will be no hydrodynamic wave.

3.3. Laminar mean flow

For a Poiseuille-type laminar mean flow having a parabolic velocity profile, $\alpha = 0$ and $\beta = 1/3$ and, therefore, this case is included in the foregoing results given for '$1/n$'th power law turbulent mean flow. The propagation constants for this case can be deduced from Eqs. (31)–(33) and are given explicitly for reference purposes:

$$K^{\mp} = \frac{\mp 1}{\sqrt{1 + \gamma \overline{M}_0^2/3} \mp \overline{M}_0}, \quad \mu = \frac{1}{\overline{M}_0}. \tag{37}$$

Here, $\overline{M}_0 = M_0/2$, where M_0 denotes the Mach number of the mean flow velocity at the duct centre.

3.4. Core flow

For a uniform duct carrying an axially uniform core flow, the present theory can be applied by taking $\beta = S/S_c - 1$, $\chi = (S/S_c)^2 - 1$ and $\overline{M}_0 = (S_c/S)M_c$, where S and S_c denote the cross-sectional area of the duct and the core, respectively, and M_c denotes the Mach number of the core flow velocity. Elimination of the ratio S/S_c between β and χ yields the relationship $\alpha = \beta(\beta - 1)/2$ and, therefore, the results of Section 3.2 can be invoked in this case if $S/S_c = 2$. This area ratio, however, corresponds to $\beta = 1$, and the solution of Eq. (22) will not decouple for the K^+, K^- and μ waves unless the Mach number of the core flow velocity is low enough. A numerical analysis similar to that shown in Section 3.2 shows that, for $\beta = 1$, the deviation of the modal matrix from the unit matrix form is less than about 6% for $\overline{M}_0 = 0.35$, or core flow velocity Mach number of $M_c = 0.7$.

Analysis of ducts having core flow of area ratio other than $S/S_c = 2$ requires the numerical solution of Eq. (27) for the propagation constants. The results of such solutions are presented in Figs. 2 and 3 for $\beta = 0$–6 and for $\overline{M}_0 = 0$–0.8. As can be seen, whilst K^+ decreases approximately linearly with β and $\overline{M}_0$, the backward wave propagation constant, K^-, is not much sensitive to these parameters for $\beta > 2$ or 3, and the hydrodynamic wave propagation constant, μ, for $\beta < 2$ or 3.

To show the degree of coupling between the K^+, K^- and μ waves in these cases, the elements of the columns of the modal matrix $\mathbf{\Phi}$ are given in Figs. 4 and 5 as functions of β and $\overline{M}_0$. These results show that, for $\overline{M}_0 < 0.1$ or 0.2, the modal matrix can be approximated by the unit matrix for all practical purpose in the considered range of β. In this range of the parameters, solution of Eq. (22) may be approximated by Eq. (36). Some coupling is observed for relatively larger values of β and $\overline{M}_0$; however, the eigenvectors corresponding to the propagation constants K^- and μ are rather insensitive to changes in these parameters.

4. Conclusion

A quasi-1-D theory of plane sound wave propagation in a steady shear flow is presented. The governing equations are derived in a generality encompassing non-uniform ducts, compressible and

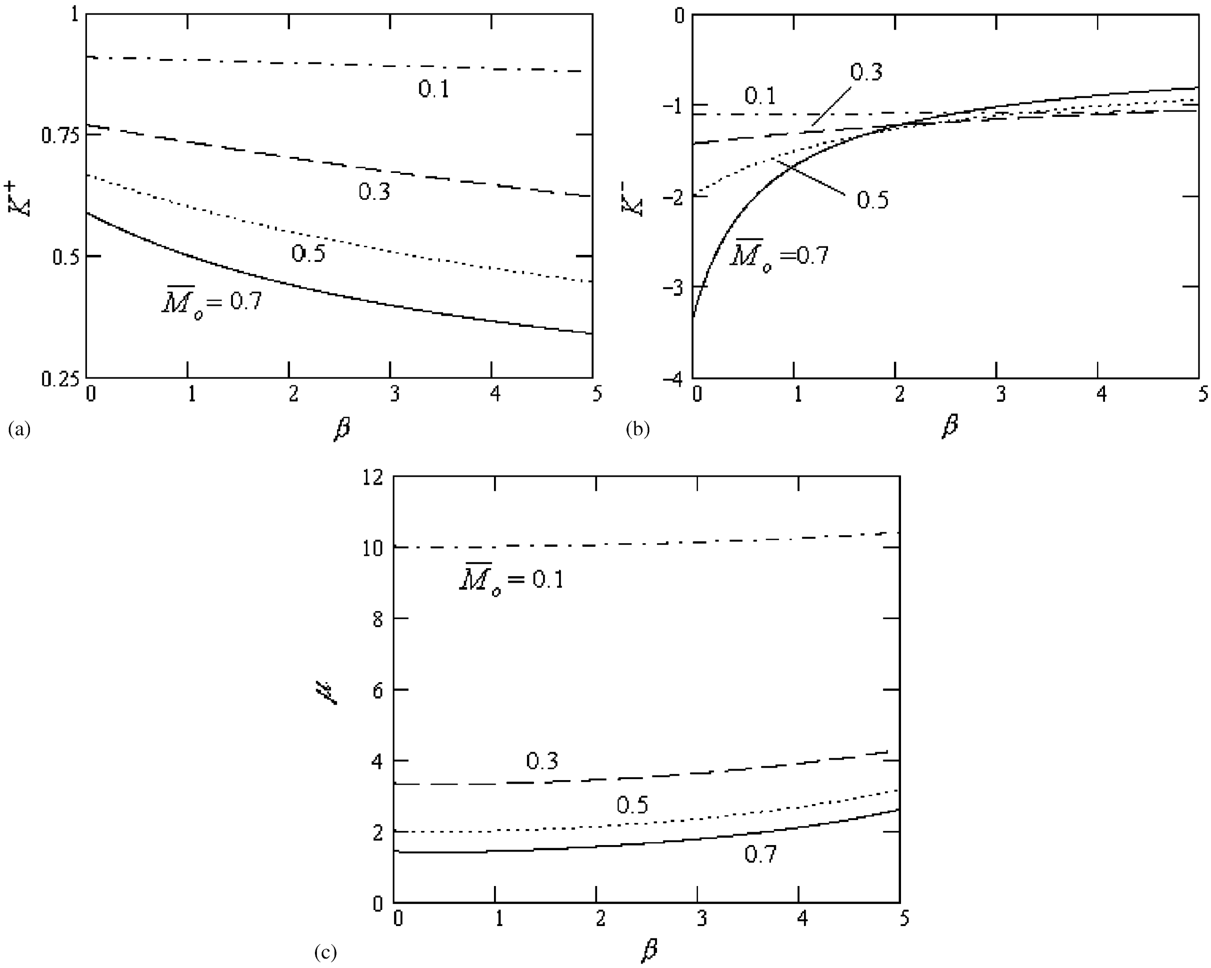


Fig. 2. Variation of the propagation constants with the parameter β: (a) K^+, (b) K^- and (c) μ.

axially non-uniform flows. An analytical solution of these equations is presented for uniform ducts carrying an incompressible mean flow. In this case, the characteristics of the plane wave field are determined by three parameters, that is, β, $\overline{M}_0$ and α, and the wave field consists of the superposition of forward and backward acoustic waves and a hydrodynamic wave, which occur as coupled waves. The propagation constants and the corresponding modal matrix that determines the degree of the coupling are analyzed with reference to applications to some standard mean flow profile shapes.

The present theory may be considered as an improvement of a previous analysis [2] which is based on the use of the isentropic state equation in place of the energy conservation law. The isentropic theory of Ref. [2] predicts an acoustic wave field consisting of uncoupled forward and backward waves, the propagation constants of which are given, for any mean flow velocity profile, by

$$K^{\mp} = \frac{\mp 1}{\sqrt{1 + \beta \overline{M}_0^2} \mp \overline{M}_0}. \tag{38}$$

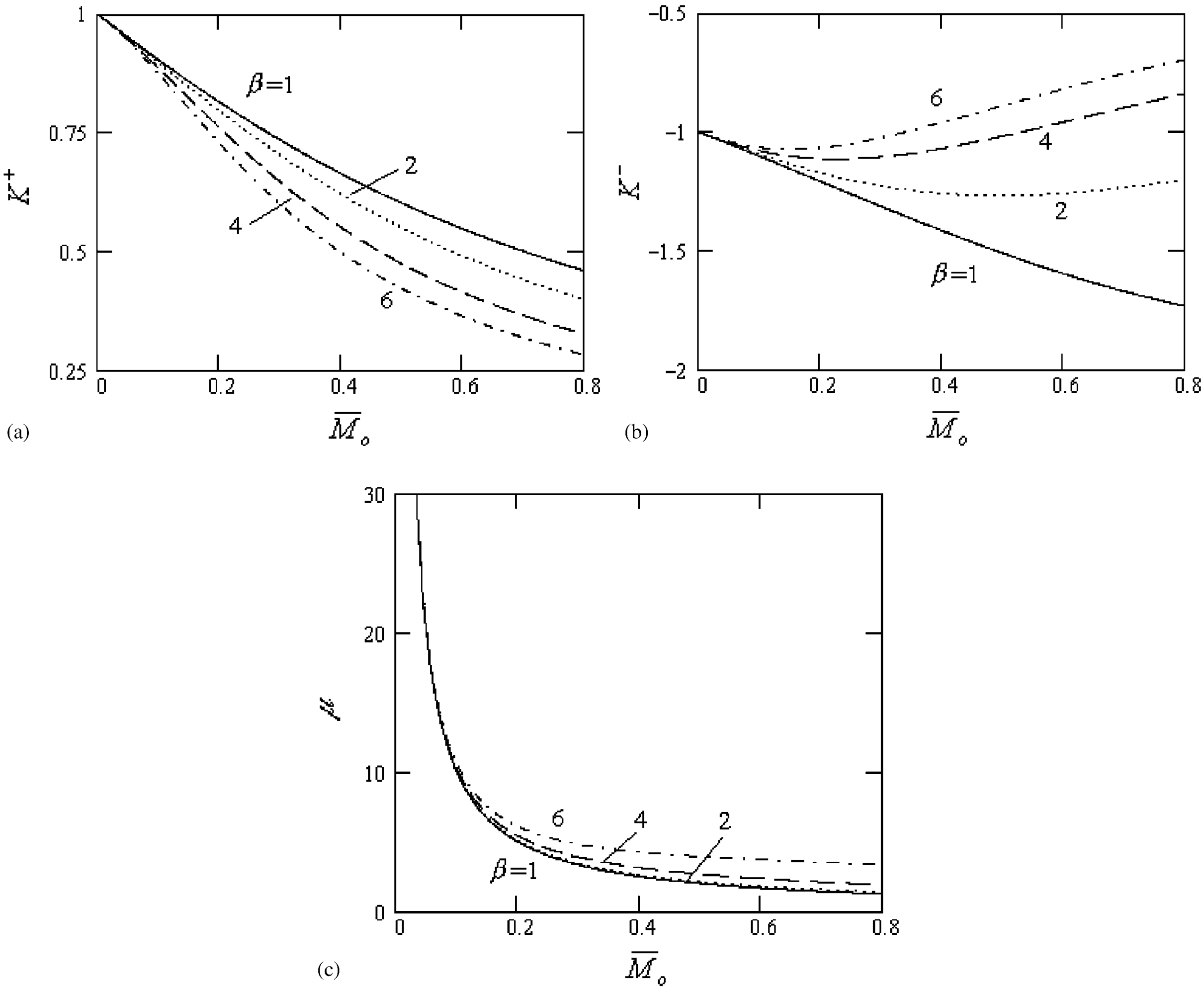


Fig. 3. Variation of the propagation constants with the Mach number of the cross-sectional average of mean flow velocity: (a) K^+, (b) K^- and (c) μ.

This is the same as Eqs. (31) and (32) with $\gamma = 1$. So, in the case of $\alpha = 0$, the improvement provided by the present theory pertains to amplification of β by about 30–40%. The effect of this on the propagation constants is negligible for relatively low subsonic $\overline{M}_0$, and barely discernible for larger Mach numbers. Therefore, for the examples considered in Ref. [2], all of which conform to the condition $\alpha = 0$, the present theory provides only slight improvements. For example, for a core flow with $\beta = 1$ and $\overline{M}_0 = 0.35$, the present theory predicts $K^+ = 0.698$ and $K^- = -1.365$, the corresponding predictions of Eq. (38) being 0.709 and -1.409, respectively. An exact value for K^+ that is based on a circular duct formulation is 0.68 [4].

In general, insofar as the prediction of the acoustic propagation constants is concerned, Eq. (38) can be considered to be a good approximation to the present theory. However, the present

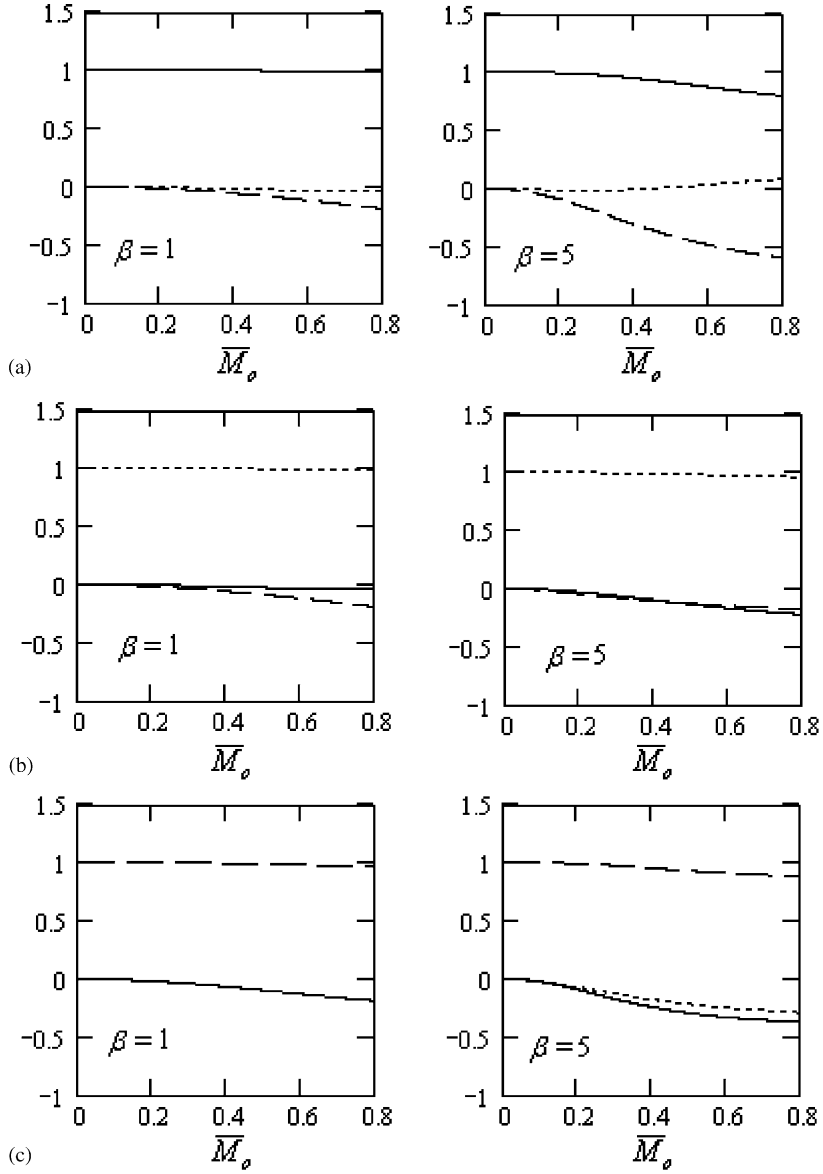
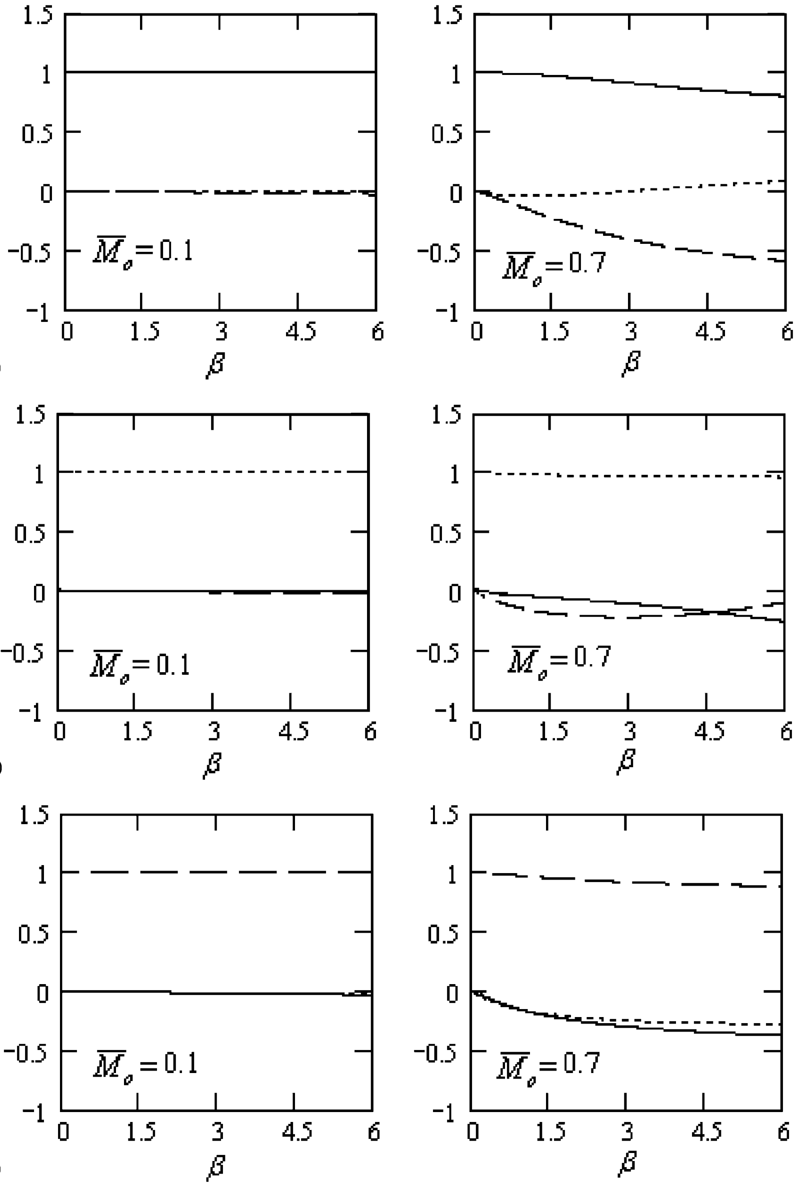


Fig. 4. Variation of the elements of the columns of modal matrix $\mathbf{\Phi}$ with the Mach number of the cross-sectional averaged mean flow velocity: (a) K^+ column, (b) K^- column, (c) μ column; ______ K^+ element, - - - - K^- element, – – – – μ element.

quasi-1-D theory is more accurate and provides a more complete representation of the plane wave field in a shear flow.

One of the referees has pointed out that the scale which influences the degree to which plane waves remain plane in a shear flow is the ratio of wavelength to something like boundary layer thickness, and queried the range of this ratio over which a plane wave assumption is reasonable.



(a) (b) (c)

Fig. 5. Variation of the elements of the columns of modal matrix $\mathbf{\Phi}$ with β: (a) K^+ column, (b) K^- column, (c) μ column; ______ K^+ element, - - - - K^- element, – – – μ element.

A rigorous treatment of this issue is strictly out of scope of the present analysis. However, the question is stimulating and warrants a comment beyond the statement that the propagation constants of the present theory are in close agreement with the results of 3-D solutions for the fundamental mode propagation. Refraction due to shear flow changes the uniform mean flow acoustic pressure distribution, with the pressure at the wall becoming higher than that at the center of the duct for forward propagation and lower for backward (against mean flow) propagation. A problem which can be adopted to study this effect in a reasonably simple setting is the problem of 2-D acoustic wave propagation in a linear boundary layer above a plane wall. This problem is considered by Pridmore-Brown [6]. His approximate solution enables calculation of the level difference between the sound pressure at the wall and at a distance L from the wall for the forward propagation of the lowest mode for a given Helmholtz number kL, when the Mach number is increasing linearly from 0 at the wall to M_1 at distance L from the wall. Strictly speaking, the approximate solution of Ref. [6] is valid asymptotically for large kL/M_1; however, the results given for level differences can be extrapolated to $kL = 0$. Thus extrapolated results indicate that, for the level difference to be less than 0.5 dB, the Helmholtz number must be less than about unity ($kL<1$) for $M_1 = 0.1$, and less than about 0.2 ($kL<0.2$) for $M_1 = 0.5$. So, it appears that the range of the ratio of the wavelength to boundary layer thickness over which the plane wave assumption is reasonable may be stated, approximately, as $\lambda/L>2\pi$ for $M_1 = 0.1$, and $\lambda/L>10\pi$ for $M_1 = 0.5$. The frequencies that are of interest in most fluid machinery are in general within these ranges; however, these preliminary results should be treated with caution, as they are based on the assumption that the results of Ref. [6] can be extrapolated to $kL = 0$.

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